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Some topological groups with, and some without, proper dense subgroups

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Abstract

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Continuing earlier investigations into the question of the existence of a proper dense subgroup of a given topological group, the authors obtain some positive and some negative results, as follows:

(a) Every non-degenerate connected Abelian group contains a proper dense subgroup.

(b) Every infinite pseudocompact group contains a proper dense subgroup.

(c) If G is a totally bounded Abelian group with $wG=|G|=\alpha>\omega$, and if in addition G is torsion-free (or more generally, if $|G/\text{tor } G|=\alpha$), then G has a proper dense subgroup.

(d) For every strong limit cardinal α such that $\text{cf}(\alpha)=\omega$ there is a totally bounded Abelian torsion group G such that $wG=|G|=\alpha$ and G has no proper dense subgroup.

Among the questions left unsettled is this: Can the conditions in (d) on α be relaxed or even omitted?

Keywords: Totally bounded group, proper dense subgroup, torsion-free group, pseudocompact group, connected group.

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Notation and conventions

Each ordinal number (in particular, each cardinal number) is identified with its set of ordinal predecessors. That is, for each ordinal number ξ we write

$$\xi=\{\eta:\ \eta<\xi\}=\{\eta:\ \eta\in\xi\}.$$

The smallest infinite cardinal is denoted ω; the symbols α, β, γ, κ denote infinite cardinals.

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The cardinality of a set X is written $|X|$. The weight and density of a space X are written wX and dX, respectively.

The symbols $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{T}$ denote the integers, the reals and the circle, respectively; in each case the usual algebraic operations and the usual topology are understood. In dealing with $\mathbb{T}$ we use multiplicative notation and we denote the identity by 1; for every other Abelian group we use additive notation and we denote the identity by 0.

The torsion subgroup of an Abelian group G is written tG, and $\phi: G \to G/tG$ is the canonical homomorphism.

The topological groups we consider are all assumed to satisfy the Hausdorff separation axiom. As is well known (see for example [11, 8.4], this condition guarantees that our groups are completely regular spaces, i.e., Tychonoff spaces.

1. Background

The naive question "Does every topological Abelian group have a proper dense subgroup?" is too general to be interesting. Here we indicate some facts which suggest reasonable constraints that lead to non-trivial specializations of that question.

For reference to the Remarks cited in 1.1, 1.2, 1.3, and 1.4, the reader may consult [1, 3.1, 3.8 and 7.4, and 7.2, respectively].

Remark 1.1. Clearly, a discrete group (in particular, a finite group) has no proper dense subgroup.

Remark 1.2. Every infinite compact group G with $wG = \alpha$ satisfies $|G| = 2^\alpha$. Since $dX \leq wX$ for every space X, each such G has a proper dense subgroup.

Remark 1.3. The equality $|G| = 2^\alpha$ holds also for every non-discrete, locally compact, σ-compact group G, so as in Remark 1.2 every such group has a proper dense subgroup.

Remark 1.4. It is tempting to conjecture on the basis of Remark 1.3 that every non-discrete, locally compact group admits a proper dense subgroup, but Rajagopalan and Subrahmanian [12] have described in detail a number of (Abelian, divisible) counterexamples.

Remark 1.5. A topological group G is said to be *totally bounded* (by some authors: *pre-compact*) if, for every non-empty open subset U of G, G is covered by a finite number of translates of U. It is a theorem of Weil [13] that a topological group G is totally bounded if and only if G is a dense subgroup of a compact group. When these conditions hold the enveloping compact group is unique in the obvious sense. It is denoted $\bar{G}$ and is called the *Weil completion* of G.

