Open Problems in Topology
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Chapter 7

Open problems on $\beta\omega$

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1. Introduction

The aim of this paper is to collect some open problems on $\beta\omega$, the Čech-Stone compactification of the integers. It is recognized that a few of the problems listed below may be inadequately worded, be trivial or be known. Of many of the problems the origin is not known. For that reason we do not credit anybody for posing a certain problem. We would like to thank our colleagues who brought many of the problems to our attention.

In addition to this, we also comment on the status of the problems in the list from the second author's paper in the Handbook of Set-Theoretic Topology: van Mill [1984], hereafter referred to as the $\beta\omega$-Handbook Paper.

2. Definitions and Notation

Many notions are defined in the list, but many occur so frequently that we collect them in this section.

If f is a function from ω to ω then $\beta f\colon \beta\omega \to \beta\omega$ is its Stone extension, i.e., $\beta f(p) = \{P : f^{\leftarrow}[P] \in p\}$.

The *Rudin-Keisler* (pre-)order $\leq_{\mathbf{RK}}$ on $\beta\omega$ is defined as follows: $p \leq_{\mathbf{RK}} q$ iff there is $f \in {}^{\omega}\omega$ such that $\beta f(q) = p$. Two points $p, q \in \beta\omega$ are called **RK**-equivalent, in symbols $p \simeq_{\mathbf{RK}} q$, if there is a permutation $\pi\colon \omega \to \omega$ such that $\beta\pi(p) = q$. We denote the **RK**-equivalence class of p by $[p]_{\mathbf{RK}}$.

Let $\mathcal{P}(\omega)$ denote the power set of ω. Let fin denote the ideal of finite subsets of ω. The quotient algebra $\mathcal{P}(\omega)/fin$ is naturally isomorphic to the Boolean algebra of clopen subsets of $\omega^* = \beta\omega \setminus \omega$.

If $A, B \subseteq \omega$, then A is *almost contained in* B, abbreviated $A \subseteq^* B$, if $|A \setminus B| < \omega$.

We denote the character of a point $p \in \omega^*$ by $\chi(p)$, thus

$$\begin{aligned} \chi(p) &= \min\{\, |\mathcal{B}| : \mathcal{B} \text{ is a local base at } p \text{ in } \omega^* \,\} \\ &= \min\{\, |\mathcal{B}| : \mathcal{B} \subseteq p \text{ generates } p \,\}. \end{aligned}$$

A point $p \in \omega^*$ is called a P-point, if for every function f from ω into itself, there is an element $P \in p$ on which f is finite-to-one, or constant. Equivalently, the intersection of countably many neighborhoods of p in ω^* is again a neighborhood of p. A *simple* P-point is a point with a linearly ordered local base. If in the above definition P can always be chosen so that f is one-to-one or constant on P, then p is called *selective*. Let κ be an infinite cardinal. A subset A of a space X is called a *P_κ-set* if the intersection of fewer than κ many neighborhoods of A is again a neighborhood of A. If A consists of one point then that point is called a *P_κ-point*. A *P-set* is a P_{ω_1}-set.

For a discussion of **MA** and **PFA**, see Weiss [1984] and Baumgartner [1984]. For information on the cardinals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ and others, see the contribution of Vaughan in this volume.

3. Answers to older problems

In this section we collect the problems from the $\beta\omega$-Handbook Paper that have been solved. We list them, with their answers, using the original numbering.

2 Are there points p and $q \in \omega^*$ such that if $f\colon \omega \to \omega$ is any finite-to-one map, then $\beta f(p) \neq \beta f(q)$?

Let us abbreviate the following statement by **NCF** (Near Coherence of Filters):

> for every p and $q \in \omega^*$ there is a finite-to-one $f\colon \omega \to \omega$ such that $\beta f(p) = \beta f(q)$.

So the question is whether **NCF** is false. Under **MA** it is. But not in **ZFC**: Shelah produced models such that for all p and $q \in \omega^*$ there is a finite-to-one map $f\colon \omega \to \omega$ such that $\beta f(p) = \beta f(q)$ is a P-point of character ω_1 (observe that in this model **CH** is false, so that a point with character ω_1 has small character), see the papers BLASS and SHELAH [1987, 19∞]. The latter model is the model obtained by iterating rational perfect set forcing (MILLER [1984]) ω_2 times; the former model is also obtained in an ω_2-step iteration but the poset used is somewhat more difficult to describe. This iteration however can be modified to produce a model in which there are p and q in ω^* with linearly ordered bases and with $\chi(p) = \omega_1$ and $\chi(q) = \omega_2$. This answered a question of Nyikos who showed that if p and q in ω^* are simple P-points with $\chi(p) < \chi(q)$ then $\chi(p) = \mathfrak{b}$ and $\chi(q) = \mathfrak{d}$ and asked whether this situation is actually possible.

The consistency of **NCF** implies among other things that the Čech-Stone-remainder $\mathbb{H}^*$ of the half-line $\mathbb{H} = [0, \infty)$, which is an indecomposable continuum (see BELLAMY [1971] and WOODS [1968]), has (consistently) only one composant. For details, see e.g., RUDIN [1970]. In fact **NCF** is equivalent to the statement that $\mathbb{H}^*$ has exactly one composant (MIODUSZEWSKI [1978]). See the papers BLASS [1986] and [1987] for more information on **NCF**.

6 Is there a **ccc** P-set in ω^*?

In [1989] FRANKIEWICZ, SHELAH and ZBIERSKI announced the consistency of a negative answer.

Now a **ccc** subset of ω^* is topologically quite small (it is nowhere dense for example), and it is also interesting to know what nowhere dense P-sets can look like. By way of an example one may wonder whether ω^* can be realized as a nowhere dense P-subset of itself. The answer to this question is in the negative. JUST [19∞] recently showed the consistency of the statement that no nowhere dense P-set in ω^* is homeomorphic to ω^*. In fact, Just showed that "if $A \subseteq \omega^*$ is a nowhere dense P-set and a continuous image of ω^* then A is **ccc**" is consistent with **ZFC**.

8 Is the autohomeomorphism group of ω^* algebraically simple?

This problem was motivated by the question in van Douwen, Monk and Rubin [1980] whether the automorphism group $\mathrm{Aut}(B)$ of a homogeneous Boolean algebra B is algebraically simple. The Boolean algebra of clopen subsets of ω^* is clearly isomorphic to the quotient algebra $\mathcal{P}(\omega)/fin$, which is easily seen to be homogeneous. It can be shown that under **CH**, $\mathrm{Aut}\,(\mathcal{P}(\omega)/fin)$ is algebraically simple, see Štěpánek and Rubin [1989]. However,van Douwen showed in [1990] that the group of *trivial* (see below) automorphisms of $\mathcal{P}(\omega)/fin$ is not algebraically simple. It follows that in Shelah's model (see Shelah [1982]) where every automorphism of $\mathcal{P}(\omega)/fin$ is trivial, $\mathrm{Aut}\,(\mathcal{P}(\omega)/fin)$ is not algebraically simple. Independently, Koppelberg [1985] constructed a different example of a homogeneous Boolean algebra the automorphism group of which is not simple, under **CH**.

Let us take this opportunity to correct a small mistake in the $\beta\omega$-Handbook Paper which caused some confusion. On page 537, line 8 it states:

> As remarked in Section 2.2, Shelah [1978] has shown it to be consistent that every autohomeomorphisms of ω^* is induced by a permutation of ω.

This is not true of course. If F and G are finite subsets of ω then each bijection $\pi\colon \omega \setminus F \to \omega \setminus G$ induces a homeomorphism $\bar{\pi}$ of ω^* and Shelah proved that consistently, all homeomorphisms of ω^* are of this form. Let us call such homeomorphisms trivial or induced, and let Triv denote the subgroup of $\mathrm{Aut}\,(\mathcal{P}(\omega)/fin)$ consisting of all trivial automorphisms. So van Mill misquoted Shelah's result. All results in the $\beta\omega$-Handbook Paper depending on Shelah's result (such as Theorem 2.2.1) are correct, as can be seen by making trivial modifications to the proofs given.

Van Douwen's argument is now easy to summarize: if $h \in \mathsf{Triv}$ is represented as above then the parity of $|F|+|G|$ depends only on h; and the automorphisms for which the parity is even form a subgroup of Triv of index 2.

10 Is every first countable compactum a continuous image of ω^*?

To put this question into perspective, note that by Arkhangel'skiĭ's result from [1969], every first countable compactum has cardinality at most $\mathfrak{c}$ and hence has weight at most $\mathfrak{c}$, and hence is—under **CH**—a continuous image of ω^* by Parovichenko's result in [1963]. Pertinent to this question is also the result of Przymusiński [1982] that every perfectly normal compact space is a continuous image of ω^*. Problem 10 was recently answered in the negative (necessarily consistently) by Bell [19∞] who modified an older construction of his, Bell [1982], to obtain the desired counterexample.

12 Is there a separable closed subspace of $\beta\omega$ which is not a retract of $\beta\omega$?

Such spaces were constructed by Shapiro [1985] and Simon [1987]. Simon, using heavy machinery from independent linked families, directly constructed a closed separable subspace of $\beta\omega$ which is not a retract; Shapiro constructed a certain compact separable space and showed that not every copy of its absolute in $\beta\omega$ could be a retract of $\beta\omega$. See question 24 for more information.

