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Admissibility, homeomorphism extension and the AR-property in topological linear spaces

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Abstract

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We consider topological linear spaces (without local convexity) and their convex subsets. We investigate relations between admissibility, the AR-property and the possibility of extending homeomorphisms between compacta.

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1. Introduction

All spaces considered are *separable* and *metrizable* topological spaces. By a *linear space* we mean a (separable metrizable) topological vector space over $\mathbb{R}$.

Let $\mathcal{K}$ denote a topological class of compacta. In particular, let $\mathcal{K}_{\text{all}}$ and $\mathcal{K}_{\text{fd}}$ denote the (fixed) classes of all compacta and of all finite-dimensional compacta, respectively.

(i) We say that a space X has the *$\mathcal{K}$-extension property*, abbreviated $\mathcal{K}$-EP, if for every space Y, every compact subspace $A \in \mathcal{K}$ of Y and every continuous map $f: A \to X$ there is a continuous extension $g: Y \to X$ of f.

(ii) A convex subset C of some linear space L is said to be *admissible for the class* $\mathcal{K}$ if for every compact subset $K \in \mathcal{K}$ of C and every $\varepsilon > 0$ there is a continuous map $f: K \to C$ such that

$$d(f, \mathrm{id}) = \sup_{x \in K} d(f(x), x) < \varepsilon$$

and $f(K)$ is contained in a finite-dimensional linear subspace (i.e., a linear subspace, having a finite Hamel basis) of L.

(iii) The space X has the *homeomorphism extension property* (HEP) for the class $\mathcal{K}$ if for every homeomorphism $h: K \to L$ between subcompacta $K, L \in \mathcal{K}$ of X there is a homeomorphism of X, extending h.

(iv) A convex subset of a linear space is *admissible* if it is admissible for the class $\mathcal{K}_{\text{all}}$. A space has the CEP or the HEP if it has the $\mathcal{K}$-EP or the HEP, respectively, for the class $\mathcal{K} = \mathcal{K}_{\text{all}}$.

Besides the introduction this paper contains three sections. We describe in short their contents.

Section 2: Our starting point is the question, whether every linear space is an AR. In [6, 17] it is shown that the so-called admissibility is an interesting motion insofar, that it turns out to be equivalent to the CEP. For arbitrary (not necessarily linear) spaces the CEP is strictly weaker than the AR-property, cf. [18]. For linear spaces, the CEP does not imply the AR-property, unless every linear space is an AR, [17, Theorem 3.8]. In the restricted case of σ-compact linear spaces the implication "admissible$\Rightarrow$AR" holds. The question whether every σ-compact linear space is an AR, is equivalent to the question "Is every linear space admissible?", cf. [17]. On the other hand we know that any $\aleph_0$-dimensional linear space is homeomorphic to the subspace

$$l_2^f = \{x \in l_2 \mid \exists N \in \mathbb{N}: n > N \Rightarrow x_n = 0\}$$

of l_2, see [5]. The space l_2^f is an AR. We have taken a look at linear spaces a little bit bigger than the $\aleph_0$-dimensional ones. The latter are the linear hull of a convergent sequence; we prove that the question "Is every linear space, which is spanned by a Cantor set, an AR?" is again equivalent to "Is every linear space admissible?".

Section 3: A third notion, which appears to be interesting, is the HEP. In [7] it is proved that every infinite-dimensional locally convex linear space has the HEP. Infinite dimensionality is easily seen to be necessary for linear spaces to have the HEP for even the class $\mathcal{K}_{\mathrm{fd}}$. Recall [9] that every locally convex linear space is an AR. In [7] it is remarked without proof that every complete linear space with the HEP is admissible. We prove this fact and we show that the HEP does not imply AR, unless every linear space is an AR. The HEP for the class $\mathcal{K}_{\mathrm{fd}}$ is found in every infinite-dimensional complete linear space, cf. [7]. We weaken the completeness condition to non-σ-compactness. For the same linear spaces we prove an estimated version of the HEP ($\mathcal{K}_{\mathrm{fd}}$).

Section 4: As might be expected from the list of definitions in the head of this introduction, we consider the properties HEP, AR and admissibility for convex subsets of linear spaces as well. The equivalence "CEP$\Leftrightarrow$admissible" holds in this case, too, [6]. In particular we study infinite-dimensional compact subsets C of linear spaces. This case stands in severe contrast with the case of a (full) linear space. We already saw that the CEP and admissibility are equivalent; by compactness the same holds for the CEP and the AR-property. In fact these three properties are equivalent with C being homeomorphic to the Hilbert cube. We cannot expect the full HEP to hold in this case, because arbitrary compacta in C could be very "big" with respect to C. We prove that C has the HEP for the class of Z-sets in C iff C is homeomorphic to the Hilbert cube. For such spaces, the considered notions turn out to coincide. The above-mentioned linear spaces with a Cantor set as spanning set could be viewed as a case in between the very small compact infinite-dimensional convex sets and the (full) linear spaces. In this light we saw that admissibility is not a very helpful property. The HEP is shown to be not much nicer: we show that the question "Does every linear space, spanned by a Cantor set and containing a Keller cube, have the HEP?" is again equivalent to "Is every linear space admissible?".