The groups of Rajagopalan and Subrahmanian [12] cannot be totally bounded: A locally compact, totally bounded topological group is compact, hence does have a proper dense subgroup by Remark 1.2. It is then natural to modify the naive question posed above as follows: Does every infinite totally bounded Abelian group have a proper dense subgroup? The answer to this question is a strong "No". Indeed, given an arbitrary infinite Abelian group G, let $G^{\#}$ denote the group G with the weakest topology making each element of $\mathrm{Hom}(G, \mathbb{T})$ continuous. That is, $G^{\#}$ has the topology relative to which the embedding function $i: G \to \mathbb{T}^{\mathrm{Hom}(G,\mathbb{T})}$ defined by $(ip)_h = h(p)$ is a homeomorphism. Clearly $G^{\#}$ is totally bounded. Further, every subgroup H of G is closed in $G^{\#}$: Given $x \in G \backslash H$ there is $h_x \in \mathrm{Hom}(G, \mathbb{T})$ such that $h_x \equiv 1$ on H and $h_x(x) \neq 1$, so the relation

$$H = \bigcap \{\ker(h_x) : x \in G \backslash H\}$$

expresses H as the intersection of closed subsets of $G^{\#}$.

Given Abelian groups G_0 and G_1, from

$$\mathrm{Hom}(G_0 \times G_1, \mathbb{T}) = \mathrm{Hom}(G_0, \mathbb{T}) \times \mathrm{Hom}(G_1, \mathbb{T})$$

one easily deduces $(G_0 \times G_1)^{\#} = (G_0^{\#}) \times (G_1^{\#})$. In contrast, given an infinite set $\{G_i : i \in I\}$ of non-degenerate Abelian groups, and writing $G = \prod_{i \in I} G_i$, there is no choice of topology $\mathcal{T}_i$ for G_i relative to which the product topology on the product group G is the topology of $G^{\#}$ (see [8, 2.6]). In particular for such G_i, the relation $G^{\#} = \prod_{i \in I} (G_i^{\#})$ is always false.

Remark 1.6. The inclusion $G^{\#} \subseteq \mathbb{T}^{\mathrm{Hom}(G,\mathbb{T})}$ makes it clear that

$$w(G^{\#}) \leq |\mathrm{Hom}(G, \mathbb{T})| \leq |\mathbb{T}^G| = 2^{\omega \cdot |G|} = 2^{|G|}.$$

In fact, equality holds: $w(G^{\#}) = 2^{|G|}$. This follows from (a) Pontrjagin duality; (b) the fact that every locally compact Abelian group and its dual have the same weight; and (c) the equality $|\mathrm{Hom}(G, \mathbb{T})| = 2^{|G|}$, valid for every infinite Abelian group G; see [11, 1] for proofs and references to the literature concerning these facts.

Remark 1.5 above indicates that we may reasonably specialize our search for the existence of proper dense subgroups to the context of totally bounded groups, and Remark 1.5 together with the identity $w(G^{\#}) = 2^{|G|}$ shows the futility of considering the case $wG = 2^{|G|} > |G|$. Accordingly the following question, to which this paper is principally devoted, emerges as the natural and 'correct' question in this area of inquiry.

Question 1.7. Does every totally bounded Abelian group G such that

$$\omega \leq wG \leq |G|$$

have a proper dense subgroup?

2. Totally bounded torsion-free groups

We show in this section that many torsion-free groups as in Question 1.7 contain not only a proper dense subgroup but in fact a large family of proper dense subgroups which is independent in the sense of the following definition.

Notation 2.1. For G a group and $X \subseteq G$, we denote by $\langle X \rangle$ the smallest subgroup of G containing X.

Definition 2.2. An indexed set $\{H_i : i \in I\}$ of subgroups of a group G is *independent* if

$$H_i \cap \left\langle \bigcup \{H_j : j \in I, j \neq i\} \right\rangle = \{0\} \quad \text{for each} \quad i \in I.$$

We first dispose of the case $\alpha = \omega$.

Theorem 2.3. *There are totally bounded, torsion-free Abelian groups* $(G_i, \mathcal{T}_i)$ $(i = 0, 1)$ *such that*

(a) $|G_i| = \omega$ $(i = 0, 1)$;

(b) $w(G_i, \mathcal{T}_i) = \omega$ $(i = 0, 1)$;

(c) *there is an independent set* $\{H_n : n < \omega\}$ *of proper dense subgroups of* $(G_0, \mathcal{T}_0)$; *and*

(d) $(G_1, \mathcal{T}_1)$ *has no proper dense subgroup.*

Proof. Let $\{r_n : n < \omega\}$ be a sequence of non-zero real numbers independent over the rationals, let $H_n = \{e^{2\pi i k r_n} : k \in \mathbb{Z}\} \subseteq \mathbb{T}$ and set $G_0 = \langle \bigcup_{n<\omega} H_n \rangle \subseteq \mathbb{T}$ with the topology $\mathcal{T}_0$ inherited from $\mathbb{T}$.

