Countably Compact Groups with Non-Countably-Compact Products

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0 INTRODUCTION

The question whether the product of countably compact (topological) groups is countably compact is an old and a natural one.

To put it into perspective consider the following three topological properties: compactness, countable compactness and pseudocompactness.

Of course the product of compact spaces is again compact. The other two properties do not behave so well with respect to products: in (1953) Novák constructed two countably compact spaces X and Y such that $X \times Y$ is not even pseudocompact.

For topological groups the situation is a bit better: in (1966) Comfort and Ross proved that any product of pseudocompact groups is pseudocompact, thus showing that a group structure can have considerable effect on the product behaviour of topological properties.

This left open the question whether there are countably compact groups G and H with $G \times H$ not countably compact. In (1980) van Douwen constructed such groups assuming **M**(artin's) **A**(xiom).

As pointed out by van Douwen, his construction can not work without some form of **MA**. To be precise, the combination of $2^{\omega} = 2^{\omega_1}$ and an ultrafilter on ω of character ω_1 makes his construction impossible.

We present here a new construction which needs a much weaker form of **MA**, but which is still impossible under the conditions mentioned above.

This note is organized as follows:

In section 1 we sketch the constructions by Novák and van Douwen, and we discuss their similarities. In section 2 we describe our construction and point out the differences with those in section 1. In section 3 we elaborate on the obstruction mentioned above, that prevents us from doing our construction in **ZFC**.

Undefined terms can be found in (Engelking, 1977) and (Comfort, 1984). Other notions will be defined when needed. We let $\mathfrak{c} = 2^{\omega}$, the cardinality of $\mathbb{R}$.

1 NOVÁK AND VAN DOUWEN

The key idea in these constructions is to start with a countably compact space Z without converging sequences. It is easy to see that then for every countably infinite set $F \subseteq Z$ we have $|\overline{F}| \geq \mathfrak{c}$.

One then constructs two countably compact subspaces X and Y such that $D = X \cap Y$ is countably infinite. Then $\Delta D = \{ \langle d, d \rangle : d \in D \}$ is a closed subset of $X \times Y$: it is the intersection of ΔZ and $X \times Y$. Certainly ΔD is not countably compact, being countably infinite; it follows that $X \times Y$ is not countably compact.

1.0 Novák

In (1953) Novák starts with the compact space $Z = \beta\omega$. In this case we even have $|\overline{F}| = 2^{\mathfrak{c}}$, whenever $F \subseteq Z$ is countably infinite. It is then not hard to construct $X, Y \subseteq Z$ with X and Y countably compact and with $X \cap Y = \omega$. In this case $\Delta\omega$ is even a clopen discrete subspace of $X \times Y$, ensuring that $X \times Y$ is not even pseudocompact.

1.1 van Douwen

In (1980) van Douwen constructs, assuming **MA**, a countably compact subgroup S of the group ${}^{\mathfrak{c}}2$, without converging sequences. A relatively straightforward modification of Novák's arguments then gives us in **ZFC** countably compact subgroups G and H of S with $G \cap H$ countably infinite.

Note that since we are working inside ${}^{\mathfrak{c}}2$ these groups are topological groups automatically.

1.2 Squares

Novák's construction immediately gives us a countably compact space with a non-pseudocompact square: $X \oplus Y$.

For topological groups things are not so easy: try to make a topological group out of $G \oplus H$. In (1980) van Douwen did announce the construction of a single countably compact group K with K^2 not countably compact, but this example was never published.

2 A NEW CONSTRUCTION

In this section we outline the construction from (Hart and van Mill, 1988) of two countably compact subgroups H_1 and H_2 of ${}^{\mathfrak{c}}2$ such that $H_1 \times H_2$ is not countably compact. We will also indicate how to modify this construction so as to yield a countably compact group H with H^2 not countably compact.

2.0 A picture of H_1 and H_2

Actually we construct three subgroups D, G_1 and G_2 of ${}^{\mathfrak{c}}2$ satisfying:

1) D is countably infinite,
2) G_1 and G_2 are ω-bounded,
3) if $d \in D, g_1 \in G_1$ and $g_2 \in G_2$, and if $d + g_1 + g_2 = \underline{0}$ then $d = g_1 = g_2 = \underline{0}$, and
4) if $E \subseteq D$ is infinite then it has accumulation points in G_1 and G_2.

We let $H_1 = D + G_1$ and $H_2 = D + G_2$. Then $H_1 \cap H_2 = D$ by 3), and H_1 and H_2 are countably compact by 2) and 4).

Note: a space is called ω-bounded if every countable subset of it has compact closure.

2.1 A sketch of the construction

To begin we take a collection $\mathcal{A}$ of size $\mathfrak{c}$ of subsets of $\mathfrak{c}$ satisfying:
(*) if $\mathcal{A}', \mathcal{A}'' \subseteq \mathcal{A}$ are countable and disjoint then $|\bigcap \mathcal{A}' \setminus \bigcup \mathcal{A}''| = \mathfrak{c}$.
Such a family is called ω-independent.