Some notation and conventions. A *map* is a continuous function. A *Cantor set* is a homeomorph of the Cantor middle-third set. A metric d on the linear space L, generating its topology, is called *strictly monotone* if

$$d(0, sx) < d(0, tx) \quad \text{for } 0 \leq s < t \text{ and } x \in L\backslash\{0\}.$$

Throughout this paper we assume the metric on any linear space to be translation invariant and strictly monotone, cf. [10]. We write $|x| = d(0, x)$ for $x \in X$. If A is a subset of a metric space X and $\varepsilon > 0$ then we write

$$B(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$$

and

$$D(A, \varepsilon) = \{x \in X : d(x, A) \leq \varepsilon\};$$

sometimes we append X as a subscript to B or D. If A and B are subsets of a linear space L, then

$$A-B=\{x-y\colon x\in A,\ y\in B\},$$
$$A+B=\{x+y\colon x\in A,\ y\in B\};$$

$\operatorname{lin}(A)$ and $\operatorname{conv}(A)$ denote the linear and the convex hull of A, respectively. $\mathscr{H}(X)$ is the group of homeomorphisms of the space X. $\dim X$ is the covering dimension of X.

2. Admissibility; the CEP

The main result of this section is Theorem 2.12. Before proving this theorem we give a number of useful, rather general facts about admissibility.

Proposition 2.1. *Let the space X have the $\mathscr{K}$-EP for some class $\mathscr{K}$ of compacta, let $K\in\mathscr{K}$, $f\colon K\to X$ and $\varepsilon>0$. Then there are a (finite) polyhedron P and maps $\alpha\colon K\to P$ and $\beta\colon P\to X$ such that $d(\beta\circ\alpha, f)<\varepsilon$.*

Proof. Suppose $K\subset Q$ and denote the canonical projection of Q onto $I^n\times\{0\}\times\cdots$ by π_n. For every n determine a polyhedron $P_n\subset I^n\times\{0\}\times\cdots$ such that

$$\pi_n(K)\subset P_n\subset B(\pi_n(K), 1/n).$$

Consider the subspace

$$Y=(K\times\{0\})\cup\bigcup_{n=1}^{\infty}(P_n\times\{1/n\})$$

of $Q\times I$ (with the sum metric). There is an extension $\tilde{f}\colon Y\to X$ of $f\circ i^{-1}\colon K\to X$ (where $i\colon K\to K\times\{0\}$ is the canonical map). The space Y is compact. Determine $\delta>0$ with the property

$$y, y'\in Y,\ d(y,y')<\delta \;\Rightarrow\; d(\tilde{f}(y), \tilde{f}(y'))<\varepsilon.$$

Note that the maps $g_n\colon K\times\{0\}\to P_n\times\{1/n\}$, defined by $g_n(x,0)=(\pi_n(x), 1/n)$, satisfy $d(g_n, \mathrm{id})\to 0$. The requirements are fulfilled by putting $P=P_n\times\{1/n\}$, $\alpha=g_n\circ i$, $\beta=\tilde{f}\,|\,P_n\times\{1/n\}$ for n so large that $d(g_n,\mathrm{id})<\delta$. □

Our first theorem is a generalization of [6, Note 3 after Theorem 5] and [17, Theorem 3.2].

Theorem 2.2. *Let $\mathscr{K}$ be a class of compacta and let C be a convex subset of a linear space L. Then the following assertions are equivalent:*

(i) *every map $f\colon K\to C$ defined on some $K\in\mathscr{K}$ is uniformly approximable by maps $g\colon K\to C$ such that* $\dim\operatorname{lin}(g(K))<\infty$,

(ii) *C has the $\mathscr{K}$-EP.*

Proof. The proof is analogous to the proof of [17, Theorem 3.2], in which the case $\mathcal{K} = \mathcal{K}_{\text{all}}$, $C = L$ is treated. We adopt the notation of that proof.

(i)$\Rightarrow$(ii): See [17], (1)$\Rightarrow$(2). Take $A \in \mathcal{K}$ and $f: A \to C$. We may assume $0 \in C$. Then we construct the maps $\tilde{\phi}_n: A \to E_n \cap C$ and $\phi_n: X \to C$ with the properties (i), (ii) and (iii), using the fact that $E_n \cap C$ is an AR as well. Follow the remaining part of the proof almost literally.