For $n < \omega$ let h_n be that element of $\mathrm{Hom}(\mathbb{Z}, \mathbb{T})$ such that $h_n(1) = e^{2\pi i/n}$ and define $\mathcal{T}_1$ on the group $G_1 = \mathbb{Z}$ by the requirement that the map $(\mathbb{Z}, \mathcal{T}_1) \to \mathbb{T}^\omega$ given by

$$k \to \{h_n(k) : n < \omega\} \in \mathbb{T}^\omega$$

is a homeomorphism.

Statements (a) and (b) are obvious. It is clear that the family $\{H_n : n < \omega\}$ is as required in (c). As for (d), we note that the proper subgroups of $\mathbb{Z}$ are exactly the kernels of the homomorphisms h_n. Since each of them is $\mathcal{T}_1$-closed, none of them is $\mathcal{T}_1$-dense. $\square$

We turn now to the case of totally bounded torsion-free Abelian groups of uncountable weight. In the following lemma we denote by $\phi : G \to G/tG$ the canonical homomorphism.

Lemma 2.4. *Let G be a totally bounded Abelian group such that $\omega < \alpha = wG = |G/tG|$. Let U be a non-empty open subset of G, and let H be a subgroup of G such that*

$|H| < \alpha$. *Then*

(a) $|\phi[U]| = \alpha$, *and*

(b) *there is* $x \in U$ *such that* $x \notin tG$ *and* $\langle\{x\}\rangle \cap H = \{0\}$.

Proof. (a) Let A be a subset of U such that $|A \cap \phi^{-1}(\phi(u))| = 1$ for each $u \in U$, and suppose that $|A| = \gamma < \alpha$. Since G is totally bounded there is finite $F \subseteq G$ such that $G = F + U$, and from

$$G = (F+U) + tG = F + (U + tG) = F + (A + tG) = (F + A) + tG$$

follows the contradiction

$$|G/tG| \leqslant |F + A| \leqslant \omega + \gamma < \alpha.$$

(b) With A chosen as in (a) we have $|A| = \alpha$ and $|A \cap tG| \leqslant 1$. We claim there is $x \in A \backslash tG$ as required. If the claim fails, then for every $x \in A \backslash tG$ there are $n \in \mathbb{Z}$ and $h \in H$ such that

$$nx = h \neq 0.$$

From $h \neq 0$ follows $n \neq 0$, and from $\alpha > \omega$ it follows that the map

$$A \backslash tG \to (\mathbb{Z} \backslash \{0\}) \times (H \backslash \{0\})$$

given by

$$a \to (n, h)$$

as in $(*)$ is not one-to-one. Thus for some $n \in \mathbb{Z}\backslash\{0\}$ and $h \in H\backslash\{0\}$ there are distinct x, $x' \in A \backslash tG$ such that

$$nx = nx' = h.$$

This gives $n(x - x') = 0$, and hence $x - x' \in tG$. It follows that

$$\phi(x) = \phi(x'),$$

contradicting the definition of A. $\square$

Theorem 2.5. *Let G be a totally bounded Abelian group such that $\omega < \alpha = wG = |G/tG|$. Then G has a proper dense subgroup K such that $|K| = \alpha$ and $K \cap tG = \{0\}$.*

Proof. Let $\{U_\xi : \xi < \alpha\}$ be a basis for G, fix $p \in G$ with $p \neq 0$, write $H_0 = \langle\{p\}\rangle$, notice $|H_0| < \alpha$, and use Lemma 2.4 to find $x_0 \in U_0 \backslash tG$ such that $\langle\{x_0\}\rangle \cap H_0 = \{0\}$.

