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Discrete sets and the maximal totally bounded group topology*

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Abstract

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If G is an Abelian group, then $G^{\#}$ is G with its maximal totally bounded group topology. We prove that every $A \subseteq G^{\#}$ contains a closed (in $G^{\#}$) and discrete subset B such that $|B| = |A|$. This answers a question posed by Eric van Douwen. We also present an example of a countable $G^{\#}$ having an infinite relatively discrete subset that is not closed.

0. Introduction

Let G be an Abelian group and let $G^{\#}$ be G with its maximal totally bounded group topology; this is the topology induced by the natural isomorphism of G into the compact product $\mathbf{T}^{\mathrm{Hom}(G,\mathbf{T})}$. (Here $\mathbf{T}$ denotes the circle group.) The topology of the groups $G^{\#}$ is quite mysterious: for example, it is known that $G^{\#}$ is zero-dimensional [2, 3], but it is not even known whether $\mathbb{R}^{\#}$ is strongly zero-dimensional [3]. In [3], van Douwen proved, among other things, the following remarkable result: if $D \subseteq G^{\#}$ is infinite then there exists $E \subseteq D$ with the following properties: $|E| = |D|$ and E is relatively discrete and C-embedded in $G^{\#}$. He asked

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whether $G^{\#}$ has a closed discrete subset of cardinality $|G|$ [3, Question 4.14]. The aim of this note is to answer this question in the affirmative.

Theorem 0.1. *Let G be an Abelian group. If $A \subseteq G^{\#}$ then A contains a subset B having the following properties*:

(1) *B is relatively discrete and closed in $G^{\#}$*;

(2) $|B| = |A|$.

In view of van Douwen's Theorem, our results would be trivial if every relatively discrete subset of $G^{\#}$ would be closed in $G^{\#}$. This is not true however, as the following example shows.

Example 0.2. There is a countable Abelian group G such that $G^{\#}$ contains an infinite relatively discrete subset that is not closed in $G^{\#}$.

1. Preliminaries

If κ is a cardinal number then $\mathrm{cf}(\kappa)$ denotes its cofinality. For a set X and a cardinal number κ, $[X]^{\kappa}$ denotes the collection of all subsets of X of cardinality κ.

All groups considered are Abelian and are written additively: so the identity element of G is denoted by 0, except in the circle group where we use multiplicative notation and use 1 for the identity element. If G is a group and $A \subseteq G$ then $\langle\langle A \rangle\rangle$ denotes the subgroup of G generated by A. For A a singleton, say $A = \{a\}$, we write $\langle\langle a \rangle\rangle$ instead of $\langle\langle \{a\} \rangle\rangle$. We also put $\langle\langle \emptyset \rangle\rangle = \{0\}$. If G is a group and $x \in G$ then $o(x)$ denotes the *order* of x, i.e., the smallest natural number n for which $n \cdot x = 0$ if such a natural number exists, and ∞ otherwise. The torsion subgroup of G is denoted by tG, and for each n, $t_n G = \{x \in G : nx = 0\}$. Note that for every n, $t_n G$ is a subgroup of G and that for all $n, m, t_n G \subseteq t_{nm} G$. A subset $A \subseteq G \setminus \{0\}$ is called *independent* if for every $B \subseteq A$,

$$\langle\langle B \rangle\rangle \cap \langle\langle A \setminus B \rangle\rangle = \{0\}.$$

The following two results follow straight from the definition: their easy proofs are included for the sake of completeness.