(ii)$\Rightarrow$(i): See [17], (2)$\Rightarrow$(1). Instead of an arbitrary compact K we take $K = f(A)$, where $A \in \mathcal{K}$ and $f: A \to C$. By Proposition 2.1 there are a polyhedron P and maps $\xi: A \to P$ and $\eta: P \to C$ such that $d(f, \eta \circ \xi)$ is as small as desired. Then we approximate η *by* ψ as in [17], noting that ψ, and so g as well, becomes a map with image in C and we finish the proof almost unaltered. $\square$

Corollary 2.3. *Let C be a convex subset of some linear space and suppose that continuous images of spaces in the class $\mathcal{K}$ of compacta are again in $\mathcal{K}$. Then C has the $\mathcal{K}$-EP iff C is admissible for $\mathcal{K}$.*

Lemma 2.4. *Let C and C' be convex subsets of some linear space L, such that $C' \subset C$ is dense in C. Furthermore, let K be a finite-dimensional compact space. Then every map $f: K \to C$ can be approximated uniformly by maps $f': K \to C'$ satisfying* $\dim \operatorname{lin}(f'(K)) < \infty$.

Proof. Let $\varepsilon > 0$. Suppose $\dim K \leq n$, $1 \leq n < \infty$. Choose an open cover $\mathcal{W}$ of C such that

$$\operatorname{mesh}(\operatorname{St}(\mathcal{W})) < \frac{\varepsilon}{4n}.$$

Since $\dim K \leq n$, there is a finite open cover $\mathcal{U}$ of K such that the nerve P of $\mathcal{U}$ is at most n-dimensional, and moreover $f(\mathcal{U})$ refines $\mathcal{W}$, see [19, Chapter 4]. For the definition of nerve see [19, § 3.6]. Define $\kappa: K \to P$ to be the standard barycentric map, i.e., the map given by

$$\kappa(x) = \sum_{U \in \mathcal{U}} \frac{d(x, K \setminus U)}{\sum_{V \in \mathcal{U}} d(x, K \setminus V)} v_U$$

(v_U is the vertex of P, corresponding to $U \in \mathcal{U}$). We shall define a map $\alpha: P \to L$, and let $\alpha \circ \kappa$ be the required map f'. For every $U \in \mathcal{U}$ choose $W_U \in \mathcal{W}$ with $f(U) \subset W_U$ and pick $y_U \in W_U \cap C'$ arbitrarily. We determine α by putting $\alpha(v_U) :=$ y_U and requiring that $\alpha | \sigma$ is an affine map for every simplex σ of P. Then by convexity of C' we have $\alpha(K) \subset C'$.

Claim 1. *For each simplex σ we have* $\operatorname{diam} \alpha(\sigma) < \frac{1}{2}\varepsilon$.

Proof. Take $\sigma = (v_{U_0} \cdots v_{U_k})$. Then

$$W_{U_0} \cap \cdots \cap W_{U_k} \supset f(U_0) \cap \cdots \cap f(U_k) \neq \emptyset,$$

so $W := W_{U_0} \cup \cdots \cup W_{U_k}$ is contained in a member of $\mathrm{St}(\mathscr{W})$ and so has diameter $<\varepsilon/(4n)$. Now $\{y_{U_0}, \ldots, y_{U_k}\}$ has diameter $<\varepsilon/(4n)$ as well because it is contained in W. Since d is invariant and monotone, and $k \leqslant n$, we have

$$\operatorname{diam} \alpha(\sigma) \leqslant \operatorname{diam} \operatorname{conv}\{y_{U_0}, \ldots, y_{U_k}\} < 2k \cdot \frac{\varepsilon}{4n} \leqslant \tfrac{1}{2}\varepsilon.$$

Claim 2. $d(\alpha \circ \kappa, f) < \varepsilon$.

Proof. Pick $x \in K$. Choose a simplex σ with $\kappa(x) \in \operatorname{Int} \sigma$ (here Int denotes geometric interior), say $\sigma = (v_{U_0} \cdots v_{U_k})$. Then

$$d(\alpha \circ \kappa(x), y_{U_0}) \leqslant \operatorname{diam} \alpha(\sigma) < \tfrac{1}{2}\varepsilon$$

and $x \in U_0$, hence

$$d(f(x), y_{U_0}) \leqslant \operatorname{diam} W_{U_0} < \frac{\varepsilon}{4n} \leqslant \tfrac{1}{4}\varepsilon.$$

The triangle inequality gives $d(\alpha \circ \kappa, f) \leqslant \frac{3}{4}\varepsilon$. □

Proposition 2.5. *Any map $f : K \to C$ from a finite-dimensional compact space K to an arbitrary convex subset C of a linear space is uniformly approximable by maps $g : K \to C$ satisfying* $\dim \operatorname{lin}(g(K)) < \infty$; *in particular: every convex subspace of a linear space is admissible for $\mathscr{K}_{\mathrm{fd}}$.*

Proof. This follows immediately from Lemma 2.4. □

The lemma hereafter shows that the finite-dimensional linear subspaces in the definition of admissibility can be replaced by bigger subsets. The assertions (iii) and (iv) of this lemma will not be applied in the sequel; we only mention them for completeness sake. For the definition of property C see [1, 11].