If $0 < \xi < \alpha$ and x_η has been defined for all $\eta < \xi$, set

$$H_\xi = \langle\{p\} \cup \{x_\eta : \eta < \xi\}\rangle,$$

notice $|H_\xi| < \alpha$, and again use Lemma 2.4 to find $x_\xi \in U_\xi \backslash tG$ such that

$$\langle\{x_\xi\}\rangle \cap H_\xi = \{0\}.$$

With x_ξ having been thus defined for all $\xi < \alpha$, set

$$K = \langle\{x_\xi : \xi < \alpha\}\rangle.$$

Since $x_\xi \notin H_\xi$, the map $\xi \to H_\xi$ from α into K is one-to-one; thus $|K| = \alpha$. From $x_\xi \in U_\xi \cap K$ it follows that K is dense in G.

To see that $K \cap tG = \{0\}$, suppose that

$$0 \neq \sum_{i=0}^{k} n_i x_{\xi(i)} \in tG$$

with each $n_i \in \mathbb{Z}\backslash\{0\}$. Since $x_{\xi(0)} \notin tG$ we have $k > 0$. Assuming without loss of generality that $\xi(k) > \xi(i)$ for $i < k$ we have

$$n_k x_{\xi(k)} = -\sum_{i=0}^{k-1} n_i x_{\xi(i)} \in H_{\xi(k)}.$$

Then

$$n_k x_{\xi(k)} = 0$$

since

$$n_k x_{\xi(k)} \in \langle\{x_{\xi(k)}\}\rangle \cap H_{\xi(k)} = \{0\},$$

but

$$n_k x_{\xi(k)} \neq 0$$

since

$$n_k \neq 0 \quad \text{and} \quad x_{\xi(k)} \notin tG.$$

This contradiction completes the proof that $K \cap tG = \{0\}$.

Essentially the same argument shows $p \notin K$ (so that K is a proper subgroup of G). Indeed, suppose that

$$p = \sum_{i=0}^{k} n_i x_{\xi(i)}$$

with each $n_i \in \mathbb{Z}$. Since $p \neq 0$ we may assume without loss of generality that each $n_i \neq 0$; the relation $p \notin \langle\{x_{\xi(0)}\}\rangle$ gives $k > 0$. Assuming as before that $\xi(k) > \xi(i)$ for $i < k$ gives

$$n_k x_{\xi(k)} = p - \sum_{i=0}^{k-1} n_i x_{\xi(i)} \in H_{\xi(k)},$$

which together with $n_k \neq 0$ and $x_{\xi(k)} \notin tG$ contradicts the relation

$$\langle\{x_{\xi(k)}\}\rangle \cap H_{\xi(k)} = \{0\}. \qquad \square$$

Corollary 2.6. *Let G be a torsion-free, totally bounded Abelian group such that $\omega < \alpha = wG = |G|$. Then G has a proper dense subgroup.*

Remark 2.7. We initiated the argument of the proof of Theorem 2.5 with an element $p \neq 0$ in order to be able to show that the group K is proper. In case G has torsion elements this initial step and all reference to the element p may be omitted, since the fact that $K \cap tG = \{0\}$ will then guarantee that $K \neq G$.

The reasoning used in Theorem 2.5 yields the following stronger result. To avoid excessive repetition we indicate the necessary modifications briefly.

Theorem 2.8. *Let G be a totally bounded Abelian group such that $\omega < \alpha = wG = |G/tG|$. Then there is an independent family $\{K_\eta : \eta < \alpha\}$ of proper dense subgroups of G, with $K_\eta \cap tG = \{0\}$ for each $\eta < \alpha$.*

Proof. Let $\{U_\xi : \xi < \alpha\}$ be a basis for G indexed in such a way that each basic element appears α-many times, and set $H_0 = \{0\}$. If $\xi < \alpha$ and x_η has been defined for all $\eta < \xi$, define

$$H_\xi = \langle\{x_\eta : \eta < \xi\}\rangle$$

and as in the Proof of Theorem 2.5 find $x_\xi \in U_\xi \backslash tG$ such that $\langle\{x_\xi\}\rangle \cap H_\xi = \{0\}$.

