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ON THE EXISTENCE OF WEAKLY n-DIMENSIONAL SPACES

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ABSTRACT. Using a certain method for constructing peculiar large-dimensional spaces in every compactum with sufficiently large dimension, we present for every n an easy example of a weakly n-dimensional space.

1. INTRODUCTION

For a space X and a point $x \in X$, $\operatorname{ind}_x X$ denotes the dimension of X at the point x (cf. [1, Problem 1.1.B]). If X is n-dimensional then its *dimensional kernel* is the set $\{x \in X\colon \operatorname{ind}_x X = n\}$. It is known that the dimensional kernel of an n-dimensional space X, where $n \geq 1$, is an F_σ subset of X of dimension at least $n - 1$. Also, if in addition X is *compact*, then its dimensional kernel is n-dimensional. For more information see [1, Problem 1.5.C]. A space X is called *weakly n-dimensional*, where $n \geq 1$, if it is n-dimensional, but its dimensional kernel is of dimension $n - 1$. Clearly, a weakly n-dimensional space contains no compact subspace of dimension n. The first examples of weakly n-dimensional spaces were given by Sierpiński [15] and Mazurkiewicz [9]. A simpler construction can be found in Tomaszewski [16]; he also proved that if X is weakly n-dimensional and if Y is weakly m-dimensional then

$$\dim(X \times Y) \leq n + m - 1 = \dim X + \dim Y - 1.$$

In this note we present an application of a certain method for constructing peculiar large-dimensional subspaces in every compactum with sufficiently large dimension. The technique goes back to Mazurkiewicz [9] and Knaster [3] and has been used by several authors: see Lelek [8], Zarelua [17], Rubin, Schori, and Walsh [14], Kulesza [6], Ivanov [2], Pol [13], and Krasinkiewicz [4, 5] (These papers contain further references.) We use the technique for the construction of easy weakly n-dimensional spaces.

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2. Preliminaries

Our terminology in dimension theory follows Engelking [1], Nagata [12], and van Mill [11].

We denote the closed interval [0, 1] by I. A *compactum* is a compact metrizable space. A family $\tau = \{(A_1, B_1), (A_2, B_2), \dots, (A_n, B_n)\}$ of pairs of disjoint closed sets in a space X is called *essential* if for every family $\{L_i : i \leq n\}$, where L_i is a partition between A_i and B_i for every i, we have $\bigcap_{i \leq n} L_i \neq \emptyset$; if τ is not essential then it is called *inessential*. Recall that X is at least n-dimensional if and only if X contains an essential family of size n.

We shall need the following well-known fact.

2.1. **Lemma.** *Let K be a $(k+1)$-dimensional compactum and A, B a pair of disjoint closed subsets of K. Then there exists a continuous function $f: K \to I$ taking A to 0 and B to 1 such that $\dim f^{-1}(t) \leq k$ for each $t \in [\frac{1}{3}, \frac{2}{3}]$.*

This lemma follows immediately from Hurewicz's theorem (Kuratowski [7, §45]) that the zero-dimensional maps $g: K \to I^{k+1}$ are dense in the function space $C(K, I^{k+1})$ (if $p: I^{k+1} \to I$ is the projection and $g: K \to I^{k+1}$ is zero-dimensional then $\dim[(p \circ g)^{-1}(t)] \leq k$ for all $t \in I$). Alternatively, one can use Nagata's metric on K [12, Theorem V.4]. For the purpose of §2 it is enough to know that $\dim f^{-1}(t) \leq k$ for t belonging to a Cantor set C in I and this can be proved directly by a standard Urysohn construction as the points t with $\dim f^{-1}(t) \leq k$ form a set of type G_δ.

We will also need the following triviality: if X is an n-dimensional σ-compact space, where $n \geq 0$, then there exists a zero-dimensional σ-compact set $N \subseteq X$ such that $\dim(X \backslash N) \leq n-1$. This can be verified easily by induction on n: note that $\dim X = \operatorname{ind} X$ and apply [11, Theorem 4.7.3].

The following lemma is probably well known: its easy proof is included for the sake of completeness.

2.2. **Lemma.** *Let X and Y be compact spaces, and let $f: X \to Y$ be a continuous surjection. If Y is zero-dimensional at $y \in Y$ and if $\dim f^{-1}(y) = 0$ then X is zero-dimensional at every point of $f^{-1}(y)$.*

Proof. Pick an arbitrary $x \in f^{-1}(y)$ and let U be a neighborhood of x in X. Since $\dim f^{-1}(y) = 0$ there is an open and closed subset C of $f^{-1}(y)$ such that $x \in C \subseteq U$. Pick disjoint open subsets E and F in X such that

$$E \cap f^{-1}(y) = C \quad \text{and} \quad F \cap f^{-1}(y) = f^{-1}(y) \backslash C.$$

It is clear that without loss of generality we may assume that $E \subseteq U$. Observe that $E \cup F$ is a neighborhood of $f^{-1}(y)$ in X. Consequently, since f is a closed map, there is a neighborhood V of y in Y such that $f^{-1}(V) \subseteq E \cup F$. Since Y is zero-dimensional at y, we may assume without loss of generality that V is open and closed. Then $f^{-1}(V) \cap E$ is an open and closed neighborhood of x in X which is contained in U. □

3. Easy weakly n-dimensional spaces

The aim of this section is to present new examples of weakly n-dimensional spaces.