Lemma 1.1. *Let G be a group, and let $A \subseteq G$ be independent. If $x \in G$ is such that $\langle\langle x \rangle\rangle \cap \langle\langle A \rangle\rangle = \{0\}$, then $A \cup \{x\}$ is independent.*

Proof. Suppose that there exist disjoint $F, G \subseteq A$ and $p \in \langle\langle F \cup \{x\} \rangle\rangle \cap \langle\langle G \rangle\rangle$ such that $p \neq 0$. Then there exist $n \in \mathbb{Z}$, $a \in \langle\langle F \rangle\rangle$ such that

$$0 \neq p = n \cdot x + a.$$

Consequently, $n \cdot x = p - a \in \langle\langle A \rangle\rangle$, so $n \cdot x = 0$. This implies that $p = a$, but this contradicts the fact that A is independent. $\square$

Lemma 1.2. *Let G be a group, and let $\mathscr{K}$ be a chain (with respect to inclusion) of independent subsets of G. Then $\bigcup\mathscr{K}$ is independent.*

Proof. Put $A=\bigcup\mathscr{K}$, and let $B\subseteq A$. Suppose that there exists an

$$x\in\langle\langle B\rangle\rangle\cap\langle\langle A\setminus B\rangle\rangle \quad \text{such that } x\neq 0.$$

There are finite $F\subseteq B$ and $G\subseteq A\setminus B$ such that

$$x\in\langle\langle F\rangle\rangle\cap\langle\langle G\rangle\rangle. \tag{$*$}$$

Since $\mathscr{K}$ is a chain with respect to inclusion, there is a $K\in\mathscr{K}$ such that $F\cup G\subseteq K$. But now $(*)$ and $x\neq 0$ contradict the fact that K is independent. □

Lemma 1.3. *Let G be a group and let $A\subseteq G$ be independent. Suppose that $f: A\to\mathbf{T}$ is a function such that for every $a\in A$,*

$$[f(a)=1] \quad \text{or} \quad [o(a)=\infty] \quad \text{or} \quad [o(a)<\infty\wedge o(f(a))\mid o(a)].$$

Then f can be extended to a homomorphism $\bar{f}: G\to\mathbf{T}$.

Proof. First observe that we can extend f to a function $h:\bigcup_{a\in A}\langle\langle a\rangle\rangle\to\mathbf{T}$ such that for every $a\in A$, $h|_{\langle\langle a\rangle\rangle}$ is a homomorphism. Next observe that for every $x\in\langle\langle A\rangle\rangle\setminus\{0\}$ there exist for some $n_x\in\mathbb{N}$, $a_1^x,\dots,a_{n_x}^x\in A$ and $b_1^x\in\langle\langle a_1^x\rangle\rangle\setminus\{0\},\dots, b_{n_x}^x\in\langle\langle a_{n_x}^x\rangle\rangle\setminus\{0\}$ such that $x=\sum_{i=1}^{n_x} b_i^x$. The independence of A easily implies that the $b_1^x,\dots,b_{n_x}^x$ depend uniquely on x. Consequently, the function $\bar{h}:\langle\langle A\rangle\rangle\to\mathbf{T}$ defined by

$$\bar{h}(0)=1, \qquad \bar{h}(x)=\sum_{i=1}^{n_x} h(b_i^x) \quad (x\neq 0)$$

is well defined. Also, it extends h so it restricts to a homomorphism on every $\langle\langle a\rangle\rangle$, $a\in A$. This easily implies that $\bar{h}$ is a homomorphism. Now since $\mathbf{T}$ is divisible, there exists a homomorphism $\bar{f}: G\to\mathbf{T}$ that extends $\bar{h}$ [5, A.7]. □

Lemma 1.4. *Let G be an Abelian group, and let $A\subseteq G$ be independent. Then A is closed and discrete in $G^{\#}$.*

Proof. Since every subset of an independent set is independent, it suffices to prove that every independent set in G is closed in $G^{\#}$. So let an independent $A\subseteq G$ be given. We first prove that $0\notin\bar{A}$. For every $a\in A$ pick an element $f(a)\in\mathbf{T}$ such that

(1) $o(a)=o(f(a))$; and

(2) $f(a)\in\{z\in\mathbf{T}: \operatorname{Re} z<0\}$.

By Lemma 1.3, we can extend $f: A\to\mathbf{T}$ to a homomorphism $\bar{f}: G^{\#}\to\mathbf{T}$. Since $\bar{f}$ is continuous, and $\bar{f}(0)\notin\overline{\bar{f}[A]}$, we get $0\notin\bar{A}$, as required.