Now write $X = \{x_\xi : \xi < \alpha\}$ and for $\xi < \alpha$ set

$$A(\xi) = \{x \in X : x \in U_\xi\}.$$

The repetitive condition on the listing $\{U_\xi : \xi < \alpha\}$ assures us that $|A(\xi)| = \alpha$ for each $\xi < \alpha$. Since $\{A(\xi) : \xi < \alpha\}$ is a listing of α-many (not necessarily distinct) sets, each of cardinality α, there is by the "disjoint refinement lemma" (as given, for example, in [4, Section 7.5]) for each $\xi < \alpha$ a set $B(\xi)$ such that

$$B(\xi) \subseteq A(\xi),$$

$$|B(\xi)| = \alpha, \quad \text{and}$$

$$B(\xi) \cap B(\zeta) = \emptyset \quad \text{for } \xi < \zeta < \alpha.$$

Write

$$B(\xi) = \{y_{\xi,\eta} : \eta < \alpha\} \quad \text{for } \xi < \alpha,$$

and for $\eta < \alpha$ define

$$K_\eta = \langle\{y_{\xi,\eta} : \xi < \alpha\}\rangle.$$

It is then easy to see, much as in the Proof of Theorem 2.5, that the family $\{K_\eta : \eta < \alpha\}$ is as required. □

Remark 2.9. Since each of the groups K_η just constructed is essentially disjoint from the group $\langle\{\bigcup K_{\eta'} : \eta' \neq \eta\}\rangle$, each group K_η is proper in G (even if $tG = \{0\}$). This explains why we did not need in the proof of Theorem 2.8 to begin the construction by fixing $p \in G$ and arranging that $p \notin K_\eta$.

3. Totally bounded torsion groups

As with the torsion-free case just considered, it is convenient to treat first the case $\alpha = \omega$.

Here is the analogue of Theorem 2.3 for torsion groups.

Theorem 3.1. *There are totally bounded Abelian torsion groups G_0 and G_1 such that*

(a) $|G_i| = \omega$ $(i = 0, 1)$;

(b) $w(G_i) = \omega$ $(i = 0, 1)$;

(c) *there is an independent set* $\{H_n : n < \omega\}$ *of proper dense subgroups of* G_0; *and*

(d) G_1 *has no proper dense subgroup.*

Proof. Take for G_0 the torsion subgroup $t\mathbb{T}$ of $\mathbb{T}$; the groups $H_n = \mathbb{Z}(p^\infty)$ with $\{p_n : n < \omega\}$ an enumeration of the prime numbers in $\mathbb{Z}$ constitute a family as in (c).

For G_1 one may take any one of the groups $\mathbb{Z}(p^\infty)$; it is known [10, pp. 15–16] that each proper subgroup of G_1 is finite. □

Remark 3.2. For any infinite cardinal α it is easy to find a totally bounded torsion group G such that $\alpha = wG = |G|$ and G has a proper dense subgroup D. For example, choose a dense subgroup D of $\{\pm 1\}^\alpha$ such that $|D| = \alpha$ and then choose G so that $D \subset G \subset \{\pm 1\}^\alpha$ with both inclusions proper and with $|G| = \alpha$. (That $wD = wG = \alpha$ in this setting follows from the relation $\bar{D} = \bar{G} = \{\pm 1\}^\alpha$ and the fact, easily proved, that $wH = w\bar{H}$ for every totally bounded topological group [6, 1.5].) In the following result we show that for certain uncountable cardinals α there are totally bounded torsion groups G such that $wG = \alpha$ and G has no proper dense subgroup.

For an infinite cardinal α we write $2^{<\alpha} = \sum \{2^\beta : \beta < \alpha\}$.