13 Let $(*)$ denote the statement that every Parovichenko space is coabsolute with ω^*. Is $(*)$ equivalent to **CH**?

DOW [1984] answered this question in the negative by establishing the interesting fact that $(*)$ follows from the continuum having cofinality ω_1. This suggests a question that will be posed later on.

14 Let X be the Stone space of the reduced measure algebra of $[0,1]$. Is it consistent that X is not a continuous image of ω^*?

This question was answered in the affirmative by FRANKIEWICZ [1985], using the oracle-cc method of SHELAH [1982].

17 Is there a $p \in \omega^*$ such that $\omega^* \setminus \{p\}$ is not C^*-embedded in ω^* ?

Recall that X is C^*-embedded in Y if every continuous function from X to $[0,1]$ can be extended over Y. This question has been solved completely now. By an old result of GILLMAN [1966] it follows that under **CH**, for every $p \in \omega^*$ the space $\omega^* \setminus \{p\}$ is not C^*-embedded in $\beta\omega$. However, by a result of VAN DOUWEN, KUNEN and VAN MILL [1989] it is consistent with $\mathbf{MA} + \mathfrak{c} = \omega_2$ that for every $p \in \omega^*$ the space $\omega^* \setminus \{p\}$ *is* C^*-embedded in $\beta\omega$. In [1987] MALYKHIN announced the result that if one adds $\mathfrak{c}^+$ Cohen reals to any model of set theory then one obtains a model in which $\omega^* \setminus \{p\}$ is C^*-embedded in ω^* for every $p \in \omega^*$. A proof may be found in DOW [1988a].

22 Is there a point $p \in \omega^*$ such that some compactification of $\omega \cup \{p\}$ does not contain a copy of $\beta\omega$?

RYLL-NARDZEWSKI and TELGÁRSKY [1970] showed that if p is a simple P-point then the space $\omega \cup \{p\}$ has a scattered (= every subspace has an isolated point) compactification. Thus the answer to this problem is in the affirmative under **MA**. On the other hand in [1987] MALYKHIN also announced that in the same model as mentioned above for every $p \in \omega^*$ every compactification of $\omega \cup \{p\}$ contains a copy of $\beta\omega$.

24 Is there a point $p \in \omega^*$ such that if $f\colon \omega \to \omega$ is any map, then either $\beta f(p) \in \omega$ or $\beta f(p)$ has character $\mathfrak{c}$ in $\beta\omega$?

Since under **MA** each point in ω^* has character $\mathfrak{c}$ in ω^*, the answer to this question is trivially YES under **MA**. However, it is not YES in **ZFC** because in answer **2** we already remarked that Shelah has constructed models in which $\mathfrak{c} = \omega_2$ and in which (in particular) for every $p \in \omega^*$ there is a finite-to-one function $f\colon \omega \to \omega$ such that $\beta f(p)$ has character ω_1 in ω^*. In fact **NCF** is equivalent to the statement that for every $p \in \omega^*$ there is a finite-to-one $f\colon \omega \to \omega$ such that $\chi(\beta f(p)) < \mathfrak{d}$, so that every model for **NCF** provides a negative answer to this question.

4. Autohomeomorphisms

We consider the autohomeomorphism group of the space ω^* or equivalently Aut $(\mathcal{P}(\omega)/fin)$. Recall answer **8**:

> The autohomeomorphism group of ω^* may, but need not, be algebraically simple.

This prompts us to add a few problems about Aut $(\mathcal{P}(\omega)/fin)$, the solutions of which may shed some light on the possible algebraic structure of this group. See also the contribution by Steprāns to this volume.

Question 1. *Can* Triv *be a proper normal subgroup of* Aut $(\mathcal{P}(\omega)/fin)$, *and if* yes *what is (or can be) the structure of the factor group* Aut $(\mathcal{P}(\omega)/fin)$ /Triv; *and if* no *what is (or can be)* [Triv : Aut $(\mathcal{P}(\omega)/fin)$]? **200. ?**

For $h \in$ Aut $(\mathcal{P}(\omega)/fin)$ we let $I(h) = \{ A \subseteq \omega : h|A \text{ is trivial} \}$, where "$h|A$ is trivial" means that there are a finite set $F \subseteq A$ and a one-to-one $f\colon A \setminus F \to \omega$ such that $h(X) = f[X \setminus F]$ for $X \subseteq A$. Let us observe that h is trivial iff $\omega \in I(h)$, and that $I(h)$ is an ideal.

To make the statements of some of the following questions a bit easier we shall call $h \in$ Aut $(\mathcal{P}(\omega)/fin)$: *totally non-trivial* if $I(h) = fin$, *somewhere trivial* if $I(h) \neq fin$ and *almost trivial* if $I(h)$ is a tall ideal. Recall that an ideal I on ω is *tall* if every infinite subset of ω contains an infinite element of I.

It is not hard to show that under **CH** there is a totally non-trivial automorphism. Recently it was shown by Shelah and Steprāns in [1988] that **PFA** implies that every $h \in$ Aut $(\mathcal{P}(\omega)/fin)$ is trivial, they also mention that Velickovic showed it to be consistent with **MA** + ¬**CH** that a non-trivial automorphism exists.

We ask the following questions:

Question 2. *Is it consistent with* **MA** + ¬**CH** *that a totally non-trivial automorphism exists?* **201. ?**

Question 3. *Is it consistent to have a non-trivial automorphism, while for every* $h \in$ Aut $(\mathcal{P}(\omega)/fin)$ *the ideal* $I(h)$ *is the intersection of finitely many prime ideals?* **202. ?**

Question 4. *Does the existence of a totally non-trivial automorphism imply that* Aut $(\mathcal{P}(\omega)/fin)$ *is simple?* **203. ?**

Question 5. *Does the existence of a non-trivial automorphism imply that* Aut $(\mathcal{P}(\omega)/fin)$ *is simple?* **204. ?**

The proof in SHELAH and STEPRĀNS [1988] suggests the following questions:

? **205.** **Question 6.** *If every automorphism is somewhere trivial, is then every automorphism trivial?*

? **206.** **Question 7.** *Is every ideal $I(h)$ a P-ideal (if every automorphism is somewhere trivial)?*

An ideal I on ω is said to be a P-ideal if whenever $\{X_n : n \in \omega\}$ is a subfamily of I there is an X in I such that $X_n \subseteq^* X$ for all $n \in \omega$.

For the next group of questions we make the following definitions: if κ is a cardinal and $h \in \mathrm{Aut}\,(\mathcal{P}(\omega)/fin)$ call h

κ-weakly trivial if $|\{\, p : h(p) \not\simeq_{\mathbf{RK}} p \,\}| < \kappa$ and,

κ-quasi trivial if for every $p \in \omega^*$ there is $S_p \subseteq [p]_{\mathbf{RK}}$ such that

$$|S_p| < \kappa \wedge \forall q, q' \in [p]_{\mathbf{RK}} \setminus S_p : h(q) \simeq_{\mathbf{RK}} h(q') \text{ and } |\{\, p : S_p \neq \emptyset \,\}| < \kappa$$

Let $W_\kappa = \{\, h : h, h^{-1} \kappa\text{-weakly trivial} \,\}$ and $Q_\kappa = \{\, h : h, h^{-1} \kappa\text{-quasi trivial} \,\}$. It is known that W_κ is a normal subgroup of Q_κ.

? **207.** **Question 8.** *Is it consistent to have a cardinal κ such that every automorphism is κ-weakly trivial?*

? **208.** **Question 9.** *Is it consistent to have a cardinal κ such that every automorphism is κ-quasi trivial?*

? **209.** **Question 10.** *Is it consistent to have $W_\kappa \neq Q_\kappa = \mathrm{Aut}\,(\mathcal{P}(\omega)/fin)$ for some regular $\kappa \leq \mathfrak{c}$?*

? **210.** **Question 11.** *(**MA** + ¬**CH**) if p and q are $P_{\mathfrak{c}}$-points is there an h in $\mathrm{Aut}\,(\mathcal{P}(\omega)/fin)$ such that $h(p) = q$?*

Note: say $p \equiv q$ iff $\omega = \dot{\bigcup}_n P_n = \dot{\bigcup}_n Q_n$ (finite sets) such that

$$\forall A \in p\, \exists B \in q\, \forall n\, |A \cap P_n| = |B \cap Q_n|$$

and conversely. Clearly **RK**-equivalent points are $\equiv$-equivalent. As a partial answer to Problem 11 the following was shown to be consistent:
MA + ¬**CH**+ "for all $P_{\mathfrak{c}}$-points p and q, if $p \equiv q$ then there is an $h \in \mathrm{Aut}\,(\mathcal{P}(\omega)/fin)$ such that $h(p) = q$."

We ask:

? **211.** **Question 12.** *Is $\equiv$ different from **RK**-equivalence in **ZFC**?*

5. Subspaces

In this section we deal with subspaces of $\beta\omega$ and ω^*. The following question is well-known.