Now let $x\in G\setminus A$ be arbitrary. We will prove that $x\notin\bar{A}$. By what we just proved, we may assume that $x\neq 0$. Since $\langle\langle A\rangle\rangle$ is closed in $G^{\#}$ [1, 2.1], we may also assume that $x \in \langle\langle A \rangle\rangle$. Pick $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $\xi_1, \ldots, \xi_n \in \mathbb{Z} \setminus \{0\}$ such that $x = \sum_{i=1}^{n} \xi_i \cdot a_i$. We may assume without loss of generality that for every $i \leq n$, if $o(a_i) < \infty$ then $0 < \xi_i < o(a_i)$. For every $i \leq n$ pick an element $z_i \in \mathbf{T}$ such that $o(z_i) = o(a_i)$. By Lemma 1.3 there exists, for every $i \leq n$, a homomorphism $f_i : G \to \mathbf{T}$ such that

$$f_i|_{A \setminus \{a_i\}} \equiv 1 \quad \text{and} \quad f_i(a_i) = z_i.$$

Then $\phi = f_1 \times \cdots \times f_n : G^{\#} \to \mathbf{T}^n$ is a homomorphism and is therefore continuous.

In case $n = 1$, we clearly have $\xi_1 \neq 1$ so that $\phi(x) = z_1^{\xi_1} \notin \{0, z_1\} = \phi[A]$.

In case $n > 1$, we see that *no* coordinate of $\phi(x)$ is equal to 1, whereas for *every* $a \in A$ *some* coordinate of $\phi(a)$ is equal to 1.

We see that in both cases $\phi(x) \notin \overline{\phi[A]}$, so that $x \notin \bar{A}$. □

By noting that $G^{\#}$ is zero-dimensional ([3, Theorem 1.1] and [2, Theorem 2.1]), the following result is Theorem 1.3(b) from [3].

Theorem 1.5. *Let G be an Abelian group. If $x \in G^{\#}$ and if $A \subseteq G^{\#}$ is uncountable then x has a clopen neighborhood U such that $|U \setminus A| = |A|$.*

We conclude this section with the following fact, which can be proved straight from the definition:

Fact. *If $f : G \to H$ is a homomorphism of groups, then $f : G^{\#} \to H^{\#}$ is continuous.*

We will use this fact often without mentioning it.

2. Proof of Theorem 0.1

Let G be a group. First observe that the theorem is trivial if A is finite, for then A is a finite discrete space. The theorem is also trivial if A is countably infinite, for then $\langle\langle A \rangle\rangle$ is a countable space every compact subspace of which is finite [4] (see also [2, Theorem 4.7] and [3, Theorem 1.3(a)], and which moreover is closed in $G^{\#}$ [1, 2.1]. So in the remaining part of this section it suffices to consider uncountable subsets of groups.

The following result is probably well known; its easy proof included for the sake of completeness.

Proposition 2.1. *Suppose that G is an Abelian group which is either torsion free or has the property that every point different from 0 has order p, for some fixed prime number p. If $E \subseteq G$ is uncountable, then there is an independent $F \subseteq E$ such that $|F| = |E|$.*

Proof. Suppose E is uncountable and $B \subseteq G$ is independent such that $|B| < |E|$. Let $K = \{x \in E: B \cup \{x\}$ is not independent$\}$ and let o be ∞ or p. For every $x \in K$ fix ξ_x such that $\xi_x \cdot x \in \langle\langle B \rangle\rangle$ and $0 < \xi_x < o$. For $\xi < o$ let $K_\xi = \{x \in K: \xi_x = \xi\}$; since the map $x \mapsto \xi \cdot x$ is one-to-one, it follows that $|K_\xi| \leq |\langle\langle B \rangle\rangle| \leq |B| \cdot \omega$. Since $|B| \cdot \omega < |E|$, we conclude that $|K| < |E|$. From this the statement of the proposition readily follows. □

Corollary 2.2. *Suppose that G is an Abelian group which is torsion free or is such that every point different from 0 has order p for some fixed prime number p. If $E \subseteq G^\#$ is uncountable, then there is a closed (in $G^\#$) and discrete $F \subseteq E$ such that $|F| = |E|$.*

Proof. Combine Proposition 2.1 and Lemma 1.4. □

So this result proves Theorem 0.1 for groups that are torsion-free. We will now in two steps prove the theorem for torsion groups. Then we piece everything together, and present a proof of the general result.