Theorem 3.3. *Let α be an uncountable cardinal such that $cf(\alpha) = \omega$. There is a totally bounded group Abelian torsion group G such that $wG = \alpha$, $|G| = 2^{<\alpha}$, and G has no proper dense subgroup.*

Proof. Choose a sequence $\{\alpha_n : n < \omega\}$ of infinite cardinals such that

$$\alpha_n < \alpha \quad \text{for } n < \omega \quad \text{and}$$

$$\sum_{n<\omega} \alpha_n = \alpha.$$

Define $k_n < \omega$ by the rule

$$k_0 = 2, \quad k_{n+1} = (k_0 \cdot k_1 \cdot \ldots \cdot k_n) + 1,$$

let $\mathbb{Z}(k_n)$ denote the cyclic group of order k_n, and define

$$A_n = \left(\bigoplus_{\alpha_n} \mathbb{Z}(k_n) \right)^{\#};$$

that is, A_n is the direct sum (=weak product) of α_n-many copies of $\mathbb{Z}(k_n)$, with its largest totally bounded topological group topology (as described in Remark 1.5). Define

$$G = \bigoplus_{n<\omega} A_n.$$

Here G has the topology inherited from the full product $\prod_{n<\omega} A_n$; since the latter is totally bounded, so is G. Clearly

$$|G| = \sum_{n<\omega} |A_n| = \sum_{n<\omega} \alpha_n = \alpha;$$

from Remark 1.6 it follows that $w(A_n) = 2^{\alpha_n}$ and hence

$$wG = \sum_{n<\omega} 2^{\alpha_n} = 2^{<\alpha}.$$

It remains to show that every (proper) subgroup H of G is closed. For $n < \omega$ set

$$G_n = \prod_{m\leq n} A_m = \bigoplus_{m\leq n} A_m,$$

note that the topology of G_n is the topology of $G_n^{\#}$, and let $\pi_n: G \to G_n$ be the natural projection. Given a subgroup H of G and $x \in \bar{H}\backslash H$, we claim first there is n such that $\pi_n(x) \notin \pi_n[H]$. Indeed, choose n so that $x_m = 0$ for all $m > n$, and suppose there is $a \in H$ such that $\pi_n(a) = \pi_n(x)$. There is $j < \omega$ such that $a_m = 0$ for all $m > n+j$; we define

$$k = k_{n+1} \cdot k_{n+2} \cdot \ldots \cdot k_{n+j}.$$

From $a \in H$ follows $ka \in H$. For $n+1 \leq m \leq n+j$, from $a_m \in A_m$ we have $k_m a_m = 0$ and therefore $ka_m = 0$. For $1 \leq m \leq n$ and $1 \leq i \leq j$ we have

$$k_{n+i} \equiv 1 \bmod k_m$$

and therefore $k \equiv 1 \bmod k_m$, so from $k_m a_m = 0$ follows $ka_m = 1 \cdot a_m = a_m = x_m$. The upshot is that

$$ka = x,$$

which together with $ka \in H$ and $x \notin H$ is a contradiction completing the claim.

Since the topology of G_n is the topology of $G_n^{\#}$, it follows from Remark 1.5 that every subgroup of G_n is closed in $G_n^{\#}$. The continuity of π_n then yields the contradiction

$$\pi_n(x) \in \pi_n[\bar{H}] \subseteq \overline{(\pi_n[H])} = \pi_n[H].$$

It follows that $H = \bar{H}$ for every subgroup H of G, as required. □

We note in passing that the group $G = \langle G, \mathcal{T}\rangle$ of Theorem 3.3 is an example of a totally bounded Abelian group which, although each of its subgroups is closed, does not carry its finest totally bounded topological group topology (that is, $\langle G, \mathcal{T}\rangle \neq G^{\#}$). To verify that $\langle G, \mathcal{T}\rangle \neq G^{\#}$ it is enough to find a $\mathcal{T}$-discontinuous homomorphism $h: G \to \mathbb{T}$ (for, as indicated above in Remark 1.5, every homomorphism from G to $\mathbb{T}$ is continuous when G carries the topology of $G^{\#}$). To do this one may argue much as in [5]: identify the group $\mathbb{Z}(k_n)$ with the multiplicative group of k_nth roots of 1 in $\mathbb{T}$, use the relation $\alpha = \sum_{n<\omega} \alpha_n$ to write

$$G = \bigoplus_{n<\omega} A_n \subseteq \bigoplus_{\xi<\alpha} \mathbb{T}_\xi = \mathbb{T}^\alpha,$$

and for $x \in G$ define

$$h(x) = \prod_{\xi<\alpha} x_\xi$$

(a finite product for each $x \in G$); it is easy to see that h is not $\mathcal{T}$-continuous on G.