Lemma 2.6. *Let C be a convex subset of a linear space L and let $K \subset C$ a compact subset. The following assertions are equivalent:*

(i) *for every $\varepsilon > 0$ there is a map $f : K \to C$ such that* $\dim \operatorname{lin} f(K) < \infty$ *and* $d(f, \mathrm{id}) < \varepsilon$,

(ii) *for every $\varepsilon > 0$ there is a map $f : K \to C$ such that* $\dim f(K) < \infty$ *and* $d(f, \mathrm{id}) < \varepsilon$,

(iii) *for every $\varepsilon > 0$ there is a map $f : K \to C$ such that $f(K)$ is a countable union of closed finite-dimensional sets and $d(f, \mathrm{id}) < \varepsilon$,*

(iv) *for every $\varepsilon > 0$ there is a map $f : K \to C$ such that $f(K)$ is a C-space and* $d(f, \mathrm{id}) < \varepsilon$.

Proof. For (i)⇒(ii) it is enough to remember that any finite-dimensional linear space is homeomorphic to one of the spaces $\mathbb{R}^n$.

For (ii)$\Rightarrow$(i) use $\frac{1}{2}\varepsilon$ to squeeze K into something topologically finite dimensional and use another $\frac{1}{2}\varepsilon$ to further squeeze this last set with the help of Lemma 2.4 into a finite-dimensional linear subspace of L.

(ii)$\Rightarrow$(iii) is trivial; (iii)$\Rightarrow$(iv) is immediately clear after [11, Proposition 1].

(iv)$\Rightarrow$(i): By (iv) we may assume that K is a C-space itself. Suppose $K \subset Q$ and let π_n, P_n, Y and i be as in the proof of Proposition 2.1. The space Y has property C as well, [1, Theorem 2.7]. Furthermore, C is contractible and locally contractible, so by [2, Corollary C.5.10] there is an extension $g: Y \to C$ of $i^{-1}: K \times \{0\} \to C$. Determine $\delta > 0$ with

$$y, y' \in Y,\ d(y, y') < \delta \quad \Rightarrow \quad d(g(y), g(y')) < \varepsilon.$$

Consider maps g_n as in the earlier-mentioned proof. Then, for n big enough to satisfy $d(g_n, \mathrm{id}) < \delta$, we have an approximation $g \circ g_n \circ i$ of $\mathrm{id}: K \to C$. By finite dimensionality of the polyhedron P_n we may apply Lemma 2.5 to approximate $g \,|\, P_n \times \{1/n\}: P_n \times \{1/n\} \to C$ by a map h with $\dim \operatorname{lin} h(P_n \times \{1/n\}) < \infty$. The required map f shall be $h \circ g_n \circ i$. $\square$

By compactness we may change the words "for every $\varepsilon > 0$ there is a map f with $d(f, \mathrm{id}) < \varepsilon$" in one or both of (i) and (ii) of this lemma by "for every open cover $\mathcal{U}$ of L there is a map f with $\{f(x), x\}$ contained in some $U \in \mathcal{U}$ for every x" without changing the equivalence of the assertions. If the compact subset K of C satisfies one or all of the conditions of Lemma 2.6, then we say that K is *admissible in C*.

Part of the next proposition is contained in [14, Proposition 2.1], but the proof presented here (for the separable metrizable case) is more elementary.

Proposition 2.7. *Let C_2 be a convex subset of some linear space and C_1 a dense convex subspace of C_2. The following are equivalent:*

(i) *C_1 is admissible and for every compact subset K of C_2 and every $\varepsilon > 0$ there is a map $f: K \to C_1$ such that $d(f, \mathrm{id}) < \varepsilon$.*

(ii) *C_2 is admissible.*

Proof. (i)$\Rightarrow$(ii): Pick $K \subset C_2$ compact and $\varepsilon > 0$. There is a map $f: K \to C_1$ with $d(f, \mathrm{id}) < \frac{1}{2}\varepsilon$ and there is a map $g: f(K) \to C_1$ with $d(g, \mathrm{id}) < \frac{1}{2}\varepsilon$ and $\dim \operatorname{lin} g(f(K)) < \infty$. The map $g \circ f$ satisfies $d(g \circ f, \mathrm{id}) < \varepsilon$.

(ii)$\Rightarrow$(i): For the admissibility of C_1, pick $K \subset C_1$ compact and $\varepsilon > 0$. There is an $f: K \to C_2$ with $\dim \operatorname{lin} f(K) < \infty$ and $d(f, \mathrm{id}) < \frac{1}{2}\varepsilon$. Secondly, by Lemma 2.4 there is a map $g: f(K) \to C_1$ with $\dim \operatorname{lin} gf(K) < \infty$ and $d(g, \mathrm{id}) < \frac{1}{2}\varepsilon$. We have $d(gf, \mathrm{id}) < \varepsilon$. By a second application of Lemma 2.4 the other assertion is verified. $\square$

We say that a subset A of some linear space L is *complete* if there is an invariant metric on L, generating its topology, of which the restriction on A is a complete metric.

