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AN INFINITE-DIMENSIONAL HOMOGENEOUS INDECOMPOSABLE CONTINUUM

JAN VAN MILL

Abstract. We prove that every homogeneous continuum is an open retract of a non-metric homogeneous indecomposable continuum.

0. Introduction. A continuum X is *indecomposable* if it cannot be written as the union of two proper subcontinua. Examples of 1-dimensional *homogeneous* indecomposable continua are the pseudo-arc and the solenoids. J. T. Rogers asked whether there is an example of a homogeneous indecomposable *metric* continuum of dimension greater than 1 (see [**1**]). The aim of this note is to show that every homogeneous continuum is an open retract of a *non-metric* homogeneous indecomposable continuum. Consequently, we leave Rogers' question unanswered but prove that the condition on metrizability in his question is essential.

1. The spaces dX. Throughout, X denotes a compact Hausdorff space. In this section we shall associate to X a certain compactum dX that will be the first step in an inverse limit construction later.

For every function $f \in \{0,1\}^X$ and $x \in X$, let $f_x : X \to \{0,1\}$ be the function defined by

$$f_x(y) = f(y), \quad \text{for } y \neq x,$$

and

$$f_x(x) = 0 \quad \text{iff } f(x) = 1.$$

Notice that $(f_x)_x = f$ for every $f \in \{0,1\}^X$. Now consider $Y = X \times \{0,1\}^X$ and put $\mathcal{G} = \{\{\langle x, f\rangle, \langle x, f_x\rangle\} : x \in X\}$. We shall prove that $\mathcal{G}$ is an upper semicontinuous decomposition of Y.

1.1. LEMMA. *$\mathcal{G}$ is a decomposition of Y. In addition, if U and N are disjoint subsets of X and if $f \in \{0,1\}^N$, then $U \times \{g \in \{0,1\}^X : g \mid N = f\}$ is $\mathcal{G}$-saturated.*

PROOF: Let $\{\langle x, f\rangle, \langle x, f_x\rangle\}$ and $\{\langle y, g\rangle, \langle y, g_y\rangle\}$ be elements of $\mathcal{G}$ that intersect. Clearly, $x = y$. If there exists $p \in X\backslash\{x\}$ such that $f(p) \neq g(p)$, then $\{\langle x, f\rangle, \langle x, f_x\rangle\} \cap \{\langle y, g\rangle, \langle y, g_y\rangle\} = \emptyset$, which is not the case. Consequently, $f \mid X\backslash\{x\} = g \mid X\backslash\{x\}$, which easily implies that $\{\langle x, f\rangle, \langle x, f_x\rangle\} = \{\langle y, g\rangle, \langle y, g_y\rangle\}$.

Now let $U, N \subseteq X$ be disjoint and $f \in \{0,1\}^N$. Take a point $\langle x, g\rangle \in V = U \times \{g \in \{0,1\}^X : g \mid N = f\}$ arbitrarily. Since $x \in U$, $x \notin N$, so that $g_x \mid N = g \mid N = f$; it easily follows that $\langle x, g_x\rangle \in V$. ∎

1.2. LEMMA. *$\mathcal{G}$ is upper semicontinuous.*

PROOF: Let $A \subseteq Y$ be closed. We claim that $\mathcal{G}(A) = \bigcup\{G \in \mathcal{G} : G \cap A \neq \emptyset\}$ is a closed subset of Y. To this end, take $\langle x, f\rangle \notin \mathcal{G}(A)$. Observe that this implies that $\{\langle x, f\rangle, \langle x, f_x\rangle\} \cap A = \emptyset$. Since $\langle x, f\rangle \notin A$ and A is closed, there is a neighborhood of U_0 of x and a finite subset N_0 of X such that

$$\langle x, f\rangle \in U_0 \times \{g \in \{0,1\}^X : f \mid N_0 = g \mid N_0\}, \tag{1}$$

$$(U_0 \times \{g \in \{0,1\}^X : f \mid N_0 = g \mid N_0\}) \cap A = \emptyset. \tag{2}$$

Similarly, since $\langle x, f_x\rangle \notin A$, there is a neighborhood U_1 of x and a finite subset N_1 of X such that

$$\langle x, f_x\rangle \in U_1 \times \{g \in \{0,1\}^X : f_x \mid N_1 = g \mid N_1\}, \tag{3}$$

$$(U_1 \times \{g \in \{0,1\}^X : f_x \mid N_1 = g \mid N_1\}) \cap A = \emptyset. \tag{4}$$

Put $N = (N_0 \cup N_1)\backslash\{x\}$ and $U = (U_0 \cap U_1)\backslash N$. Observe that U is a neighborhood of x. Finally, put $E = U \times \{g \in \{0,1\}^X : f \mid N = g \mid N\}$. By Lemma 1.1, E is $\mathcal{G}$-saturated.