Other examples of totally bounded Abelian groups $G = \langle G, \mathcal{T}\rangle$ for which $\langle G, \mathcal{T}\rangle \neq G^{\#}$ but every subgroup of G is $\mathcal{T}$-closed are easy to find. Perhaps the simplest are the subgroups $\mathbb{Z}(p^\infty)$ of $\mathbb{T}$. In these, as indicated above in the proof of Theorem 3.1, every proper subgroup is finite and hence closed.

A cardinal α is said to be a *strong limit cardinal* if $2^\beta < \alpha$ whenever $\beta < \alpha$. Clearly the generalized continuum hypothesis (GCH) implies that every limit cardinal is a strong limit cardinal, but strong limit cardinals α can be shown (without invoking GCH) to exist in every model of ZFC. Indeed, given a cardinal β, write $\beta_0 = \beta$, $\beta_{n+1} = 2^{\beta_n}$ for $n < \omega$, and $\alpha = \sup_{n<\omega} \beta_n$. This construction gives a strong limit cardinal α with $\alpha > \beta$ and with $\mathrm{cf}(\alpha) = \omega$. A similar argument shows in ZFC that for any cardinal β and any regular cardinal κ there is a strong limit cardinal α such that $\alpha > \beta$ and $\mathrm{cf}(\alpha) = \kappa$.

Corollary 3.4. *For every strong limit cardinal α such that* $\mathrm{cf}(\alpha) = \omega$ *there is a totally bounded Abelian torsion group G such that $wG = \alpha$, $|G| = \alpha$, and G has no proper dense subgroup.*

Proof. The case $\alpha > \omega$ is given by Theorem 3.3. For $\alpha = \omega$, as in the proof of Theorem 3.1 one may take for G one of the groups $\mathbb{Z}(p^\infty)$. Alternatively in the case $\alpha = \omega$ one may use the proof of Theorem 3.3 with k_n as given there and, say, with $\alpha_n = n$; the groups A_n are now discrete, and the conditions $|A_n| = \alpha_n$ and $w(A_n) = 2^{\alpha_n}$ of Theorem 3.3 now become $|A_n| = w(A_n) = (k_n)^n$; from this follows $|G| = wG = \omega$, as desired. $\square$

Some questions arise naturally from Theorem 3.3. We collect these and some other questions in Section 5 below.

4. Connected groups; pseudocompact groups

We have considered the question of the existence of proper dense subgroups by considering separately two antipodal classes: the torsion groups, and the torsion-free groups. The same question can be reasonably approached through a different antipodal pair: the connected groups and the totally disconnected groups. The torsion groups considered in Section 3 are easily seen to be totally disconnected; indeed it is known [9] that all groups of the form $G^{\#}$, and hence every product of such groups as in Theorem 3.3, are zero-dimensional. Thus there are some totally bounded, zero-dimensional Abelian groups which admit a proper dense subgroup, and others which do not. It is surprising that one can show easily, in contrast, that every connected Abelian group does have a proper dense subgroup.

We emphasize that in the following theorem the condition that G be totally bounded, and restrictions on the size of wG and $|G|$, have been abandoned.