3.1. **Theorem.** *Let K be an $(n+1)$-dimensional compactum, where $n \geq 1$. Then K contains a weakly n-dimensional G_δ-subset X (hence X is completely metrizable).*

Proof. Let $\tau = \{(A_0, B_0), \ldots, (A_n, B_n)\}$ be an essential family in K. By Lemma 2.1 there exists a continuous function $f : K \to I$ taking A_0 to 0 and A_1 to 1 such that there exists a Cantor set $\Delta \subseteq (0, 1)$ such that $\dim f^{-1}(t) \leq n$ for every $t \in \Delta$.

We now closely follow a construction in Rubin, Schori, and Walsh [14]. Denote the hyperspace of K by 2^K and put

$$\mathscr{C} = \{C \in 2^K : C \text{ is a continuum from } A_0 \text{ to } B_0\}.$$

Then $\mathscr{C}$ is closed in 2^K and hence is a compact space (see [11, Claim 1 of Theorem 4.7.10]). Consequently, there is a continuous surjection $\varphi : \Delta \to \mathscr{C}$. Put

$$Z = \bigcup\{f^{-1}(t) \cap \varphi(t) \colon t \in \Delta\}.$$

Then Z is closed in K, hence is compact, and $f[Z] = \Delta$ ([11, Claim 2 of Theorem 4.7.10]). Then $\dim Z \geq n$ because Z intersects every continuum from A_0 to B_0 ([11, Corollary 4.7.9]). In addition, $\dim Z \leq n$ because the fibers of the restriction g of f to Z are at most n-dimensional and Δ is zero-dimensional [1, Theorem 1.12.4]. We conclude that $\dim Z = n$.

Now let Λ denote the dimensional kernel of Z. Then because Z is compact, Λ is an n-dimensional σ-compact subset of Z, cf. the remarks in §2. There exists consequently a σ-compact zero-dimensional subset N of Λ such that $\dim(\Lambda \backslash N) \leq n-1$, cf. the remark preceding Lemma 2.2. Now put $X = Z \backslash N$. Then X is clearly a G_δ-subset of X and we claim that it is weakly n-dimensional.

Let $C \in \mathscr{C}$. We will prove that C meets X. Pick $t \in \Delta$ such that $\varphi(t) = C$. Observe that $g^{-1}(t) \subseteq C \cap Z$. If $\dim g^{-1}(t) > 0$ then $g^{-1}(t)$ intersects X because the complement of X in Z is zero-dimensional. We may therefore assume that $\dim g^{-1}(t) = 0$. But since Δ is zero-dimensional, we now obtain that Z is zero-dimensional at all points of $g^{-1}(t)$ (Lemma 2.2). Consequently, $g^{-1}(t) \cap \Lambda = \varnothing$, that is, $\varnothing \neq g^{-1}(t) \subseteq X$. We conclude from [11, Corollary 4.7.9] that $\dim X \geq n$. However, because X is a subspace of Z, we also have $\dim X \leq n$. Consequently, $\dim X = n$, as required.

We next prove that X is weakly n-dimensional. This is however a triviality. Simply observe that if x is a point of X at which X is n-dimensional, then Z is n-dimensional at x, which implies that $x \in X \cap \Lambda$. Since by construction, $\dim(X \cap \Lambda) \leq n-1$, we are done. □

3.2. *Remark.* Observe that the spaces X constructed in Theorem 3.1 are *rim-compact*, that is, have a basis consisting of open sets with compact boundaries. This is clear because the complement of X in K is zero-dimensional.

3.3. *Remark.* If one takes $K = I^{n+1}$ in the above theorem then there is no need to use Lemma 2.1 in the proof. It seems that this gives us the easiest known examples of weakly n-dimensional spaces.

3.4. *Remark.* It seems useful to recall that the main points of the original construction of Mazurkiewicz [10]. Let C be the Cantor set and consider $K = C \times I^n$. Let $\mathscr{I}$ denote the subspace of $2^K \times 2^K \times \cdots$ consisting of all sequences $(A_1, A_2, \dots)$ such that $A_1 \subseteq A_2 \subseteq \cdots$. Then $\mathscr{I}$ is compact, so there is a continuous surjection $\varphi\colon C \to \mathscr{I}$. For $t \in C$ we write $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots)$. Let $\mathscr{C}_i = \{t \in C\colon \varphi_i(t) \cap (\{t\} \times I^n) \neq \varnothing\}$, $D_1 = C_1$, $D_{i+1} = \mathscr{C}_{i+1} \backslash \mathscr{C}_i$, and let

$$M = \{(t, x)\colon t \in D_i, x \in \varphi(t) \cap (\{t\} \times I^n)\}.$$

Then $M \subseteq C \times I^n$ has the following property: for each $S \subseteq M$ which projects onto C, each G_δ-set G in $C \times I^n$ containing S, also contains some section $\{t\} \times I^n$. This can be seen as follows. If $(C \times I^n) \backslash G = F_1 \cup F_2 \cup \cdots$, $F_i \in 2^K$, $F_1 \subseteq F_2 \subseteq \cdots$, then pick $t \in C$ such that $\varphi(t) = (F_1, F_2, \dots)$. Then $t \notin \mathscr{C}_i$ for all i, hence $\{t\} \times I^n \subseteq G$; in particular, $\dim S \geq n$.

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