Question 13. *For what p are $\omega^*\backslash\{p\}$ and (equivalently) $\beta\omega\backslash\{p\}$ non-normal?* **212. ?**

There are several simple proofs that, under **CH**, for any $p \in \omega^*$, the spaces $\beta\omega\backslash\{p\}$ and $\omega^*\backslash\{p\}$ are not normal, see e.g., Rajagopalan [1972], Warren [1972], and van Mill [1986]. It was shown by Beslagic and Van Douwen that **CH** may be relaxed to the equality $\mathfrak{r} = \mathfrak{c}$, here $\mathfrak{r}$ is the least cardinality of a "reaping" family; this is a family $\mathcal{R}$ of subsets of ω such that for every subset X of ω there is an $R \in \mathcal{R}$ such that $R \subseteq X$ or $R \cap X = \emptyset$.

However, for years no significant progress has been made on Problem 13. The best result so far is that if $p \in \omega^*$ is an accumulation point of some countable discrete subset of ω^*, then $\omega^*\backslash\{p\}$ is not normal. In [1982] Gryzlov showed that also points that are not an accumulation point of any countable subset of ω^* may have this property.

Related to this question is the following:

Question 14. *Is it consistent that there is a non-butterfly point in ω^*?* **213. ?**

A butterfly point is a point p for which there are sets D and E such that $\overline{D} \cap \overline{E} = \{p\}$. For a non-butterfly point p the space $\omega^*\backslash\{p\}$ is normal.

In connection with Answer **17** one may also ask

Question 15. *Is it consistent that $\omega^*\backslash\{p\}$ is C^*-embedded in ω^* for some but not all $p \in \omega^*$?* **214. ?**

Question 16. *What spaces can be embedded in $\beta\omega$?* **215. ?**

In [1973] Louveau proved that under **CH**, a compact space X can be embedded in $\beta\omega$ if and only if X is a compact zero-dimensional F-space of weight at most $\mathfrak{c}$ (or, equivalently, the Stone space of a weakly countably complete Boolean algebra of cardinality at most $\mathfrak{c}$). There is a consistent example in van Douwen and van Mill [1980] of a compact zero-dimensional F-space of weight $\mathfrak{c}$ that cannot be embedded in any compact extremally disconnected space (that is, the Stone space of a complete Boolean algebra). So the **CH** assumption in Louveau's result is essential. These remarks have motivated Problem 16.

Question 17. *Is* **CH** *equivalent to the statement that every compact zero-dimensional F-space of weight $\mathfrak{c}$ is embeddable in ω^*?* **216. ?**

A compact space is *basically disconnected* if it is the Stone space of a σ-complete Boolean algebra.

? **217. Question 18.** *Is there (consistently) a basically disconnected compact space of weight $\mathfrak{c}$ that is not embeddable in ω^*?*

A natural candidate for such an example would be the Čech-Stone compactification of an appropriate P-space (= a space in which every G_δ-set is open). However, DOW and VAN MILL showed in [1982] that such an example does not work. For more information, see the remarks following Problem 16

Motivated by Answer **6** we specialize Problem 16 to:

? **218. Question 19.** *Describe the P-sets of ω^**

Finally, to finish the questions on embeddings we ask:

? **219. Question 20.** *Is there a copy of ω^* in ω^* not of the form $\overline{D} \setminus D$ for some countable and discrete $D \subseteq \omega^*$?*

Of course Just's result cited after Answer **6** blocks an "easy" way out of this problem: a P-set homeomorphic to ω^* would certainly do the trick. However it may still be possible for example, to realize ω^* as a weak P-set in ω^*.

? **220. Question 21.** *Is every subspace of ω^* strongly zero-dimensional?*

We have no information on this problem.

? **221. Question 22.** *Is every nowhere dense subset of ω^* a $\mathfrak{c}$-set?*

A set D in a topological space is called a κ-set, where κ is a cardinal number, if there is a disjoint family $\mathcal{U}$ of size κ of open sets such that $D \subseteq \overline{U}$ for every U in $\mathcal{U}$. One may think of κ-sets as providing an indication of how non-extremally disconnected a space is (in an extremally disconnected space *no* nowhere dense set is even a 2-set). As is well-known, ω^* is not extremally disconnected and we are asking whether it is, in a way, totally non-extremally disconnected ($\mathfrak{c}$ is the largest cardinal we can hope for of course). It should be noted however that, as far as we know, it is also unknown whether every nowhere dense set in ω^* is a 2-set. The reason we ask about $\mathfrak{c}$-sets is that all partial answers seem to point into the direction of $\mathfrak{c}$: BALCAR and VOJTÁŠ showed in [1980] that every one-point set is a $\mathfrak{c}$-set. This was improved in BALCAR, DOČKALKOVÁ and SIMON [1984] to sets of density less than $\mathfrak{c}$. Furthermore the answer to the general question is known to be YES if either $\mathfrak{a} = \mathfrak{c}$, $\mathfrak{d} = \mathfrak{c}$, $\mathfrak{d} = \omega_1$ or $\mathfrak{b} = \mathfrak{d}$. More information, including proofs of the above YES can be found in BALCAR and SIMON [1989].

A positive answer to question 22 would provide a negative answer to the following, natural sounding, question:

Question 23. *Is there a maximal nowhere dense subset in ω^*?* **222. ?**

Here 'maximal' means that it is not nowhere dense in a larger nowhere dense subset of ω^*. One can see with a little effort that no $\mathfrak{c}$-set can be a maximal nowhere dense set.

A compact space X is *dyadic* if it is a continuous image of a power of $\{0,1\}$. The *absolute* of a compact space X is the Stone space of the Boolean algebra of regular open subsets of X.

Question 24. *Is there an absolute retract of $\beta\omega$ that is not the absolute of a dyadic space?* **223. ?**

Recall from answer **12** that not every separable closed subspace of $\beta\omega$ is a retract of $\beta\omega$. On the other hand it was shown by Maharam in [1976] that such a subspace can be reembedded into $\beta\omega$ in such a way that it is a retract of $\beta\omega$. This motivates the notion of an absolute retract of $\beta\omega$. A closed subspace X of $\beta\omega$ is an Absolute Retract (**AR**) of $\beta\omega$ if for every embedding $h\colon X \to \beta\omega$, $h[X]$ is a retract of $\beta\omega$. It can be shown that if $X \subseteq \beta\omega$ is the absolute of a dyadic space then X is an **AR** of $\beta\omega$ and Shapiro [1985] established the converse in case X is the absolute of some (compact separable) space of weight at most ω_1.

6. Individual Ultrafilters

In this section we collect some questions that ask for individual points in $\beta\omega$ or for special ultrafilters.

Question 25. *Is there a model in which there are no P-points and no Q-points?* **224. ?**

Recall that $p \in \omega^*$ is a Q-point if for every finite-to-one function $f\colon \omega \to \omega$ there is an element $E \in p$ such that f is one-to-one on E. We already mentioned that Shelah produced a model in which there are no P-points (Wimmers [1982]). The continuum is ω_2 in this model. On the other hand there is also a model in which there are no Q-points (Miller [1980]). In this model—Laver's model for the Borel Conjecture (Laver [1976])—the continuum is also ω_2. The interest in Problem 25 comes from the fact that, by results from Ketonen [1976] and Mathias [1978], if $\mathfrak{c} \le \omega_2$ then there is either a P-point or a Q-point.

An ultrafilter p is called *rapid* if for every $f \in {}^\omega\omega$ there exists a $g \in {}^\omega\omega$ such that $\forall n f(n) < g(n)$ and $g[\omega] \in p$ (in words: the counting functions of the elements of p form a dominating family). Every Q-point is rapid.

? **225. Question 26.** *Is there a model in which there is a rapid ultrafilter but in which there is no Q-point?*

The reason we ask this question is that in every model without Q-points that we know of there are also no rapid ultrafilters.

In [1982] SHELAH showed that it is consistent that there is, up to permutation, only one selective ultrafilter. Of course one may then also ask:

? **226. Question 27.** *Is it consistent that there is, up to permutation, only one P-point in ω^*?*

? **227. Question 28.** *Is there a model in which every point of ω^* is an R-point?*

A point $p \in \omega^*$ is an R-point if there is an open F_σ-set $U \subseteq \omega^*$ such that
(i) $p \in \overline{U}$, and
(ii) $\forall A \in [U]^{<\mathfrak{c}} : p \notin \overline{A}$.

Note that an R-point is not a P-point. So we are asking for a special model without P-points. R-points were introduced in VAN MILL [1983] but have played no role of importance so far. So this question is probably not very much of interest.

? **228. Question 29.** *Is there $p \in \omega^*$ such that every compactification of $\omega \cup \{p\}$ contains $\beta\omega$?*

This question was motivated by an example in VAN DOUWEN and PRZYMUSIŃSKY [1979]: there is a countable space with only one non-isolated point, every compactification of which contains a copy of $\beta\omega$. Compare this question also with Answer **22**: it is consistent that for every $p \in \omega^*$ every compactification of $\omega \cup \{p\}$ contains a copy of $\beta\omega$. There is however, as far as we know, no **ZFC**-construction of a point with these properties.

For the next question identify ω with $\mathbb{Q}$.

? **229. Question 30.** *Is there $p \in \omega^*$ such that $\{ A \in p : A$ is closed and nowhere dense in $\mathbb{Q}$ and also homeomorphic to $\mathbb{Q} \}$ is a base for p?*

This question arose in the study of remote points. A point p in $\beta X \setminus X$ is called a remote point of X if for every nowhere dense subset D of X one has $p \notin \overline{D}$. For us it is important to know that $\mathbb{Q}$ has remote points (VAN DOUWEN [1981b] and CHAE and SMITH [1980]). $\mathbb{Q}$ also has non-remote points: simply take $p \in \overline{\mathbb{N}}$, such a point is also a real ultrafilter on the set $\mathbb{Q}$.