Theorem 4.1. *Let G be an infinite connected topological Abelian group. Then G has a proper dense subgroup.*

Proof. It is well known and easy to prove from standard structure theorems (see for example [11, p. 227]) that there is a subgroup H of G such that $|G/H| = \omega$. We claim that H is dense in G. Indeed otherwise, since the canonical homomorphism $\psi : G \to G/\bar{H}$ is continuous, the quotient group $G/\bar{H}$ is a connected (Hausdorff, Tychonoff) space such that $1 < |G/\bar{H}| \leq \omega$; this contradiction completes the proof. □

A topological space is said to be *pseudocompact* if each continuous real-valued function on it is bounded. It is easy to see that a pseudocompact topological group is totally bounded, and it is known [7] that a totally bounded topological group G is pseudocompact if and only if G is G_δ-dense in its Weil completion $\bar{G}$. Every zero-dimensional pseudocompact Abelian group of uncountable weight admits a proper dense subgroup [6, (7.3)], and recently we [2] have listed several conditions sufficient to ensure that a connected, pseudocompact Abelian group have a proper dense pseudocompact subgroup. While we have been unable to determine whether every pseudocompact group of uncountable weight admits a proper, dense, pseudocompact subgroup (see Section 5 below), the following weaker statement is easily established.

Theorem 4.2. *Let G be an infinite pseudocompact topological group. Then G has a proper dense subgroup.*

Proof. Let N be a closed, normal, G_δ-subgroup of the Weil completion $\bar{G}$ of G such that N has infinite index in $\bar{G}$ (cf. [11, (8.7)]) and let ϕ be the canonical homomorphism from $\bar{G}$ onto $\bar{G}/N$. Since $\bar{G}/N$ is metrizable (loc. cit.) it contains a (proper) countable dense subgroup D; we set $H = G \cap \phi^{-1}(D)$. The containment $H \subseteq G$ is proper since $\phi[H]$ is countable while $\phi[G]$, like every infinite pseudocompact group, has cardinality at least $\mathfrak{c}$ [6, 2]. To see that H is dense in G let U be a nonempty open subset of G, choose V open in $\bar{G}$ such that $V \cap G = U$, and choose $p \in \phi[V] \cap D$. (Such p exists since $\phi : \bar{G} \to \bar{G}/N$ is an open function.) Since $V \cap \phi^{-1}(\{p\})$ is a nonempty G_δ-subset of $\bar{G}$ we have

$$\emptyset \neq G \cap V \cap \phi^{-1}(\{p\}) \subseteq U \cap H,$$

as required. □

Remark 4.3. In a preliminary version of this manuscript circulated privately to friends in 1988, and in our abstract [3], we had left unsolved the question settled by the present Theorem 4.2; the result itself was announced at the annual meeting of the American Mathematical Society in Louisville, Kentucky in January, 1990. Shortly thereafter we received a letter from Dimitrii B. Shakhmatov containing a proof of Theorem 4.2 due to E. A. Reznichenko. It should be acknowledged that Professor Reznichenko proved this result independently of us, and approximately simultaneously.

5. Some unsolved questions

The theorems of the preceding sections leave the following questions unsettled.

Question 5.1. Does every connected pseudocompact Abelian group of uncountable weight have a proper, dense, pseudocompact subgroup?

Question 5.2. Does every pseudocompact Abelian group of uncountable weight have a proper dense pseudocompact subgroup?

The theorems of [6, 2] describe several classes of pseudocompact groups which contain proper dense pseudocompact subgroups. Perhaps the simplest or most accessible context not covered in [6, 2] is the case of a connected pseudocompact group G such that $|G| = \mathfrak{c}$ and $wG = \mathfrak{c}^+$. Specifically, one may ask:

Question 5.3. Is there a dense (necessarily connected) pseudocompact subgroup G of $\mathbb{T}^{(\mathfrak{c}^+)}$ such that $|G| = \mathfrak{c}$ and G has no proper dense pseudocompact subgroup?

In Section 3 we showed that for suitably restricted cardinals α there are totally bounded Abelian torsion groups G such that $wG = |G| = \alpha$ and G has no proper dense subgroup. Are our restrictions on α necessary? Specifically, we ask:

Question 5.4. Let α be an infinite cardinal number. Are there totally bounded Abelian torsion groups G_0 and G_1 such that G_i has no proper dense subgroup and

(a) $w(G_0) = |G_0| = \alpha$?

(b) $w(G_1) = |G_1| = 2^{<\alpha}$?

What if α is assumed to be a (strong) limit cardinal?

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