Theorem 2.8. *The following assertions are equivalent:*
(i) *every convex subspace of every linear space is admissible,*
(ii) *every σ-compact convex subspace of every linear space is admissible,*
(iii) *every σ-compact convex subspace of every linear space is an AR,*
(iv) *every complete convex subspace of every linear space is admissible.*

Proof. Note that (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(ii), (i)$\Rightarrow$(iv) are trivialities modulo the equivalence of admissibility and the CEP.

(ii)$\Rightarrow$(iii) follows from [6, Corollary 1].

For (ii)$\Rightarrow$(i), take an arbitrary convex subspace C of some linear space L, a compact subset K of C and $\varepsilon > 0$. The convex subspace $\mathrm{conv}(K)$ of C is σ-compact and $K \subset \mathrm{conv}(K)$, so there is a map $f: K \to \mathrm{conv}(K) \subset C$ such that $d(f, \mathrm{id}) < \varepsilon$ and $f(K)$ is contained in a finite-dimensional linear subspace of L.

For (iv)$\Rightarrow$(i), take an arbitrary convex subspace C of some linear space L. There is a complete linear space L' in which L can be embedded densely, [16, I.1.5 and I.6.1]. The closure C' of C in L' is complete, so it is admissible. By Proposition 2.7, C is admissible as well. □

For linear spaces we have the following analogue. Its proof is similar to the proof of the preceding theorem.

Theorem 2.9. *The following assertions are equivalent:*
(i) *every linear space is admissible,*
(ii) *every σ-compact linear space is admissible,*
(iii) *every σ-compact linear space is an AR,*
(iv) *every topologically complete linear space is admissible.*

As already mentioned, we know that every linear space, which is spanned by a countable subset (equivalently: by a compact, countable subset, for we can shorten the spanning vectors so as to let them converge to the origin) is an AR. Therefore we are interested in whether the linear hull of a compact, zero-dimensional set is admissible (or an AR). The class, formed by these linear spaces is strictly larger than the class of the $\aleph_0$-dimensional ones, e.g. consider $L = \mathrm{lin}(\alpha(C))$, where $\alpha: C \to l^2$ is the embedding of the Cantor set $C \subset [0, 1]$, given by

$$\alpha(x) = (\tfrac{1}{2}x, (\tfrac{1}{2}x)^2, (\tfrac{1}{2}x)^3, \ldots);$$

the image set $\alpha(C)$ is uncountable and it is known that it is linearly independent. This class even contains strongly infinite-dimensional spaces. For example, take the countably infinite product $K = C^\infty$ and its linear hull $L = \mathrm{lin}(K)$ in $\mathbb{R}^\infty$. From $C - C = [-1, 1]$, cf. [19], we immediately see that $L \supset [-1, 1]^\infty$. This observation inspired us to the next proposition.

Proposition 2.10. *Let L be a non-σ-compact linear space; let K be a compact subset of L and let $\varepsilon > 0$. Then there exist two Cantor sets C_K and C_0 in L such that*

$$C_K - C_0 \supset K,$$

$$C_K \subset B(K, \varepsilon)$$

and

$$C_0 \subset B(0, \varepsilon).$$

Proof. L is not σ-compact, so we can pick

$$v \in L \backslash \operatorname{lin}(K), \quad d(v, 0) < \varepsilon.$$

Furthermore there exists a surjective map $\alpha : C \to K$ of a Cantor set $C \subset (0, 1)$ to K. Define

$$C_K = \{\alpha(t) + t \cdot v \mid t \in C\}$$

and

$$C_0 = \{t \cdot v \mid t \in C\}.$$

Obviously, C_K and C_0 are homeomorphic images of C, so Cantor sets, and they satisfy the requirements. □

Corollary 2.11. *Let L be a non-σ-compact linear space and let $E \subset L$ be a σ-compact linear subspace of L. Then there exists a Cantor set $K \subset L$ such that $E \subset \operatorname{lin}(K)$.*

Proof. Write E as the union $E = \bigcup_{n=1}^{\infty} K_n$ of compacta K_n. For every n determine Cantor sets C_n^1 and C_n^2 such that $C_n^1 - C_n^2 \supset K_n$. Then $C_n = C_n^1 \cup C_n^2$ is a Cantor set as well, and $\operatorname{lin}(C_n) \supset K_n$. The union $C = \bigcup_{n=1}^{\infty} \lambda_n C_n$ for some small enough positive numbers λ_n is again a Cantor set and its linear hull contains all of E. □

Theorem 2.12. *The following assertions are equivalent:*

(i) *every linear space is admissible,*

(ii) *every linear space, which is spanned by a Cantor set, is admissible, and*

(iii) *every linear space, which is spanned by a Cantor set, is an AR.*

Proof. (i)$\Rightarrow$(ii) is a triviality.

(ii)$\Leftrightarrow$(iii) follows from the σ-compactness.