Problem 30 asks for a not-so-simple non-remote point, which is still a real ultrafilter on $\mathbb{Q}$. The answer is known to be YES under $\mathbf{MA}_{\text{countable}}$. A related question is the following:

Question 31. *Is there a $p \in \omega^*$ such that whenever $\langle x_n : n \in \omega \rangle$ is a sequence in $\mathbb{Q}$ there is an $A \in p$ such that $\{ x_n : n \in A \}$ is nowhere dense?* **230. ?**

For the next question let

$$\mathbb{G} = \Big\langle \langle f_\alpha : \alpha \in \omega_1 \rangle, \langle g_\alpha : \alpha \in \omega_1 \rangle \Big\rangle$$

be a Hausdorff Gap in ${}^\omega\omega$, and let

$$I_{\mathbb{G}} = \{ M : \exists h \in {}^M\omega \, \forall \alpha \; f_\alpha | M <^* h <^* g_\alpha | M \}$$

Under $\mathbf{MA} + \neg\mathbf{CH}$ this ideal is tall.

Question 32. *($\mathbf{MA} + \neg\mathbf{CH}$) Are there $\mathbb{G}$ and p (P-point, selective) such that $p \subseteq I_{\mathbb{G}}^+$?* **231. ?**

This question is more delicate than it may seem: it is a theorem of Woodin, see Dales and Woodin [1987], that under $\mathbf{MA}+\neg\mathbf{CH}$ one can find for every $p \in \omega^*$ an $h \in {}^\omega\omega$ such that for all $\alpha \in \omega_1$ $f_\alpha <_p h <_p g_\alpha$, where $f <_p g$ means that $\{ n : f(n) < g(n) \} \in p$.

Now let p be a $P_{\mathfrak{c}}$-point, and find $A \in p$ such that $A \subseteq^* \{ n : f_\alpha(n) < h(n) < g_\alpha(n) \}$ for all α. It follows that $A \in I_{\mathbb{G}}$. Loosely speaking one can say that to a $P_{\mathfrak{c}}$-point ω_1 seems countable. What we are asking for here is an ultrafilter with some strong properties that, in spite of $\mathbf{MA}+\neg\mathbf{CH}$, considers ω_1 to be uncountable.

7. Dynamics, Algebra and Number Theory

For the next question identify ω with $\mathbb{Z}$, and consider the shift $\sigma: \mathbb{Z} \to \mathbb{Z}$ defined by $\sigma(n) = n + 1$. Denote its extension to $\beta\mathbb{Z}$ also by σ, and likewise its restriction to $\mathbb{Z}^*$. For $p \in \mathbb{Z}^*$ we let O_p denote its orbit $\{ \sigma^n(p) : n \in \mathbb{Z} \}$ and C_p is the closure of O_p. C_p is called an orbit closure. An orbit closure is called *maximal* if it is not a proper subset of any other orbit closure.

Question 33. *Is there a point in ω^* that is not an element of any maximal orbit closure?* **232. ?**

It would also be of interest to know the answer to the following, related, question:

Question 34. *Is there an infinite strictly increasing sequence of orbit closures?* **233. ?**

The next question is related to Furstenberg's multiple recurrence theorem, FURSTENBERG [1981]. A convenient way to state this theorem is for us: if f and g are commuting continuous selfmaps of the Cantor set ${}^{\omega}2$ then there are a $p \in \omega^*$ and an $x \in {}^{\omega}2$ such that $p\text{-}\lim f^n(x) = p\text{-}\lim g^n(x) = x$. The question is whether one can switch quantifiers, i.e.

? 234. **Question 35.** *Is there a $p \in \omega^*$ such that for every pair of commuting continuous maps $f, g : {}^{\omega}2 \to {}^{\omega}2$ there is an $x \in {}^{\omega}2$ such that $p\text{-}\lim f^n(x) = p\text{-}\lim g^n(x) = x$?*

There are ultrafilters p such that for every f there is an x such that $x = p\text{-}\lim f^n(x)$: take an idempotent in the semigroup $\langle \omega^*, + \rangle$ (see below), pick y arbitrary and let $x = p\text{-}\lim f^n(y)$. An equivalent question is whether the Cantor cube ${}^{\mathfrak{c}}2$ satisfies the conclusion of Furstenberg's theorem. The answer is known to be yes under **MA**.

Let S_ω denote the permutation group of ω, it acts on ω^* in the obvious way.

? 235. **Question 36.** *For what nowhere dense sets $A \subseteq \omega^*$ do we have $\bigcup_{\pi \in S_\omega} \pi[A] \neq \omega^*$?*

Let $\mathfrak{n}$ be the smallest number of nowhere dense sets needed to cover ω^*. In BALCAR, PELANT and SIMON [1980] it is shown that $\mathfrak{c} < \mathfrak{n}$ is consistent, hence we can say "for all nowhere dense sets" in models for this inequality. However also $\mathfrak{n} \leq \mathfrak{c}$ is consistent, and in such models we do not have an easy answer. Some nowhere dense sets satisfy the inequality in **ZFC**: singletons work since $|\omega^*| > |S_\omega|$. In addition GRYZLOV has shown in [1984] that in **ZFC** the following nowhere dense P-set also works:

$$A = \bigcap \{ X^* : X \subseteq \omega \text{ and } \delta(X) = 1 \},$$

where

$$\delta(X) = \lim_{n \to \infty} \frac{|X \cap n|}{n}$$

if this limit exists ($\delta(X)$ is called the density of X in ω).

There is a natural nowhere dense set in ω^* the permutations of which consistently cover ω^*: identify ω and ω^2, and for every $n \in \omega$ and $f \in {}^{\omega}\omega$ put $U(f, n) = \{ \langle k, l \rangle : k \geq n, l \geq f(k) \}$. The set $B = \bigcap \{ U(f, n)^* : f \in {}^{\omega}\omega$ and $n \in \omega \}$ is nowhere dense and $\bigcup_{\pi \in S_\omega} \pi[B] = \omega^* \setminus \{ p : p \text{ is a } P\text{-point} \}$. A probably more difficult question is:

? 236. **Question 37.** *For what nowhere dense sets $A \subseteq \omega^*$ do we have $\bigcup \{ h[A] : h \in \mathcal{H}(\omega^*) \} \neq \omega^*$?*

Again singletons work, but now the reason is deeper: ω^* is not homogeneous. Clearly the set B of the question 36 also satisfies $\bigcup\{\, h[B] : h \in \mathcal{H}(\omega^*) \,\} = \omega^* \setminus \{\, p : p \text{ is a } P\text{-point} \,\}$. As a start one may investigate the set A defined above.

For the next set of questions we consider binary operations on $\beta\omega$. If $*$ is any binary operation on ω then one can extend it in a natural way to $\beta\omega$ as follows: first define $p * n$ for $p \in \omega^*$ and $n \in \omega$ by $p * n = p\text{--}\lim m * n$; then if $q \in \omega^*$ we define $p*q = q\text{--}\lim p*n$. It is not hard to verify that this operation is continuous in the second coordinate. We shall be especially interested in the cases $* = +$ and $* = \times$. In these cases $\langle \beta\omega, * \rangle$ is a right-continuous semigroup. A lot is known about these semigroups, see Hindman [1979] and van Douwen [19∞b], but some questions still remain.

Question 38. *Can $\langle \beta\mathbb{N}, + \rangle$ be embedded in $\langle \mathbb{N}^*, + \rangle$?* **237. ?**

Question 39. *Are there p, q, r and s in $\mathbb{N}^*$ such that $p + q = r \times s$?* **238. ?**

Question 40. *What are the maximal subgroups of $\langle \beta\mathbb{N}, + \rangle$ and $\langle \beta\mathbb{N}, \times \rangle$?* **239. ?**

The next two questions are from van Douwen [1981a], where one can find much more information on the topic of these problems. To begin a definition: a map $h\colon X \times Y \to S \times T$ is said to be elementary if it is a product of two mappings or a product composed with (if possible) a reflection on $S \times T$. Let X be ω^* or $\beta\omega$.

Question 41. *If $h\colon X^2 \to X^2$ is a homeomorphism, is there a disjoint open cover $\mathcal{U}$ of X such that for all U, $V \in \mathcal{U}$ the map $h|U \times V$ is elementary?* **240. ?**

Question 42. *If $\varphi\colon X^2 \to X$ is continuous, is there a disjoint open cover $\mathcal{U}$ of X such that for all U, $V \in \mathcal{U}$ the map $\varphi|U \times V$ depends on one coordinate?* **241. ?**

8. Other

The following question is one of the most, if not *the* most, interesting problems about $\beta\omega$.

Question 43. *Are ω^* and ω_1^* ever homeomorphic?* **242. ?**

In spite of its simplicity and the general gut reaction: NO!, it is still unanswered. A nice touch to this question is its Boolean Algebraic variant:

are the Boolean Algebras $\mathcal{P}(\omega)/fin$ and $\mathcal{P}(\omega_1)/[\omega_1]^{<\omega}$ ever isomorphic?

