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A HOMEOMORPHISM ON s NOT CONJUGATE TO AN EXTENDABLE HOMEOMORPHISM

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Dedicated to Profesor Yukihiro Kodama on his sixtieth birthday

ABSTRACT. Consider $s = \prod_{i=1}^{\infty}(-1, 1)_i$ and its compactification $Q = \prod_{i=1}^{\infty}[-1, 1]_i$. Anderson and Bing asked whether for every homeomorphism $f: s \to s$ there is a homeomorphism $\phi: s \to s$ such that $\phi^{-1}f\phi$ is extendable to a homeomorphism $\overline{\phi^{-1}f\phi}: Q \to Q$. The aim of this note is to construct a counterexample to this question.

1. INTRODUCTION

Consider $s = \prod_{i=1}^{\infty}(-1, 1)_i$ and its compactification

$$Q = \prod_{i=1}^{\infty}[-1, 1]_i.$$

A homeomorphism $f: s \to s$ is said to be *conjugate to an extendable homeomorphism* if there is a homeomorphism $\phi: s \to s$ such that $\phi^{-1}f\phi$ is extendable to a homeomorphism $\overline{\phi^{-1}f\phi}: Q \to Q$. In [1], Anderson and Bing asked whether every homeomorphism on s is conjugate to an extendable homeomorphism. Sakai and Wong [5] recently presented several conditions that imply that a homeomorphism on s satisfying one of them is conjugate to an extendable homeomorphism.

The corresponding finite-dimensional problem, replacing s by $\{x \in \mathbf{R}^n: \|x\| < 1\}$ and Q by $\{x \in \mathbf{R}^n: \|x\| \leq 1\}$, has a simple negative answer for every $n \geq 2$. For details, see [5]. We remark that Eric van Douwen communicated a similar solution to us at least 5 years ago.

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The aim of this note is to present an example of a homeomorphism on s which is not conjugate to an extendable homeomorphism, thereby answering the Anderson and Bing question in the negative.

2. The construction

Let X be a space. An *isotopy* on X is a homotopy $H: X \times I \to X$ such that the function $\phi: X \times I \to X \times I$ defined by

$$\phi(x, t) = (H(x, t), t), \qquad (x \in X, t \in I)$$

is a homeomorphism. Observe that if $H: X \times I \to X$ is an isotopy then for every $t \in I$ the function $H_t: X \to X$ defined by

$$H_t(x) = H(x, t), \qquad (x \in X)$$

is a homeomorphism.

A *Hilbert cube* is a space homeomorphic to Q. Let M be a Hilbert cube. A *capset* of M is a subset $A \subseteq M$ for which there is a homeomorphism $f: M \to Q$ such that $f(A) = Q \backslash s$. In addition, a closed set B of M is called a *Z-set* if for every $\varepsilon > 0$ there is a map $f: M \to M \backslash B$ with $d(f, 1_M) < \varepsilon$ (1_X means the identity function on X). It is known that if $A \subseteq M$ is a capset and $B \subseteq M$ is a Z-set then $A \backslash B$ is a capset (for details see [2] and [3]).

We shall now describe a special isotopy on s that shall be important later in the construction of the example.

2.1. **Proposition.** *There are an isotopy $H: s \times I \to s$ on s and two compact sets $A, B \subseteq s$ such that*

(1) $H_0 = 1_s$;
(2) *if $E \subseteq s$ is closed and misses A and if U is a neighborhood of B in s then there is an $n \in \mathbf{N}$ such that $(H_1)^n(E) \subseteq U$.*

Proof. There is an isotopy $F: I \times I \to I$ such that

(3) $F_0 = 1_I$,
(4) for every $t \in I$, $F_t(0) = 0$ and $F_t(1) = 1$,
(5) $\forall t \in (0, 1]$, $\lim_{n\to\infty}(F_1)^n(t) = 1$.

For example, F can be defined by $F(s, t) = (1-t)s + t\sqrt{s}$.

Put $\vec{Q} = Q \times I$. The isotopy F induces an isotopy $G: \vec{Q} \times I \to \vec{Q}$ as follows:

$$G((x, s), t) = (x, F_t(s)), \qquad (x \in Q, t, s \in I).$$

Since $Q \times \{0, 1\}$ is a Z-set in $\vec{Q}$ and $\vec{Q} \backslash (s \times (0, 1))$ is clearly a capset of $\vec{Q}$, by the remarks preceding Proposition 2.1, $X = (s \times (0, 1)) \cup (Q \times \{0, 1\})$ is the complement of a capset in $\vec{Q}$, and hence is homeomorphic to s. We shall construct the required isotopy on X.