Claim: $E \cap A = \emptyset$. Let $\langle y, g\rangle \in (U \times \{g \in \{0,1\}^X : f \mid N = g \mid N\})$. Since $g(x) \in \{f(x), f_x(x)\}$, there are two cases:

Case 1: $g(x) = f(x)$. Then $g \mid N_0 = f \mid N_0$ which implies that $\langle y, g\rangle \notin A$ by (2).

Case 2: $g(x) = f_x(x)$. Since $g \mid N_1\backslash\{x\} = f \mid N_1\backslash\{x\} = f_x \mid N_1\backslash\{x\}$, we obtain $g \mid N_1 = f_x \mid N_1$ which implies $\langle y, g\rangle \notin A$ by (4). ∎

In view of Lemma 1.2, $dX = Y/\mathcal{G}$ is a compact Hausdorff space; let $\rho : Y \to dX$ be the decomposition map.

Let $\sigma = \{f \in \{0,1\}^X : f(x) = 0$ for all but infinitely many $x \in X\}$. It is well-known, and easy to prove, that σ is dense in $\{0,1\}^X$. For $f \in \sigma$, let $\lambda(f) = |\{x \in X : f(x) \neq 0\}|$.

1.3. Lemma. *If X is a continuum, then so is dX.*

Proof: We shall prove that $\rho(X \times \sigma)$ is contained in a connected subset of dX. This clearly implies that dX is connected. Let $O : X \to \{0,1\}$ be the function with constant values 0 and let $x_0 \in X$. Let $n \geq 0$ and consider the following statement:

$S(n)$: for all $y \in X$ and $f \in \sigma$ with $\lambda(f) \leq n$, there exists a subcontinuum of dX containing $\rho(\langle x_0, O\rangle)$ and $\rho(\langle y, f\rangle)$.

We shall prove $S(n)$ by induction on n. If $n = 0$, then there is nothing to prove since O is the only function f in σ with $\lambda(f) = 0$ and $\rho(X \times \{O\})$ is connected. Therefore, assume that $S(n-1)$ is true, $n \geq 1$. We shall prove $S(n)$. Indeed, take $y \in X$ and $f \in \sigma$ such that $\lambda(f) = n$. Since $n \geq 1$, there exists $z \in X$ with $f(z) = 1$. Define $g : X \to \{0,1\}$ by

$$g(p) = f(p) \text{ for } p \neq z, \text{ and } g(z) = 0.$$

By our inductive assumption, there exists a subcontinuum of dX containing the points $\rho(\langle O, x_0\rangle)$ and $\rho(\langle y, g\rangle)$. It therefore suffices to construct a subcontinuum of dX containing the points $\rho(\langle y, g\rangle)$ and $\rho(\langle y, f\rangle)$; we claim that $L = \rho(X \times \{f, g\})$ is the desired continuum. Clearly, L contains $\rho(\langle y, g\rangle)$ and $\rho(\langle y, f\rangle)$. Observe that $g = f_z$. Consequently, $\rho(\langle z, g\rangle) = \rho(\langle z, f\rangle)$. We conclude that $\rho(X \times \{f, g\})$ is connected, as required. ∎

Let $\pi : Y \to X$ be the projection. It is clear that, for every $p \in dX$, $\pi(\rho^{-1}(p))$ consists of precisely one point, which we shall denote by $\kappa(p)$. Consequently, the diagram below commutes. We conclude that κ is continuous. It is easy to see that κ in fact is an open retraction.

$$\begin{tikzcd} X \times \{0,1\}^X \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & X \\ & \mathrm{dX} \arrow[ur, "\kappa"'] & \end{tikzcd}$$

1.4. PROPOSITION. *If X is a continuum and K is a subcontinuum of dX such that $\kappa(K) \neq X$, then K has empty interior in dX.*

PROOF: Put $L = \rho^{-1}(K)$. Since $\pi(L) = \kappa(K) \neq X$, by compactness of $\pi(L)$ and by connectivity of X, $X \setminus \pi(L)$ is infinite (in fact, even uncountable). Fix $x \in X \setminus \pi(L)$ arbitrarily. Define

$$V_0 = \{\langle y, f \rangle \in Y : y \in X \setminus \{x\} \text{ and } f(x) = 0\},$$
$$V_1 = \{\langle y, f \rangle \in Y : y \in X \setminus \{x\} \text{ and } f(x) = 1\}.$$