Split $\mathcal{A}$ into two disjoint parts $\mathcal{A}_1$ and $\mathcal{A}_2$ both of size $\mathfrak{c}$.

We let for $i = 1,2$ G_i be the smallest ω-bounded subgroup of ${}^{\mathfrak{c}}2$ containing $\mathcal{A}_1$ (here we identify a set $X \subseteq \mathfrak{c}$ with its characteristic function $\chi_X : \mathfrak{c} \to 2$). Then 3) already holds for G_1 and G_2.

The main problem then is to construct D. The points of D are constructed in an induction of length $\mathfrak{c}$, one coordinate at a time.

We let $I = \{ x \in {}^{\omega}2 : |x^{\leftarrow}(1)| < \omega \}$ and we will have $D = \{ d_x : x \in I \}$ with $d_x \restriction \omega = x$ and $d_{x+y} = d_x + d_y$, for all x and y in I.

At every stage $\alpha \geq \omega$ we must find $d_x(\alpha)$ for all $x \in I$. Note that the map $x \mapsto d_x(\alpha)$ is to be a homomorphism from I to 2. At the same time we begin to make sure that 4) will hold for one more infinite $E \subseteq D$. This gives us each time fewer than $\mathfrak{c}$ conditions to satisfy. Every such condition determines a dense open set in the space *Hom* of homomorphisms from I to 2. Now *Hom* is homeomorphic to the Cantor set, so we may apply the weakest form of **MA**:

"The real line can not be covered by fewer than $\mathfrak{c}$ nowhere dense sets,"

to find a homomorphism satisfying all conditions.

Some minor modifications have to be made to achieve 3) (and in all honesty also 4)), but this is the main idea.

2.2 Towards one group

The idea here is to consider the subgroups $H_1' = H_1 \times \{\underline{0}\}$ and $H_2' = \{\underline{0}\} \times H_2$ of ${}^{\mathfrak{c}}2 \times {}^{\mathfrak{c}}2$. Now $H_1' + H_2'$ being homeomorhic to $H_1 \times H_2$ is not a countably compact group, but we can throw in an extra ω-bounded subgroup G_3 of ${}^{\mathfrak{c}}2 \times {}^{\mathfrak{c}}2$ with $G_3 \cap (H_1' + H_2') = \{\underline{0}\}$ and such that $H = G_3 + H_1' + H_2'$ is countably compact.

Then $H_1 \times H_2$ is a closed subgroup of H^2 so that H^2 is not countably compact.

Again we have sketched the main idea of the construction; the details of 2.1 and 2.2 appear in (Hart and van Mill, 1988).

2.3 Differences

Here we want to point out the differences between van Douwen's example and ours.

To begin the strategies of the respective construction are different: van Douwen begins with a countable group D and in an induction of length $\mathfrak{c}$ extends this group to two countably compact groups whose intersection is D. We on the other hand have our two groups almost ready, except for the countable intersection.

Also the resulting groups are quite different: van Douwen's groups have no non-trivial converging sequences, whereas our groups are full of them. Every countably infinite subset of either G_1 or G_2 is contained in a compact subgroup of G_1 or G_2 respectively, and compact groups contain many non-trivial converging sequences.

We see that, although countably compact spaces with a non-countably compact product must contain sequences without converging subsequences, they still may contain lots of non-trivial converging sequences.

3 WHY MA?

Both in the construction by van Douwen and in ours some form of **MA** seems to be necessary:

First define a space to be initially ω_1-compact if every open cover of it of size at most ω_1 has a finite subcover.

If one assumes $2^{\omega} = 2^{\omega_1}$ in addition to the form of **MA** needed for the construction then in both cases one can construct the groups in such a way that they become initially ω_1-compact, without any essential modifications. Thus we get two initially ω_1-compact groups whose product is not even countably compact.

On the other hand it is consistent with all cardinal arithmetic that every product of initially ω_1-compact spaces is countably compact. We sketch the argument (see [vD]):

Assume u is an ultrafilter on ω of character ω_1. By Kunen (1980; VIII.A10) the existence of such an ultrafilter is consistent with any consistent cardinal arithmetic.

Let X be initially ω_1-compact and let $\langle x_n : n \in \omega \rangle$ be a sequence in X; since u has character ω_1 and since X is initially ω_1-compact

$$\bigcap_{U \in u} \overline{\{ x_n : n \in U \}} \neq \emptyset.$$

Actually since we tacitly have been assuming that all our spaces are Hausdorff there is exactly one point x in this intersection. We write $x = u\text{-}\lim x_n$.

Now if $\{ X_i : i \in I \}$ is any family of initially ω_1-compact spaces and $\langle x_n : n \in \omega \rangle$ is a sequence in $\prod_{i \in I} X_i$ then $u\text{-}\lim x_n$ exists and is an accumulation point of the set $\{ x_n : n \in \omega \}$.

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