Lemma 2.3. *If G is Abelian and if $G = t_n G$ for some n then every (uncountable) subset A of $G^\#$ contains a closed (in $G^\#$) and discrete subset of size $|A|$.*

Proof. Associate with G the following sequence of groups: $G_0 = G$; if G_i is known and non-trivial let $p_i = \min\{k: \exists x \in G_i \setminus \{0\}\ o(x) = k\}$ and $G_{i+1} = G_i / t_{p_i} G_i$; if G_i is trivial stop. Note that every p_i is prime and that the sequence must stop somewhere. Let us call the index i for which G_i is trivial the depth of G. We prove the lemma by induction on the depth of G.

If the depth of G is 1 then every element of G has prime order p_0 and we can apply Corollary 2.2.

If the depth of G is $i > 1$ consider the natural homomorphism $\phi: G \to G_1$. ϕ is continuous by the Fact from Section 1. Fix a subset B of A such that ϕ is one-to-one on B and $\phi[B] = \phi[A]$. For $b \in B$ we put $A_b = (A - b) \cap t_{p_0} G$; observe that $A_b + b = \phi^{-1}(\phi(b)) \cap A$.

By Corollary 2.2 we may find for every $b \in B$ a subset A'_b of A_b such that A'_b is closed and discrete in $G^\#$ and such that $|A'_b| = |A_b|$.

Case 1: $|A_b| = |A|$ for some b. Then $A'_b + b$ is the desired subset of A.

Case 2: $|A_b| < |A|$ for all b.

Subcase 2a: $\sup_{b \in B} |A_b| = |A|$. First observe that $|B| \geq \mathrm{cf}(|A|)$. Now we can thin out B to a subset B' of size $\mathrm{cf}(|A|)$ such that for *every* subset B'' of B' of cardinality $\mathrm{cf}(|A|)$ we have $|A| = \sup_{b \in B''} |A_b|$. Then we may find by our inductive assumption a subset C of B' such that $\phi[C]$ is closed and discrete in $(G_1)^\#$ and $|A| = \sup_{b \in C} |A_b|$. Then $\bigcup_{b \in C} (A'_b + b)$ is the desired closed and discrete subset of A.

Subcase 2b: $\sup_{b \in B} |A_b| < |A|$. Now we know that $|B| = |A|$ and by the inductive assumption we can find a subset D of B such that $|D|=|B|=|A|$ and $\phi[D]$ is closed and discrete in $(G_1)^{\#}$; then D is closed and discrete in $G^{\#}$. $\square$

Proposition 2.4. *If G is Abelian and if $G=tG$ then every (uncountable) subset A of $G^{\#}$ contains a closed (in $G^{\#}$) and discrete set of size $|A|$.*

Proof. For convenience, put $\kappa=|A|$. Since t_nG is a subgroup of G for every n, by [1, p. 41], the identity id_{t_nG} is a closed embedding of $(t_nG)^{\#}$ into $G^{\#}$. Consequently, by Lemma 2.3 we are done if for some n, $|A\cap t_nG|=\kappa$. We therefore assume without loss of generality that for every n,

$$|A\cap t_nG|<\kappa. \tag{**}$$

Observe that (**) implies that κ has countable cofinality.

Claim. *If $\hat{A}\subseteq A$ has cardinality κ, then for every $n\in\mathbb{N}$ there exists a clopen neighborhood V_n of t_nG such that $|\hat{A}\setminus V_n|=\kappa$.*

Proof. Consider the natural homomorphism $\phi: G\to G/t_nG$ and let $\psi=\phi\restriction_{\hat{A}}$.