Observe that by (4), $G(X \times I) = X$. Consequently, $H = G|(X \times I)$ is an isotopy on X. Define $A = Q \times \{0\}$ and $B = Q \times \{1\}$, respectively. We shall prove that H, A and B satisfy (1) and (2). Of course, only (2) needs verification. To this end, let $E \subseteq X$ be closed such that $E \cap A = \varnothing$, and let U be an open neighborhood of B in X. Then $\bar{E} \cap A = \varnothing$ ($\bar{E}$ is the closure of E in $\vec{Q}$) because A is compact. So there exists an element $t \in (0,1]$ with $\bar{E} \subseteq Q \times [t,1]$. Let $U' \subseteq \vec{Q}$ be open such that $U' \cap X = U$. There is an element $s \in (0,1)$ such that $Q \times [s,1] \subseteq U'$. By (5) there exists $n \in \mathbf{N}$ with $(F_1)^n(t) \geq s$. Then clearly $(G_1)^n(\bar{E}) \subseteq U'$, i.e. $(H_1)^n(E) \subseteq U$.

Now for $n \in \mathbf{N}$ let $x_n = (1, 1/n) \in \mathbf{R}^2$. The straight line segment connecting $(0,0)$ and x_n will be denoted by T_n. Let T denote the subspace $\bigcup_{n=1}^{\infty} T_n$ of $\mathbf{R}^2$. We define a homeomorphism $\xi: T \times s \to T \times s$ as follows. On $\{(0,0)\} \times s$, ξ is the homeomorphism H_1 of Propostion 2.1 (we make an obvious identification here). For every n, $\xi | \{x_n\} \times s$ is the identity. We use the isotopy of Proposition 2.1 to "connect" $\xi | \{(0,0)\} \times s$ and $\xi | \{x_n\} \times s$ for every n. Clearly, the resulting function is a homeomorphism.

The space T is contractible, locally contractible and 1-dimensional. Consequently, T is an **AR** ([4]). Since T is also topologically complete, by a theorem of Toruńczyk [6], $T \times s$ is homeomorphic to s; let $h: s \to T \times s$ be any homeomorphism. Then

$$\eta = h^{-1} \circ \xi \circ h : s \to s$$

is a homeomorphism and we claim that it is as desired.

2.2. **Proposition.** *η is not conjugate to an extendable homeomorphism.*

Proof. To the contrary, assume that there exists a homeomorphism $f: s \to s$ such that $f^{-1}\eta f = f^{-1}h^{-1}\xi h f$ is extendable to a homeomorphism $\rho: Q \to Q$. It will be convenient to let g denote the composition $h \circ f : s \to T \times s$. Let $\underline{0}$ be the point in s all coordinates of which are equal to 0. For every n, let $p_n = g_n^{-1}(x_n, \underline{0})$ and let $p \in Q$ be a limit point of the sequence $(p_n)_n$. Observe that $\{p_n : n \in \mathbf{N}\}$ is closed in s so that $p \notin s$. Also observe that every p_n is a fixed point of $g^{-1}\xi g$ from which it follows that p is a fixed point of ρ.

Claim. $p \notin \overline{g^{-1}(\{\underline{0}\} \times s)}$.

To the contrary, assume that $p \in \overline{g^{-1}(\{\underline{0}\} \times s)}$. Since A and B are compact and $p \notin g^{-1}(\{\underline{0}\} \times s)$ there are open neighborhoods U and V of A and B in s, respectively, such that

$$p \notin \overline{g^{-1}(\{\underline{0}\} \times (U \cup V))}.$$

Put $E = s \setminus (U \cup V)$. Then E is closed in s, misses A, and clearly has the property that $p \in \overline{g^{-1}(\{\underline{0}\} \times E)}$. By Proposition 2.1 there exists $n \in \mathbf{N}$ such that $(H_1)^n(E) \subset V$. This implies that

$$\begin{aligned}(g^{-1}\xi g)^n(g^{-1}(\{\underline{0}\} \times E)) &= g^{-1}\xi^n g(g^{-1}(\{\underline{0}\} \times E)) \\ &= g^{-1}\xi^n(\{\underline{0}\} \times E) \subseteq g^{-1}(\{\underline{0}\} \times V).\end{aligned}$$

Consequently,

$$\rho^n(p) \in \overline{\rho^n(g^{-1}(\{\underline{0}\} \times E))} \subseteq \overline{g^{-1}(\{\underline{0}\} \times V)} \subseteq Q \backslash \{p\}.$$

But this contradicts the fact that p is a fixed point of ρ.

By the claim there is an open neighborhood U of p in Q such that

(1) $U \cap (g^{-1}(\{\underline{0}\} \times s)) = \varnothing$;

(2) $U \cap s$ is connected

(simply let U be a basic open subcube of Q containing p but missing $g^{-1}(\{\underline{0}\} \times s))$. Observe that $U \cap s$ contains infinitely many p_n's. Consequently, $g(U \cap s)$ is a connected open subset of $T \times s$ which misses $\{\underline{0}\} \times s$ and contains infinitely many $(x_n, \underline{0})$'s. This is clearly a contradiction.

2.3. **Question.** Is every homeomorphism on s the composition of two conjugates of extendable homeomorphisms?

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