(ii)$\Rightarrow$(i): Theorem 2.9 allows us to assume that the linear space L is topologically complete. If $\dim L < \infty$, then there is nothing to prove because in that case L is homeomorphic to $\mathbb{R}^n$ for some n. So, in addition, assume $\dim L = \infty$. Pick a compact subset K of L. It suffices to find an admissible linear subspace L_1 of L such that $L_1 \supset K$. From Corollary 2.11, applied to $\operatorname{lin}(K)$ we deduce such a space L_1. □

We have tried to replace the linear spaces in this theorem by convex sets and the linear hull by the convex hull. In order to prove the resulting conjecture we have tried to generalize Corollary 2.11 to this situation. This amounts to the following question:

Question. Let L be a non-σ-compact linear space and let $K\subset L$ be a σ-compact convex subset of L. Does there exist a Cantor set $C\subset L$ such that $K\subset \operatorname{conv}(C)$?

Given a compactum K and two positive constants u and v we get from Proposition 2.10 a pair C, D of Cantor sets in L with $u\cdot C+v\cdot D\supset K$, for take $C=1/u\cdot C_1$ and $D=-1/v\cdot C_2$, where C_1 and C_2 are given by the mentioned proposition. We get

$$C\subset \frac{1}{u}B(K,\varepsilon) \quad\text{and}\quad D\subset \frac{1}{v}B(0,\varepsilon). \tag{$*$}$$

Write $K=\bigcup_{n=1}^{\infty}K_n$, where K_n is compact for every n. We would like to take something like $C=C_1\cup D_1\cup C_2\cup\cdots$, where C_n and D_n are to satisfy

$$K_n\subset \frac{1}{2^{2n-1}}\cdot C_n+\frac{1}{2^{2n}}\cdot D_n,$$

so

$$K\subset \tfrac12\cdot C_1+\tfrac14\cdot D_1+\tfrac18\cdot C_2+\cdots\subset \operatorname{conv} C.$$

For C to be compact, we would like to have the C_n and D_n converge to some compactum, $\{0\}$ or K for example. For the D_n, this causes no problem. Alas, by $(*)$ we see that the C_n could converge to infinity. And when looking more precise, this phenomenon seemed to us to be persistent.

3. The homeomorphism extension property

We start this section by a simple proposition, showing that finite-dimensional linear spaces are of minor interest in studying the HEP.

Proposition 3.1. *If the linear space $X\neq\{0\}$ has the HEP for the class $\mathcal{K}_{\mathrm{fd}}$, then* $\dim X=\infty$.

Proof. Suppose that X has the HEP for $\mathcal{K}_{\mathrm{fd}}$ and X is finite dimensional. We may suppose $X=\mathbb{R}^n$ for some $n\in\mathbb{N}$, cf. [16, I.3.2]. Consider the compactum K, given by

$$K=\{x\in\mathbb{R}^n \mid x=0 \text{ or } \|x\|=1 \text{ or } \|x\|=2\}$$

($\|\cdot\|$ denotes the Euclidean norm) and define $h\in\mathcal{H}(K)$ by

$$h(x)=\begin{cases}0, & x=0,\\ 2x, & \|x\|=1,\\ \tfrac12 x, & \|x\|=2.\end{cases}$$

By elementary arguments it is easily seen that h is not extendible to any homeomorphism of $\mathbb{R}^n$. $\square$

The next lemma will be very useful in the sequel.

Lemma 3.2. *Let C be a convex subset of a linear space L and let K and K' be compact subsets of C, such that there is a homeomorphism $h: C \to C$ with $h(K) = K'$. Then K is admissible in C iff K' is admissible in C.*

Proof. Let K be admissible in C. Pick an open cover $\mathcal{U}'$ of C. Choose an open cover $\mathcal{U}$ of C such that $h(\mathcal{U})$ refines $\mathcal{U}'$. There is a map $f: K \to C$ such that $\dim f(K) < \infty$ and $\{\{f(x), x\}\}_{x \in K}$ refines $\mathcal{U}$. Define the map $f': K' \to C$ by $f'(x) = h \circ f \circ h^{-1}(x)$. Then $\{\{f'(x), x\}\}_{x \in K'}$ refines $\mathcal{U}'$ and $\dim f'(K') < \infty$, Lemma 2.6. $\square$

We call a convex subspace C of a linear space L *locally complete* at $x \in C$ if there is a neighborhood of x in C, which is complete with respect to an invariant metric on L, cf. [5].

Proposition 3.3. *Let C be an infinite-dimensional convex subset of some linear space, such that C is locally complete at some point. If C has the HEP, then C is admissible.*

Proof. Pick an arbitrary compact subset K of C. There exists an embedding $i: K \to Q$. Now [5, Proposition 3.5] gives an embedding $j: Q \to C$. The Hilbert cube $j(Q)$ is admissible in C since we have the projections

$$\pi_k : Q = \prod_{i=1}^{\infty} [0,1] \to \prod_{i=1}^{k} [0,1] \times \prod_{i=k+1}^{\infty} \{0\}.$$

The subset $j \circ i(K)$ of $j(Q)$ is of course admissible in C as well. Now there is a homeomorphism h of C with $h(K) = j \circ i(K)$. Lemma 3.2 easily implies the admissibility of K in C. $\square$

The following was already stated in [7] without proof.