By Lemma 1.1, V_0 and V_1 are disjoint $\mathcal{G}$-saturated open subsets of Y. Consequently, $\rho(V_0)$ and $\rho(V_1)$ are disjoint open subsets of dX covering K. By connectivity of K, it therefore follows that, without loss of generality, $L \subseteq V_0$, i.e., L projects onto precisely one point in the x^{th} coordinate direction of $\{0,1\}^X$. Since $X \setminus \pi(L)$ is infinite, we conclude that, in infinitely many coordinate directions of $\{0,1\}^X$, L projects onto precisely one point. By definition of the product topology on $\{0,1\}^X$, this implies that L projects onto a nowhere dense closed subset of $\{0,1\}^X$. We conclude that L is nowhere dense itself, and therefore that K has empty interior in dX. ∎

We finish this section with a few easy observations on homeomorphisms of dX. For every space Y, let $\mathcal{H}(Y)$ denote the autohomeomorphism group of Y. Observe that each $\xi \in \mathcal{H}(X)$ is a permutation of X, and therefore induces a homeomorphism $\sigma(\xi)$ of $\{0,1\}^X$ as follows:

$$\sigma(\xi)(f)(p) = f(\xi(p)).$$

We say that an element $h \in \mathcal{H}(Y)$ respects $\mathcal{G}$ if, for every $H \in \mathcal{G}$, $h(H) \in \mathcal{G}$. Observe that if $h \in \mathcal{H}(Y)$ respects $\mathcal{G}$, then there is a homeomorphism $f \in \mathcal{H}(dX)$ such that $\rho \circ f = h \circ \rho$. Let id denote the identity homeomorphism on X. If, for every $x \in X$, $h_x : \{0,1\} \to \{0,1\}$ is a homeomorphism, then $\Pi_{x \in X} h_x$ is a homeomorphism of $\{0,1\}^X$. Let $\Pi(\{0,1\}^X)$ be the set of all these homeomorphisms. Observe that, for all $f, g \in \{0,1\}^X$, there exists an element $h \in \Pi(\{0,1\}^X)$ with $h(f) = g$.

1.5. PROPOSITION. *For all $\xi \in \mathcal{H}(X)$ and $h \in \Pi(\{0,1\}^X)$ the homeomorphisms $\xi^{-1} \times \sigma(\xi)$ and $\mathrm{id} \times h$ of Y both respect $\mathcal{G}$.*

PROOF: Take $\{\langle x,f\rangle,\langle x,f_x\rangle\}\in\mathcal{G}$ arbitrarily. Let $q=\xi^{-1}(x)$. Then, for $p\neq q$, we have

$$\sigma(\xi)(f)(p)=f(\xi(p))=f_x(\xi(p))=\sigma(\xi)(f_x)(p).$$

In addition,

$$\sigma(\xi)(f)(q)=f(\xi(q))=f(x),\text{ and }\sigma(\xi)(f_x)(q)=f_x(\xi(q))=f_x(x).$$

We conclude that $\sigma(\xi)(f)_q=\sigma(\xi)(f_x)$. Consequently,

$$(\xi^{-1}\times\sigma(\xi))\left(\{\langle x,f\rangle,\langle x,f_x\rangle\}\right)=\{\langle q,\sigma(\xi)(f)\rangle,\langle q,\sigma(\xi)(f)_q\rangle\},$$

as required.

Now again, take $\{\langle x,f\rangle,\langle x,f_x\rangle\}\in\mathcal{G}$ arbitrarily. Since h is a product homeomorphism, it is easy to see that $h(f)_x=h(f_x)$. From this we find that

$$(\mathrm{id}\times h)\left(\{\langle x,f\rangle,\langle x,f_x\rangle\}\right)=\{\langle x,h(f)\rangle,\langle x,h(f)_x\rangle\},$$

as required. ∎

This leads us to the following result.

1.6. THEOREM. *If $p,q\in dX$ and if $\xi\in\mathcal{H}(X)$ is a homeomorphism with $\xi(\kappa(p))=\kappa(q)$, then there exists a homeomorphism $h\in\mathcal{H}(dX)$ such that $h(p)=q$ while, moreover, $\kappa\circ h=\xi\circ\kappa$.*

PROOF: There exist points $\langle x,f\rangle$ and $\langle y,g\rangle$ in Y such that $\rho(\langle x,f\rangle)=p$ and $\rho(\langle y,g\rangle)=q$. Observe that $\kappa(p)=x$ and $\kappa(q)=y$. Consider the homeomorphism $\sigma(\xi^{-1})$ of $\{0,1\}^X$. There exists a homeomorphism $\eta\in\Pi(\{0,1\}^X)$ such that $\eta(\sigma(\xi^{-1})(f))=g$. By Proposition 1.5, the homeomorphism $(\mathrm{id}\times\eta)\circ(\xi\times\sigma(\xi^{-1}))$ of $Y$is $\mathcal{G}$-preserving. Now

$$(5)\qquad (\mathrm{id}\times\eta)\circ(\xi\times\sigma(\xi^{-1}))(\langle x,f\rangle)=(\mathrm{id}\times\eta)(\langle\xi(x),\sigma(\xi^{-1})(f)\rangle)$$
$$=\langle\xi(x),\eta(\sigma(\xi^{-1})(f))\rangle=\langle y,g\rangle.$$