This variant also makes sense in the absence of the Axiom of Choice (**AC**): the spaces ω^* and ω_1^* then need not exist (Feferman [1964/65]); but the algebras $\mathcal{P}(\omega)/fin$ and $\mathcal{P}(\omega_1)/[\omega_1]^{<\omega}$ always do. It would be very interesting indeed if on the one hand in **ZFC** the spaces ω^* and ω_1^* are not homeomorphic, while on the other hand it would be consistent with **ZF** that $\mathcal{P}(\omega)/fin$ and $\mathcal{P}(\omega_1)/[\omega_1]^{<\omega}$ are isomorphic.

In BALCAR and FRANKIEWICZ [1978] it is shown that if ω^* and κ^* are homeomorphic for some uncountable regular κ then there is a κ-scale in ${}^\omega\omega$. As a consequence one obtains that if $\mathfrak{b} < \mathfrak{d}$ the spaces ω^* and ω_1^* are not homeomorphic. The same conclusion follows from **MA**: if **CH** is true then $|\omega^*| < |\omega_1^*|$ and if **CH** is false then there is no ω_1-scale. The results from FRANKIEWICZ [1977] allow one to conclude that ω_1^* and ω_2^* are not homeomorphic and that then, in fact, κ^* and λ^* are not homeomorphic if $\omega \le \kappa < \lambda$ and $\langle \kappa, \lambda \rangle \ne \langle \omega, \omega_1 \rangle$. Some of the consequences of a positive answer have been shown to be consistent, see e.g., STEPRĀNS [1985].

Related to this question is the following:

? **243. Question 44.** *Is there consistently an uncountable cardinal κ such that ω^* and $U(\kappa)$ are homeomorphic?*

Here, $U(\kappa)$ is the subspace of $\beta\kappa$ consisting of all uniform ultrafilters. Let us observe that for such a κ we would have $\mathrm{cof}(\kappa) = \omega$ and $2^\omega = 2^\kappa$, see VAN DOUWEN [19∞a] for more information, including a proof of the following curious fact: there is at most one $n \in \omega$ for which there is a $\kappa > \omega_n$ with $U(\omega_n)$ and $U(\kappa)$ homeomorphic.

Recall from question 36 that $\mathfrak{n}$ is the minimal number of nowhere dense sets needed to cover ω^*. For any dense-in-itself topological space X one can define $n(X)$ ($wn(X)$) as the minimal cardinality of a family of nowhere dense sets that covers X (has a dense union). The number $wn(\omega^*)$ is equal to the cardinal $\mathfrak{h}$ (see the article by Vaughan). It is straightforward to show that $n(X^n) \ge n(X^m)$ and $wn(X^n) \ge wn(X^m)$ whenever $n \le m$. The general question is about the behaviour of the sequences $\langle n(\omega^*) : n \in \mathbb{N}\rangle$ and $\langle wn(\omega^*) : n \in \mathbb{N}\rangle$. Some specific questions:

? **244. Question 45.** *When do the sequences $\langle n(\omega^*) : n \in \mathbb{N}\rangle$ and $\langle wn(\omega^*) : n \in \mathbb{N}\rangle$ become constant?*

? **245. Question 46.** *Is it consistent that $n(\omega^*) > n(\omega^* \times \omega^*)$, that $wn(\omega^*) > wn(\omega^* \times \omega^*)$?*

What is known is that $\mathfrak{n} > \mathfrak{c}$ implies $n(\omega^*) = n(\omega^* \times \omega^*)$.

Here we pose the question suggested by Answer **13**:

? **246. Question 47.** *Does the statement that all Parovichenko spaces are coabsolute (with ω^*) imply that $cf(\mathfrak{c}) = \omega_1$?*

Question 48. *Let X be a compact space that can be mapped onto ω^*. Is X non-homogeneous?* **247. ?**

Van Douwen proved in [1978] proved that the answer to this question is in the affirmative provided that X has weight at most $\mathfrak{c}$. In general, the problem is unsolved. For more information, see the article by Kunen in this volume.

Question 49. *Is it consistent that every compact space contains either a converging sequence or a copy of $\beta\omega$?* **248. ?**

Under various extra-set-theoretical assumptions compact spaces have been constructed that contain neither a converging sequence nor a copy of $\beta\omega$, but no **ZFC**-example is known. It is as far as we know also unknown what the effect of **MA**$+\neg$**CH** is on this problem. Of this question there is also a Boolean Algebraic variant: is it consistent that every infinite Boolean Algebra has either a countably infinite homomorphic image or a complete homomorphic image.

Question 50. *Is there a locally connected continuum such that every proper subcontinuum contains a copy of $\beta\omega$?* **249. ?**

Question 51. *Is there an extremally disconnected normal locally compact space that is not paracompact?* **250. ?**

Kunen and Parsons proved in [1979] proved that if κ is weakly compact, then the space $\beta\kappa \setminus U(\kappa)$ is normal but not paracompact. In addition, van Douwen [1979] proved that there is a locally compact basically disconnected (= the closure of every open F_σ-set is open) space which is normal but not paracompact. This is basically all we know about this problem.

Question 52. *Is every compact hereditarily paracompact space of weight at most $\mathfrak{c}$ a continuous image of ω^*?* **251. ?**

This question is related to Answer **10**: Przymusiński showed that every perfectly normal (= hereditarily Lindelöf) compact space is a continuous image of ω^*, whereas the first-countable nonimage by Bell is hereditarily metacompact. Since perfectly normal compact spaces are (hereditarily) **ccc**, and since separable compact spaces are clearly continuous images of ω^*, we are also led to ask:

Question 53. *Is every hereditarily* **ccc** *compact space a continuous image of ω^*?* **252. ?**

The answer is yes under $\mathbf{MA}_{\omega_1}$ by Szentmiklossy's result from [1978] that then compact hereditarily **ccc** spaces are perfectly normal, and under

CH by Parovichenko's theorem, since compact hereditarily **ccc** spaces are of size at most $\mathfrak{c}$, see e.g., HODEL [1984].

Identify $\mathcal{P}(\omega)$ with ${}^{\omega}2$, and define $\leq_\alpha$ by $p \leq_\alpha q$ iff there is a map $f\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ of Baire class α such that $f(q) = p$.

? **253. Question 54.** *Suppose that $p \leq_\alpha q$ and $q \leq_\alpha p$. Are p and q* **RK**-*equivalent, or can they be mapped to each other by a Baire isomorphism of class α?*

? **254. Question 55.** *Do $\leq_\alpha$-minimal points exist, and can they be characterized?*

? **255. Question 56.** *Do $\leq_\alpha$-incomparable points exist?*

For the next question identify ω with $\mathbb{Q}$.

? **256. Question 57.** *If I is the ideal of nowhere dense subsets of $\mathbb{Q}$ can I be extended to a (tall) P-ideal?*

See DOW [1990] for more information on this problem (a YES answer implies that the space of minimal prime ideals of $C(\omega^*)$ is not basically disconnected).

The following question is probably more about forcing than about $\beta\omega$.

? **257. Question 58.** *Is there a* **ccc** *forcing extension of L, in which there are no P-points?*

Now we formulate some problems on characters of ultrafilters. It it easy to show that $\omega_1 \leq \chi(p) \leq \mathfrak{c}$ for all $p \in \omega^*$. Furthermore in [1939] POSPÍŠIL has shown that there are $2^{\mathfrak{c}}$ points in ω^* of character $\mathfrak{c}$. This is the best one can say: under **MA** we have $\chi(p) = \mathfrak{c}$ for all $p \in \omega^*$ while by exercise VII A10 in KUNEN [1980] the existence of a $p \in \omega^*$ with $\chi(p) = \omega_1$ is consistent with any cardinal arithmetic.

There are several models in which one has a $p \in \omega^*$ with $\chi(p) < \mathfrak{c}$, but these models have a few properties in common.

The first is that in all of these models there are P-points in ω^*.

These constructions fall roughly speaking into two categories: in the first of these, and the models from KUNEN [1980] fall into this one, the ultrafilter of small character is build in an iterated forcing construction and is almost unavoidably a P-point.

In the constructions of the second category one normally starts with a model of **CH** and enlarges the continuum while preserving some ultrafilters from the ground model. Again the ultrafilters that are preserved are most of the time P-points. An extreme case of this are the models for **NCF** from BLASS and SHELAH [1987, 19∞]: there the ultrafilters that are preserved are precisely the P-points and in the final model we even have $\chi(p) < \mathfrak{c}$ if and only if p is a P-point.

In Hart [1989] the first author showed that in the model obtained by adding any number of Sacks-reals side-by-side there are many types of ultrafilters of character ω_1 including very many non P-points. Unfortunately all these ultrafilters were constructed using P-points. This leads us to our first problem:

Question 59. *Is there a model in which there are no P-points, but there is an ultrafilter of character less than $\mathfrak{c}$?* **258. ?**

This is probably a very difficult problem and an answer to the following problem may be easier to give:

Question 60. *Is there a model in which there is an ultrafilter of character less than $\mathfrak{c}$ without any P-point below it in the Rudin-Keisler order?* **259. ?**

The second property of these models is maybe not so obvious: in all models that we know of there seems to be only one character below $\mathfrak{c}$. In the majority of these models we have $\mathfrak{c} = \omega_2$ so that an ultrafilter of small character automatically has character ω_1. In various other models usually nothing is known about ultrafilters other than the ones constructed explicitly. Thus we get to our second problem: let $\Xi = \{\, \chi(p) : p \in \omega^* \text{ and } \chi(p) < \mathfrak{c} \,\}$.