Corollary 3.4. *Let L be a complete linear space with the HEP. Then L is admissible.*

Now we shall prove the HEP from the CEP in a rather restricted case. An interesting corollary is, that the HEP does not imply the AR-property in linear spaces, unless every linear space is an AR. We start with recalling an old result of Klee.

Proposition 3.5. *Let $\mathcal{K}$ be a class of compacta; let X_1 and X_2 be linear spaces with the $\mathcal{K}$-EP. Furthermore, let $X = X_1 \times X_2$ contain K, $L \in \mathcal{K}$ such that*
(i) *$\pi_1 : K \cup L \to X_1$ is an embedding,*
(ii) *X_2 contains a copy of K, and*
(iii) *there is a homeomorphism $h: K \to L$.*
Then there is an $H \in \mathcal{H}(X)$ such that $H \mid K = h$.

Proof. See [13, § 3]. □

Proposition 3.6. *If the linear space Y is admissible, then $X = Y \times \mathbb{R}^\infty$ has the homeomorphism extension property.*

Proof. Pick compacta K and L in X and a homeomorphism $h: K \to L$. There are compacta M_1 and M_2 such that

$$K \cup L \subset M_1 \times M_2 \subset Y \times \mathbb{R}^\infty.$$

Noting that the pseudointerior $(0, 1)^\infty$ of the Hilbert cube is homeomorphic to $\mathbb{R}^\infty$, we get from [19, 6.3.2] a homeomorphism $\psi: \mathbb{R}^\infty \to \mathbb{R}^\infty \times \mathbb{R}^\infty$ satisfying $\psi(M_2) \subset \mathbb{R}^\infty \times \{0\}$. Identifying $Y \times (\mathbb{R}^\infty \times \mathbb{R}^\infty)$ and $(Y \times \mathbb{R}^\infty) \times \mathbb{R}^\infty$, we have a homeomorphism $\Psi = \mathrm{id}_Y \times \psi: X \to X \times \mathbb{R}^\infty$, such that $\Psi(K \cup L) \subset X \times \{0\}$. Both of X and $\mathbb{R}^\infty$ are admissible, hence by Corollary 2.3 they have the CEP. Furthermore $\mathbb{R}^\infty$ contains a copy of $\Psi(K)$, so Proposition 3.5 is applicable to the homeomorphism $\Psi h \Psi^{-1}: \Psi(K) \to \Psi(L)$. Composition with Ψ and Ψ^{-1} gives the required homeomorphism of X. □

Theorem 3.7. *If there exists a linear space which is not an absolute retract, then there exists a linear space with the homeomorphism extension property, which is not an absolute retract.*

Proof. Suppose there is a linear space which is not an AR. In [17, Theorem 3.8] an admissible linear space Y is constructed, which is not an AR. Consider the product $X = Y \times \mathbb{R}^\infty$. By the equivalence of the CEP and admissibility, Y is admissible, so by Proposition 3.6, X has the HEP. Furthermore it is not an AR, for Y is not. □

Next we shall prove the HEP for the class $\mathcal{K}_{\mathrm{fd}}$ for any non-σ-compact linear space. We need an easy proposition.

Proposition 3.8. *If the space X is contractible and locally contractible, then X has the $\mathcal{K}_{\mathrm{fd}}$-EP.*

Proof. Pick a subspace $A \in \mathcal{K}_{\mathrm{fd}}$ of an arbitrary space Y. There is an embedding $j: A \to B$ of A into the n-cube $B = I^n$ for some $n \in \mathbb{N}$. The space X is an absolute extensor for the class of finite-dimensional spaces, [12, Chapter V], so the map $f \circ j^{-1}: j(A) \to X$ has an extension $g: B \to X$. Next, B is an AR, so $j: A \to B$ has an extension $h: Y \to B$. The map $g \circ h: Y \to X$ extends f. □

Theorem 3.9. *Let X be a non-σ-compact linear space. Then X has the homeomorphism extension property for the class $\mathcal{K}_{\mathrm{fd}}$.*

Proof. The proof is strongly inspired by that of [7, Lemma 2.9]. Pick finite-dimensional compacta K and L in X and a homeomorphism $h: K \to L$. Let $r = 2 \cdot \dim K + 1$. Using the σ-compactness of $\mathrm{lin}(K \cup L)$ we can inductively find

$x_1, \ldots, x_r \in X$ such that

$$x_i \notin \operatorname{lin}(K \cup L \cup \{x_1, \ldots, x_{i-1}\}), \quad i = 1, \ldots, r.$$