Case 1: There exists $a\in\hat{A}$ such that $|\psi^{-1}(\psi(a))|=\kappa$. Then by (**), $a\notin t_nG$ which implies that $\psi(a)\neq 0$. Now since $(G/t_nG)^{\#}$ is zero-dimensional, [2,3], there is a clopen neighborhood C of 0 in $(G/t_nG)^{\#}$ such that $\phi(a)\notin C$. Then $V_n=\phi^{-1}(C)$ is clearly as required.

Case 2: $|\psi[\hat{A}]|=\kappa$. Then by Theorem 1.5 there exists a clopen neighborhood C of 0 in $(G/t_nG)^{\#}$ such that $|C\setminus\psi[\hat{A}]|=\kappa$. Then $V_n=\phi^{-1}(C)$ is as required.

Case 3: $[|\psi[\hat{A}]|<\kappa]\wedge[\forall a\in\hat{A}: |\psi^{-1}(\psi(a))|<\kappa]$. Then $\sup_{a\in\hat{A}}|\psi^{-1}(\psi(a))|=\kappa$ so that $\mathrm{cf}(\kappa)=\omega$ implies that there is a countable infinite set $B\subseteq\psi[\hat{A}]$ such that for every infinite $E\subseteq B$ we have $\sup_{e\in E}|\psi^{-1}(e)|=\kappa$. Again by Theorem 1.5 there exists a clopen neighborhood C of 0 in $(G/t_nG)^{\#}$ such that $C\setminus B$ is infinite. So $V_n=\phi^{-1}(C)$ is as required. $\square$

Now since κ has countable cofinality, we may pick a sequence of regular uncountable cardinals $\kappa_1<\kappa_2<\cdots<\kappa_n<\cdots$ such that $\sup_n\kappa_n=\kappa$. Put $U_0=\emptyset$. By induction on $n\in\mathbb{N}$ we will construct an integer m_n, a clopen neighborhood U_n of $t_{m_n}G$ and a closed discrete set $A_n\subseteq(A\cap t_{m_n}G)\setminus U_{n-1}$ such that

(1) $m_1<m_2<\cdots<m_n<\cdots$;
(2) the numbers m_n and $n+1$ are factors of m_{n+1} for every n;
(3) $U_0\subseteq U_1\subseteq\cdots\subseteq U_n\subseteq\cdots$;
(4) for every n, $|A_n|=\kappa_n$;
(5) for every n, $|A\setminus U_n|=\kappa$.

Since κ_1 is regular and uncountable, there exists $m_1\in\mathbb{N}$ such that $|A\cap t_{m_1}G|\geq\kappa_1$. By Lemma 2.3 there is a closed and discrete set $A_1\subseteq A\cap t_{m_1}G$ such that $|A_1|=\kappa_1$. By the Claim there is a clopen neighborhood V of $t_{m_1}G$ such that $A\setminus V$ is of cardinality κ. Put $U_1=V\cup U_0$. Now by applying the Claim inductively and by noting that if $|A \cap t_m G| \geq \kappa$ then $|A \cap t_{sm} G| \geq \kappa$ for every s, it is clear how to construct the other m_n's, U_n's and A_n's in precisely the same way.

Observe that the collection $\{U_n \setminus U_{n-1} : n \in \mathbb{N}\}$ is a clopen partition of $G^{\#}$. Since for every n, $A_n \subseteq U_n \setminus U_{n-1}$ we obtain that $\bigcup_{n \in \omega} A_n$ is a closed and discrete subset of A of size κ. □

Proposition 2.5. *If G is an abelian group and $A \subseteq G^{\#}$ then A contains a closed (in $G^{\#}$) and discrete subset B of size $|A|$.*

Proof. For convenience, put $\kappa = |A|$. As remarked at the beginning of this section, we may assume that $\kappa > \omega$. Consider the natural homomorphism $\phi : G \to G/tG$. Observe that G/tG is torsion free. So if $|\phi[A]| = \kappa$ then $\phi[A]$ contains a closed and discrete subset of cardinality κ by Corollary 2.2. By continuity of ϕ this then implies that A contains a closed (in $G^{\#}$) and discrete subset of cardinality κ. Therefore assume that $|\phi[A]| < \kappa$. Fix a subset B of A such that $\phi[B] = \phi[A]$ and ϕ is one-to-one on B. For $b \in B$ let $A_b = (A - b) \cap tG$ and again observe that $A_b + b = \phi^{-1}(\phi(b)) \cap A$. Then $A = \bigcup_{b \in B} (A_b + b)$, so that $|A| = \sup_{b \in B} |A_b|$ because $|B| < \kappa$.