Question 61. *What are the possibilities for Ξ; can Ξ be the set of all (regular) cardinals below $\mathfrak{c}$, with $\mathfrak{c}$ large; what is Ξ in the side-by-side Sacks model?* **260. ?**

In [1989] Frankiewicz, Shelah and Zbierski announce the consistency of "$\mathfrak{c} > \omega_2$ and for every regular $\kappa \le \mathfrak{c}$ there is an ultrafilter of character κ".

Here we mention a well-known question on the Rudin-Keisler order, which has some partial answers involving characters of ultrafilters.

Question 62. *Is there for every $p \in \omega^*$ a $q \in \omega^*$ such that p and q are $\le_{\mathbf{RK}}$-incomparable?* **261. ?**

It is known that there exist p and q in ω^* such that p and q are $\le_{\mathbf{RK}}$-incomparable (Kunen [1972]). However, the full answer to this problem is not known yet, some partial positive results can be found in Hindman [1988] and Butkovičová [19∞b]; for example if p is such that $\chi(r) = \mathfrak{c}$ for every $r \le_{\mathbf{RK}}$ then there is a q that is $\le_{\mathbf{RK}}$-incomparable with p, and if $2^\kappa > \mathfrak{c}$ for some $\kappa < \mathfrak{c}$ then such a q can be found for every p of character $\mathfrak{c}$. The ideas in Blass and Shelah [1987, 19∞] may shed light on this problem.

The *π-character* of a point p in a space X is the minimum cardinality of a family of nonempty open sets such that every neighborhood of p contains one of them.

? **262. Question 63.** *Is there consistently a point in ω^* whose π-character has countable cofinality?*

Under **MA** the π-character of any ultrafilter is $\mathfrak{c}$. It was shown by BELL and KUNEN in [1981] that there is always a p with $\pi\chi(p) \geq \mathrm{cof}(\mathfrak{c})$ and that it is consistent that both $\mathfrak{c} = \aleph_{\omega_1}$ and $\pi\chi(p) = \omega_1$ for all $p \in \omega^*$.

Another character problem is the following:

? **263. Question 64.** *Is it consistent that $t(p, \omega^*) < \chi(p)$ for some $p \in \omega^*$?*

The tightness $t(p, X)$ of a point p in a space X is the smallest cardinal κ such that: whenever $p \in \overline{A}$ there is a $B \subseteq A$ with $|B| \leq \kappa$ such that $p \in \overline{A}$.

For the next few questions we consider the product $\omega \times I$, the projection $\pi\colon \omega \times I \to \omega$ and its Čech-Stone-extension $\beta\pi$. For $p \in \omega^*$ we put $I_p = \beta\pi^{\leftarrow}(p)$. It is not too hard to show that I_p is a continuum, and in fact a component of the remainder of $\omega \times I$. Our first question is:

? **264. Question 65.** *Are there p and q with I_p and I_q not homeomorphic?*

I_p has many cutpoints: for every $f\colon \omega \to I$ the point $f_p = p\text{-}\lim\langle n, f(n)\rangle$ is a cutpoint of I_p provided $\{\, n : f(n) \neq 0, 1 \,\} \in p$. The question is whether there are any others.

? **265. Question 66.** *Are there cutpoints in I_p other than the points f_p for $f\colon \omega \to I$?*

Under $\mathbf{MA}_{\text{countable}}$ such points exist, and it is conjectured that there are none in Laver's model (LAVER [1976]) for the Borel Conjecture.

? **266. Question 67.** *How many subcontinua does I_p have?*

SMITH [1986] and VAN DOUWEN [1977] have a few.

The next few questions come from analysis. We refer the reader to the book by DALES and WOODIN [1987] for more information on, and references for, what follows.

It is an old problem of Kaplansky for what (if any) compact spaces X the algebra $C(X)$ admits an incomplete norm. For the moment call X *incomplete* if $C(X)$ does admit an incomplete norm. It is not overly difficult to show that if $\beta\omega$ is incomplete then so is every space X. Moreover if some space is incomplete then so is $\omega + 1$ (the converging sequence).

The problem itself is solved to a large extent. Dales and Esterle independently showed that under **CH** the space $\beta\omega$—and hence every space—is incomplete. Woodin showed that it is consistent with $\mathbf{MA}+\neg\mathbf{CH}$ that $\omega+1$—and hence every space—is not incomplete. What remains is the following question:

Question 68. *If $C(\omega+1)$ admits an incomplete norm then does $C(\beta\omega)$ admit one too?* **267. ?**

To make the question maybe a bit more managable and also to be able to pose some more specialized problems we take the following facts from Dales and Woodin [1987]: to begin observe that $C(\beta\omega)$ is the same as ℓ^∞. For $p \in \omega^*$ put $\mathcal{M}_p = \{ x \in \ell^\infty : p\text{--}\lim x = 0 \}$ and $\mathcal{I}_p = \{ x \in \ell^\infty : \{ n : x(n) = 0\} \in p \}$. The quotient algebra $\mathcal{M}_p/\mathcal{I}_p$ is denoted by A_p. Now one can show that $\beta\omega$ is incomplete iff for some $p \in \omega^*$ the algebra A_p admits a non-trivial seminorm. One can do a similar thing for $\omega+1$. We write c_0/p for the algebra $c_0/\mathcal{I}_p$, where as usual $c_0 = \{ x \in \ell^\infty : \lim x = 0\}$. Now $\omega+1$ is incomplete iff for some $p \in \omega^*$ the algebra c_0/p admits a non-trivial seminorm.

We see that if there is a $p \in \omega^*$ such that A_p is seminormable then there is a $q \in \omega^*$ such that c_0/q is seminormable. Problem 68 now becomes: "if c_0/p is seminormable for some p, is there a q such that A_q is seminormable?" A stronger question is:

Question 69. *If $p \in \omega^*$ and c_0/p is seminormable, is A_p seminormable?* **268. ?**

It would also be interesting to know the answer to the following:

Question 70. *If $p \in \omega^*$ and A_p is seminormable, is c_0/p seminormable?* **269. ?**

Finally, to end this set, we mention a question connected to Woodin's proof. First we define a partial order $\ll$ on the algebras A_p and c_0/p: say $a \ll b$ iff there is a c such that $a = bc$. If B is A_p or c_0/p we call B weakly seminormable iff there are a nonempty downward closed—wrt. $\ll$—subset S of $B \setminus \{0\}$ and a strictly increasing map of $\langle S, \ll\rangle$ into $\langle {}^\omega\omega, <^*\rangle$. It can be shown that if B is seminormable, it is also weakly seminormable. In addition if there is a p such that c_0/p is weakly seminormable, there is also a q such that A_q is weakly seminormable. Thus a positive answer to the following question would also answer question 68 positively.

Question 71. *If $p \in \omega^*$ and A_p is weakly seminormable, is A_p seminormable, or is A_q for some other q?* **270. ?**

We now turn to the Rudin-Frolík order $\le_{\mathbf{RF}}$ on ω^*, which is defined as follows: $p \le_{\mathbf{RF}} q$ iff there is an embedding $i\colon \beta\omega \to \beta\omega$ such that $i(p) = q$. A lot is known about this order but a few problems remain:

Question 72. *What are the possible lengths of unbounded $\le_{\mathbf{RF}}$-chains?* **271. ?**

In [1985, 1984] Butkovičová has shown that ω_1 and $\mathfrak{c}^+$ are both possible. Another question is related to decreasing $\le_{\mathbf{RF}}$-chains:

? **272.** **Question 73.** *For what cardinals κ is there a strictly decreasing chain of copies of $\beta\omega$ in ω^* with a one-point intersection?*

VAN DOUWEN [1985] showed that $\mathfrak{c}$ works. It is readily seen that for any κ for which there is a positive answer to this question one gets a strictly decreasing $\leq_{\mathbf{RF}}$-chain of length κ without a lower bound.

However BUTKOVIČOVÁ [19∞a] has shown that such chains exist for every infinite $\kappa < \mathfrak{c}$. What is needed, in case κ has uncountable cofinality, to produce such a chain, is a strictly decreasing sequence $\langle X_\alpha : \alpha < \kappa \rangle$ of copies of $\beta\omega$ and a point p in $K = \bigcap_{\alpha<\kappa} X_\alpha$ which is not an accumulation point of any countable discrete subset of K. Butkovičová constructed such a sequence and such a point directly, but one naturally wonders whether this can be done for every sequence of copies of $\beta\omega$.

? **273.** **Question 74.** *If $\kappa \leq \mathfrak{c}$ has uncountable cofinality and if $\langle X_\alpha : \alpha < \kappa \rangle$ is a strictly decreasing sequence of copies of $\beta\omega$ with intersection K, is there a point p in K that is not an accumulation point of any countable discrete subset of K?*

9. Uncountable Cardinals

In this section we collect some questions on ultrafilters on uncountable cardinals, and we are mainly interested in uniform ultrafilters here. We let κ denote an arbitrary infinite cardinal. To begin we ask whether ultrafilters of small character may exist.