There exists a homeomorphism $h\in\mathcal{H}(dX)$ such that

$$(6)\qquad h\circ\rho=\rho\circ((\mathrm{id}\times\eta)\circ(\xi\times\sigma(\xi^{-1}))).$$

Now (5) and (6) easily imply that $h(p)=q$.

Finally, for every $\langle z,k\rangle\in Y$,

$$(7)\qquad \left(\pi\circ(\mathrm{id}\times\eta)\circ(\xi\times\sigma(\xi^{-1}))\right)(\langle z,k\rangle)=\pi\left(\langle\xi(z),?\rangle\right)=\xi(z).$$

From (6) and (7) we therefore conclude that $\kappa\circ h=\xi\circ\kappa$, i.e., h is as required.

2. The construction. We shall now prove that any homogeneous continuum X is an open retract of an indecomposable homogeneous continuum $d^{(\infty)}X$ of the same dimension.

To this end, let X be a homogeneous continuum. For $n \geq 0$, define $d^{(0)}X = X$, and $d^{(n+1)}X = d(d^{(n)}X)$. Also, for $n \geq 0$, let $\kappa_n : d^{(n+1)}X \to d^{(n)}X$ be the canonical function defined in Section 1. Now form the following inverse sequence

$$d^{(0)}X \xleftarrow{\kappa_0} d^{(1)}X \xleftarrow{\kappa_1} d^{(2)}X \leftarrow \cdots \leftarrow d^{(n)}X \xleftarrow{\kappa_n} d^{(n+1)}X \leftarrow \cdots .$$

Let $d^{(\infty)}X$ be the inverse limit of this sequence, and, for every $n \geq 0$, let $\mu_n : d^{(\infty)}X \to d^{(n)}X$ be the standard projection. Since each κ_n is an open retraction, it follows easily that X is an open retract of $d^{(\infty)}X$. Also, it is a triviality to verify that, for every compact space X, we have $\dim dX = \dim X$. From this it easily follows that $\dim X = \dim d^{(\infty)}X$. The details of checking this are left to the reader.

We shall now prove that $d^{(\infty)}X$ is a homogeneous indecomposable continuum.

2.1. PROPOSITION. *$d^{(\infty)}X$ is indecomposable.*

PROOF: Let K be a proper subcontinuum of $d^{(\infty)}X$. Since K is proper, there exists $n \geq 0$ such that $\mu_n(K)$ is a proper subcontinuum of $d^{(n)}X$. By Proposition 1.4, it therefore follows that $\mu_{n+1}(K)$ is a nowhere dense subcontinuum of $d^{(n+1)}X$. Repeating the same observation, it follows that $\mu_m(K)$ is a nowhere dense subcontinuum of $d^{(m)}X$ for every $m \geq n+1$. We conclude that K is nowhere dense in $d^{(\infty)}X$. ∎

2.2. PROPOSITION. *$d^{(\infty)}X$ is homogeneous.*

PROOF: Let x and y be points in $d^{(\infty)}X$. By the homogeneity of X, there exists a homeomorphism h of X such that $h(x_0) = y_0$. By Theorem 1.6, there exists a homeomorphism h_1 of $d^{(1)}X$ such that $h_1(x_1) = y_1$ while, moreover, $\kappa_1 \circ h_1 = h \circ \kappa_0$. By a repeated application of Theorem 1.6, it is easy to construct a sequence $h_n \in \mathcal{H}(d^{(n)}X)$, $n \geq 0$, such that $h_0 = h$, while, moreover, for $n \geq 1$, we have

(I) $\kappa_n \circ h_n = h_{n-1} \circ \kappa_{n-1}$,
(II) $h_n(x_n) = y_n$.

Consequently, the sequence $(h_n)_n$ induces a homeomorphism $h : d^{(\infty)}X \to d^{(\infty)}X$ such that $h(x) = y$.

We have completed the proof of the following

2.3. THEOREM. *Every homogeneous continuum is an open retract of an indecomposable homogeneous continuum of the same dimension.*

REFERENCES

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Vrije Universiteit
Amsterdam, The Netherlands

and

Universiteit van Amsterdam
Amsterdam, The Netherlands

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