Case 1: For all b we have $|A_b| < |A|$. Now we may thin out B to a subset C of size $\mathrm{cf}(|A|)$ such that for *every* subset D of C of size $\mathrm{cf}(|A|)$ we have $|A| = \sup_{b \in D} |A_b|$. We take $D \subseteq C$ of size $|C|$ such that $\phi[D]$ is closed and discrete in $(G/tG)^{\#}$. It is clear that we can use Proposition 2.4 to find for every $b \in D$ a closed and discrete subset A'_b of A_b such that $|A| = \sup_{b \in D} |A'_b|$. Then $\bigcup_{b \in D} A'_b + b$ is closed and discrete and has the right cardinality.

Case 2: For some b we have $|A_b| = |A|$. Apply Proposition 2.4. □

3. Construction of Example 0.2

One of the reasons that the topology of $G^{\#}$ is difficult to deal with, is that $\mathrm{Hom}(G, \mathbf{T})$ is always big, and usually has a complicated structure. However, its structure is not always complicated. For example, let G be a *Boolean* group, i.e., a group in which every point has order at most 2. Then each homomorphism $\phi : G \to \mathbf{T}$ has finite range, and a moment's reflection proves the following:

Theorem 3.1. *Let G be a Boolean group. Then the collection*

$$\{E : E \text{ is a subgroup of } G \text{ with finite index}\}$$

is a local basis at $0 \in G^{\#}$ consisting of clopen sets. □

Of course a similar result can be derived for all groups G for which there exists an n such that $G = t_n G$. So now let G be any infinite Boolean group. Let H be a maximal independent subset of G. By Lemma 1.1, $\langle\langle H \rangle\rangle = G$, so H is infinite. We will first prove that $D = (H + H) \setminus \{0\}$ is discrete. Indeed, pick distinct elements x

and y in H. Then by the independence of H, $x+y \notin \langle\langle H\setminus\{x,y\}\rangle\rangle$. Since $\langle\langle H\setminus\{x,y\}\rangle\rangle$ is closed [1,2.1], there exists disjoint open neighborhoods U and V of x and y, respectively, such that $(U+V)\cap\langle\langle H\setminus\{x,y\}\rangle\rangle=\emptyset$. Since H is discrete, Lemma 1.4, we may assume without loss of generality that $U\cap H=\{x\}$ and $V\cap H=\{y\}$. Now put $W=(x+V)\cap(y+U)$. Then W is a neighborhood of $x+y$ and we claim that $W\cap((H+H)\setminus\{0\})=\{x+y\}$. To this end, suppose that $a+b\in W$ for $a,b\in H$. Observe that $a\neq b$. We will prove that $\{a,b\}=\{x,y\}$. If $\{a,b\}\cap\{x,y\}=\emptyset$, then

$$a+b\in\langle\langle H\setminus\{x,y\}\rangle\rangle\cap W\subseteq\langle\langle H\setminus\{x,y\}\rangle\rangle\cap(U+V)=\emptyset,$$

which is a contradiction. So we may assume without loss of generality that e.g., $a=x$. There exists $v\in V$ such that $a+b=x+v$. Consequently,

$$b=v\in V\cap H=\{y\},$$

as required.

We will next prove that $0\in\bar{D}$. This is easy. Indeed, let E be a basic neighborhood of 0 in $G^{\#}$, i.e., E is a subgroup of G with finite index. There is a translate of E, say $x+E$, that contains two distinct points of H, say a and b. Then $a+b\in E\cap D$.

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