? **274.** **Question 75.** *Is there consistently an uncountable cardinal κ with a $p \in U(\kappa)$ such that $\chi(p) < 2^\kappa$?*

Let us note that for "small" uncountable cardinals there is no easy analogue of Kunen's method mentioned above (see problem 59): to preserve the cardinals below κ one seems to need a κ-complete ultrafilter, and that brings us immediately to measurable cardinals. So we ask in particular:

? **275.** **Question 76.** *Is it consistent to have a measurable cardinal κ with a $p \in U(\kappa)$ such that $\chi(p) < 2^\kappa$?*

And to stay somewhat down to earth we also ask specifically:

? **276.** **Question 77.** *Is it consistent to have a uniform ultrafilter on ω_1 of character less than 2^{ω_1} e.g., ω_2?*

A related and intriguing question is:

Question 78. *Is it consistent to have cardinals $\kappa < \lambda$ with points $p \in U(\kappa)$ and $q \in U(\lambda)$ such that $\chi(p) > \chi(q)$?* **277. ?**

A question with a topological background is the following:

Question 79. *If $\kappa \geq \omega$ is nonmeasurable and $\mathcal{F}$ is a countably complete uniform filter on κ^+ then what is the cardinality of the set $\{ u \in U(\kappa^+) : \mathcal{F} \subseteq u \}$?* **278. ?**

If the cardinality is $2^{2^{\kappa^+}}$ then there are almost Lindelöf spaces X and Y with $X \times Y$ not even almost κ-Lindelöf. For $\kappa = \omega$ the cardinality is indeed $2^{2^{\omega_1}}$, see Balcar and Štěpánek [1986].

The next question is purely topological. To state it we must make some definitions. In general if A is a subset of a topological space X we let $[A]_{<\kappa}$ denote the set $\bigcup\{ \overline{B} : B \in [A]^{<\kappa} \}$. Furthermore if $\kappa \subseteq X \subseteq \beta\kappa$ then $\beta_X\kappa$ is the maximal subset of $\beta\kappa$ for which every (continuous) $f\colon \kappa \to X$ has a continuous extension $\bar{f}\colon \beta_X\kappa \to X$. The question is

Question 80. *Assume that κ is regular, that $\kappa \subseteq X \subseteq \beta\kappa$ is such that $[X]_{<\kappa} = X$ and $\beta_X\kappa = X$. Now if Y is a closed subspace of a power of X, is then also X a closed subspace of a power of Y?* **279. ?**

The answer is yes for $\kappa = \omega$, see Hušek and Pelant [1974] for the proof and more information.

The following question is related to the analysis-type problems from the previous section. If p is a (uniform) ultrafilter on κ then we denote by $\mathbb{R}_p$ the ultrapower of $\mathbb{R}$ modulo the ultrafilter p. On it we define an equivalence relation $\equiv$ by $a \equiv b$ iff there is an $n \in \mathbb{N}$ such that $|a| < |nb|$ and $|b| < |na|$, here $<$ is the natural linear order of $\mathbb{R}_p$. The equivalence classes under $\equiv$ are called the Archimedean classes of $\mathbb{R}_p$. The question is whether the cardinality of $\mathbb{R}_p$ can be larger than the cardinality of $\mathbb{R}_p/\equiv$, specifically:

Question 81. *Are there κ and $p \in U(\kappa)$ such that $|\mathbb{R}_p| > |\mathbb{R}_p/\equiv| = \mathfrak{c}$?* **280. ?**

Dales and Woodin have shown that a positive answer is consistent relative to the existence of a large cardinal.

We finish with two questions about ω_1.

Question 82. *Is there a C^*-embedded bi-Bernstein set in $U(\omega_1)$?* **281. ?**

A bi-Bernstein set is a set X such that X and its complement intersect every uncountable closed subset of $U(\omega_1)$.

? **282.** **Question 83.** *Are there open sets G_1 and G_2 in $U(\omega_1)$ such that $\overline{G_1} \cap \overline{G_2}$ consists of exactly one point?*

See DOW [1988b] for more information on ω_1^*.

References

ARKHANGEL'SKI , A. V.

[1969] On the cardinality of bicompacta satisfying the first axiom of countability. *Soviet Math. Doklady*, **10**, 951–955.

BALCAR, B., J. DOCKALKOVA, and P. SIMON.

[1984] Almost disjoint families of countable sets. In *Finite and Infinite Sets II*, pages 59–88. *Coll. Math. Soc. Bolyai János* 37, Budapest (Hungary).

BALCAR, B. and R. FRANKIEWICZ.

[1978] To distinguish topologically the spaces m^*, II. *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys.*, **26**, 521–523.

BALCAR, B., J. PELANT, and P. SIMON.

[1980] The space of ultrafilters on $\mathbb{N}$ covered by nowhere dense sets. *Fund. Math.*, **110**, 11–24.

BALCAR, B. and P. SIMON.

[1989] Disjoint refinement. In *Handbook of Boolean Algebras*, J. Donald Monk with Robert Bonnet, editor, chapter 9, pages 333–386. North-Holland, Amsterdam.

BALCAR, B. and P. VOJTAS.

[1980] Almost disjoint refinement of families of subsets of $\mathbb{N}$. *Proc. Amer. Math. Soc.*, **79**, 465–470.

BALCAR, B. and P. STEPANEK.

[1986] *Teorie Množin.* Academia, Praha. in Czech.

BAUMGARTNER, J. E.

[1984] Applications of the Proper Forcing Axiom. In *Handbook of Set Theoretic Topology*, K. Kunen and J. E. Vaughan, editors, chapter 21, pages 913–960. North-Holland, Amsterdam.

BELL, M. G.

[1982] The space of complete subgraphs of a graph. *Comm. Math. Univ. Carolinae*, **23**, 525–536.

[19∞] A first countable compact space that is not an $\mathbb{N}^*$ image. *Top. Appl.* to appear.

BELL, M. G. and K. KUNEN.

[1981] On the pi-character of ultrafilters. *C. R. Math. Rep. Acad. Sci. Canada*, **3**, 351–356.

BELLAMY, D.

[1971] An non-metric indecomposable continuum. *Duke Math J.*, **38**, 15–20.

Blass, A.

[1986] Near Coherence of Filters I: cofinal equivalence of models of arithmetic. *Notre Dame J. Formal Logic*, **27**, 579–591.

[1987] Near Coherence of Filters II: applications to operator ideals, the Stone-Čech remainder of a half-line, order ideals of sequences, and slenderness of groups. *Trans. Amer. Math. Soc.*, **300**, 557–581.

Blass, A. and S. Shelah.

[1987] There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed. *Ann. Pure Appl. Logic*, **33**, 213–243.

[19∞] Near Coherence of Filters III: a simplified consistency proof. *Notre Dame J. Formal Logic.* to appear.

Butkovicova, E.

[1984] Long chains in Rudin-Frolík order. *Comm. Math. Univ. Carolinae*, **24**, 563–570.

[1985] Short branches in Rudin-Frolík order. *Comm. Math. Univ. Carolinae*, **26**, 631–635.

[19∞a] Decreasing chains without lower bounds in Rudin-Frolík order. *Proc. Amer. Math. Soc.* to appear.

[19∞b] A remark on incomparable ultrafilters in the Rudin-Frolík-order. *Proc. Amer. Math. Soc.* to appear.

Chae, S. B. and J. H. Smith.

[1980] Remote points and G-spaces. *Top. Appl.*, **11**, 243–246.

Dales, H. G. and W. H. Woodin.

[1987] *An Introduction to Independence for Analysts.* London Mathematical Society Lecture Note Series 115, Cambridge University Press, Cambridge.

van Douwen, E. K.

[1977] Subcontinua and nonhomogeneity of $\beta\mathbb{R}^+ - \mathbb{R}^+$. *Notices Amer. Math. Soc.*, **24**, A559.

[1978] Nonhomogeneity of products of preimages and π-weight. *Proc. Amer. Math. Soc.*, **69**, 183–192.

[1979] A basically disconnected normal space Φ with $|\beta\Phi\backslash\Phi| = 1$. *Canad. J. Math.*, **31**, 911–914.

[1981a] Prime mappings, number of factors and binary operations. *Diss. Math.*, **199**, 1–35.

[1981b] Remote points. *Diss. Math.*, **188**, 1–45.

[1985] A $\mathfrak{c}$-chain of copies of $\beta\omega$. In *Topology, Coll. Math. Soc. Bolyai János 41 (Eger, Hungary 1983)*, pages 261–267. North-Holland, Amsterdam.

[1990] The automorphism group of $\mathcal{P}(\omega)/fin$ need not be simple. *Top. Appl.*, **34**, 97–103.

[19∞a] On question Q47. *Top. Appl.* to appear.

[19∞b] The Čech-Stone compactification of a discrete groupoid. *Top. Appl.* to appear.

VAN DOUWEN, E. K., K. KUNEN, and J. VAN MILL.
[1989] There can be proper dense C^*-embedded subspaces in $\beta\omega \setminus \omega$. *Proc. Amer. Math. Soc.*, **105**, 462–470.

VAN DOUWEN, E. K. and J. VAN MILL.
[1980] Subspaces of basically disconnected spaces or quotients of countably complete Boolean Algebras. *Trans. Amer. Math. Soc.*, **259**, 121–127.

VAN DOUWEN, E. K., J. D. MONK, and M. RUBIN.
[1980] Some questions about Boolean Algebras. *Algebra Universalis*, **11**, 220–243.

VAN DOUWEN, E. K. and T. C. PRZYMUSINSKI.
[1979] First countable and countable spaces all compactifications of which contain $\beta\mathbb{N}$. *Fund. Math.*, **102**, 229–234.

DOW, A.
[1984] Coabsolutes of $\beta\mathbb{N} - \mathbb{N}$. *Top. Appl.*, **18**, 1–15.
[1988a] *An Introduction to Applications of Elementary Submodels to Topology.* Technical Report 88–04, York University.
[1988b] PFA and ω_1^*. *Top. Appl.*, **28**, 127–140.
[1990] The space of minimal prime ideals of $C(\beta\mathbb{N} \setminus \mathbb{N})$ is probably not basically disconnected. In *General Topology and Applications, Proceedings of the 1988 Northeast Conference*, R. M. Shortt, editor, pages 81–86. Marcel Dekker, Inc., New York.

DOW, A. and J. VAN MILL.
[1982] An extremally disconnected Dowker space. *Proc. Amer. Math. Soc.*, **86**, 669–672.

FEFERMAN, S.
[1964/65] Some applications of the notions of forcing and generic sets. *Fund. Math.*, **56**, 325–345.

FRANKIEWICZ, R.
[1977] To distinguish topologically the spaces m^*. *Bull. Acad. Sci. Pol.*, **25**, 891–893.
[1985] Some remarks on embeddings of Boolean Algebras and topological spaces, II. *Fund. Math.*, **126**, 63–68.

FRANKIEWICZ, R., S. SHELAH, and P. ZBIERSKI.
[1989] Embeddings of Boolean Algebras in $\mathcal{P}(\omega)$ mod finite. *Abstracts Amer. Math. Soc.*, **10**, 399. Abstract N° 89T-03-189.

FURSTENBERG, H.
[1981] *Recurrence in Ergodic Theory and Combinatorial Number Theory.* M. B. Porter Lectures, Rice University, Princeton University Press, Princeton, New Jersey.

GILLMAN, L. J.
[1966] The space $\beta\mathbb{N}$ and the Continuum Hypothesis. In *Proceedings of the Second Prague Topological Symposium*, pages 144–146. Academia, Praha.

GRYZLOV, A. A.

[1982] On the question of hereditary normality of the space $\beta\omega \setminus \omega$. In *Topology and Set Theory*, pages 61–64. Udmurt. Gos. Univ., Izhevsk. in Russian.

[1984] Some types of points in $\beta\mathbb{N}$. In *Proc. 12-th Winterschool on Abstract Analysis (Srní 1984) Rend. Circ. Mat. Palermo Suppl. No. 6*, pages 137–138.

HART, K. P.

[1989] Ultrafilters of character ω_1. *J. Symb. Logic*, **54**, 1–15.

HINDMAN, N.

[1979] Ultrafilters and combinatorial number theory. In *Number Theory Carbondale*, M. Nathanson, editor, pages 119–184. *Lecture Notes in Mathematics* 751, Springer-Verlag, Berlin etc.

[1988] Is there a point in ω^* that sees all others? *Proc. Amer. Math. Soc.*, **104**, 1235–1238.

HODEL, R.

[1984] Cardinal Functions I. In *Handbook of Set-Theoretic Topology*, K. Kunen and J. E. Vaughan, editors, chapter 1, pages 1–61. North-Holland, Amsterdam.

HUSEK, M. and J. PELANT.

[1974] Note about atom-categories of topological spaces. *Comm. Math. Univ. Carolinae*, **15**, 767–773.

JUST, W.

[19∞] Nowhere dense P-subsets of ω^*. *Proc. Amer. Math. Soc.* to appear.

KETONEN, J.

[1976] On the existence of P-points in the Stone -Čech compactification of integers. *Fund. Math.*, **62**, 91–94.

KOPPELBERG, S.

[1985] Homogeneous Boolean Algebras may have non-simple automorphism groups. *Top. Appl.*, **21**, 103–120.

KUNEN, K.

[1972] Ultrafilters and independent sets. *Trans. Amer. Math. Soc.*, **172**, 299–306.

[1980] *Set Theory. An introduction to independence proofs.* Studies in Logic and the foundations of mathematics 102, North-Holland, Amsterdam.

KUNEN, K. and L. PARSONS.

[1979] Projective covers of ordinal subspaces. *Top. Proc.*, **3**, 407–428.

LAVER, R.

[1976] On the consistency of Borel's Conjecture. *Acta Math.*, **137**, 151–169.

LOUVEAU, A.

[1973] Charactérisation des sous-espaces compacts de $\beta\mathbb{N}$. *Bull. Sci. Math.*, **97**, 259–263.

MAHARAM, D.

[1976] Finitely additive measures on the integers. *Sankhyā Ser. A*, **38**, 44–59.

Malykhin, V. I.
[1987] $\beta\omega$ under negation of **CH**. Interim Report of the Prague Topological Symposium. 2/1987.

Mathias, A. R. D.
[1978] $0^{\#}$ and the P-point problem. In *Higher Set Theory, Oberwolfach 1977*, G. H. Müller and D. S. Scott, editors, pages 375–384. *Lecture Notes in Mathematics* 669, Springer-Verlag, Berlin etc.

van Mill, J.
[1983] A remark on the Rudin-Keisler order of ultrafilters. *Houston J. Math.*, **9**, 125–129.
[1984] An Introduction to $\beta\omega$. In *Handbook of Set-Theoretic Topology*, K. Kunen and J. E. Vaughan, editors, chapter 11, pages 503–568. North-Holland, Amsterdam.
[1986] An easy proof that $\beta\mathbb{N} - \mathbb{N} - \{p\}$ is not normal. *Ann. Math. Silesianae*, **2(14)**, 81–84.

Miller, A. W.
[1980] There are no Q-points in Laver's model for the Borel Conjecture. *Proc. Amer. Math. Soc.*, **78**, 103–106.
[1984] Rational perfect set forcing. In *Axiomatic Set Theory*, J. E. Baumgartner, D. A. Martin, and S. Shelah, editors, pages 143–159. *Contemporary Mathematics* 31, American Mathematical Society, Providence.

Mioduszewski, J.
[1978] An approach to $\beta\mathbb{R} \setminus \mathbb{R}$. In *Topology, Coll. Math. Soc. Bolyai János 23*, pages 853–854. Budapest(Hungary).

Parovicenko, I. I.
[1963] A universal bicompact of weight $\aleph$. *Soviet Math. Doklady*, **4**, 592–592.

Posp sil, B.
[1939] On bicompact spaces. *Publ. Fac. Sci. Univ. Masaryk*, **270**, 3–16.

Przymusinski, T. C.
[1982] Perfectly normal compact spaces are continuous images of $\beta\mathbb{N} - \mathbb{N}$. *Proc. Amer. Math. Soc.*, **86**, 541–544.

Rajagopalan, M.
[1972] $\beta\mathbb{N} \setminus \mathbb{N} \setminus \{p\}$ is not normal. *Journal of the Indian Math. Soc.*, **36**, 173–176.

Rudin, M. E.
[1970] Composants and $\beta\mathbb{N}$. In *Proc. Wash. State Univ. Conf. on Gen. Topology*, pages 117–119. Pullman, Washington.

Ryll-Nardzewski, C. and R. Telgarsky.
[1970] On the scattered compactification. *Bull. Acad. Polon. Sci., Sér. Sci., Math. Astronom. et Phys.*, **18**, 233–234.

Shapiro, L. B.
[1985] A counterexample in the theory of dyadic bicompacta. *Russian Math. Surveys*, **40**, 1985.

SHELAH, S.
[1982] *Proper Forcing.* Lecture Notes in Mathematics 940, Springer-Verlag, Berlin etc.

SHELAH, S. and J. STEPRANS.
[1988] **PFA** implies all automorphisms are trivial. *Proc. Amer. Math. Soc.*, **104**, 1220–1225.

SIMON, P.
[1987] A closed separable subspace of $\beta\mathbb{N}$ which is not a retract. *Trans. Amer. Math. Soc.*, **299**, 641–655.

SMITH, M.
[1986] The subcontinua of $\beta[0,\infty) - [0,\infty)$. *Top. Proc.*, **11**, 385–413.

STEPRANS, J.
[1985] Strong-Q-sequences and variations on Martin's Axiom. *Can. J. Math.*, **37**, 730–746.

SZENTMIKLOSSY, Z.
[1978] S-spaces and L-spaces under Martin's Axiom. In *Topology*, pages 1139–1145. *Coll. Math. Soc. Bolyai János* 23, Budapest(Hungary).

STEPANEK, P. and M. RUBIN.
[1989] Homogeneous Boolean Algebras. In *Handbook of Boolean Algebras*, J. M. with Robert Bonnet, editor, chapter 18, pages 679–715. North-Holland, Amsterdam.

WARREN, N. M.
[1972] Properties of Stone-Čech compactifications of discrete spaces. *Proc. Amer. Math. Soc.*, **33**, 599–606.

WEISS, W.
[1984] Versions of Martin's Axiom. In *Handbook of Set-Theoretic Topology*, K. Kunen and J. E. Vaughan, editors, chapter 19, pages 827–886. North-Holland, Amsterdam.

WIMMERS, E.
[1982] The Shelah P-point independence theorem. *Israel J. Math.*, **43**, 28–48.

WOODS, R. G.
[1968] *Certain properties of $\beta X \setminus X$ for σ-compact X.* PhD thesis, McGill University (Montreal).