Let $E = \operatorname{lin}\{x_1, \ldots, x_r\}$. Then E is a linear subspace, such that $\dim E = 2 \cdot \dim K + 1$ and $E \cap \operatorname{lin}(K \cup L) = \{0\}$. It is well known that the quotient space X/E is a metrizable linear space as well and also that there exists a map $\alpha : X/E \to X$ such that $\alpha(0) = 0$ and $\kappa \circ \alpha = \mathrm{id}_{X/E}$ (denoting the canonical map from X onto X/E by κ). Define $\phi : X \to (X/E) \times E$ by

$$\phi(x) = (\kappa(x), x - \alpha\kappa(x)).$$

Then ϕ is a homeomorphism (its inverse is $\phi^{-1}(x, y) = \alpha(x) + y$) and $\pi_1 : \phi(K \cup L) \to X/E$ is an embedding. Furthermore, by $\dim E = 2 \cdot \dim K + 1$, E contains a copy of $\phi(K)$, [19, Theorem 4.4.4], and $\phi \circ h \circ \phi^{-1} : \phi(K) \to \phi(L)$ is a homeomorphism. From Proposition 3.5 we find a $G \in \mathscr{H}((X/E) \times E)$ with

$$G \,|\, \phi(K) = \phi \circ h \circ \phi^{-1}.$$

Now $H = \phi^{-1} \circ G \circ \phi \in \mathscr{H}(X)$ satisfies $H \,|\, K = h$. □

For the deduction of an estimated version of this theorem we need an embedding result.

Lemma 3.10. *Let X be a linear space, Y compact, $A \subset Y$ closed, such that* $\dim Y \backslash A < \infty$. *Then every $f : Y \to X$ can be approximated by maps $g : Y \to X$ with $g|A = f|A$ and $g(Y \backslash A)$ contained in an $\aleph_0$-dimensional linear subspace of X.*

Proof. Pick $\varepsilon > 0$. Let $n = \dim Y \backslash A$. Fix (see for example [12]) a locally finite open cover $\mathscr{U} = \{U_s\}_{s \in S}$ of $Y \backslash A$ of order at most n such that

$$\{y, y'\} \in U_s \quad \Rightarrow \quad d(f(y), f(y')) < \tfrac{1}{2}\varepsilon, \quad s \in S,$$

$$\operatorname{diam} U_s \leq d(U_s, A), \quad s \in S.$$

Pick arbitrary $b_s \in U_s$ $(s \in S)$ and choose a locally finite partition of unity $\{\phi_s\}_{s \in S}$ on $Y \backslash A$. We define $g : Y \to X$ by

$$g(y) = \begin{cases} f(y), & y \in A, \\ \sum_s \phi_s(y) \cdot f(b_s), & y \in Y \backslash A. \end{cases}$$

The continuity of g in every point of $Y \backslash A$ is clear, so pick $y \in A$ and an open neighborhood W of 0 in X. Determine $\eta > 0$ with $B(0, 2(n-1)\eta) \subset W$ and $\delta > 0$ with $f(B(y, \delta)) \subset B(f(y), \eta)$. Continuity in y is clear after

Claim. $g(B(y, \frac{1}{2}\delta)) \subset f(y) + W$.

Proof. Take y' with $d(y, y') < \frac{1}{2}\delta$. For every $s \in S$ with $y' \in U_s$ we have $d(U_s, A) < \frac{1}{2}\delta$, so $d(b_s, y') \leq \operatorname{diam} U_s < \frac{1}{2}\delta$, so $d(b_s, y) < 2 \cdot \frac{1}{2}\delta = \delta$, from which it follows that $f(b_s) \in B(f(y), \eta)$. There are at most n of these $s \in S$, so by [17, Lemma 3.1] we have

$$g(y') \in \operatorname{conv}\{f(b_s) \mid y' \in U_s\} \subset B(f(y), 2(n-1)\eta) \subset f(y) + W.$$

Furthermore, $d(f, g) \leq \varepsilon$, for take $y \in Y \backslash A$. Then for every $s \in S$ with $y \in U_s$ we have $f(b_s) - f(y) \in B(0, \varepsilon/(2n))$, so by [17]

$$g(y) \in \operatorname{conv}\{f(b_s) \mid y \in U_s\} \subset f(y) + B(0, \varepsilon)$$

and we are done. □

Proposition 3.11. *Let X be a linear space with the HEP for $\mathcal{K}_{\mathrm{fd}}$, let Y be finite dimensional compact, and $A \subset Y$ closed. Then every map $f: Y \to X$ such that $f \mid A$ is an embedding is approximable by embeddings $g: Y \to X$ such that $g|A = f|A$.*

Proof. Pick an embedding $i: A \to X$ such that $i(A)$ is contained in some finite-dimensional linear subspace of X. By the HEP for $\mathcal{K}_{\mathrm{fd}}$ there is a $\xi \in \mathcal{H}(X)$ with $\xi \circ i = f \mid A$. If we can approximate $\xi^{-1} \circ f$ by embeddings $h: Y \to X$ with $h|A = \xi^{-1} f|A$, then the embeddings $g = \xi \circ h$ are as required. From this it follows that without loss of generality we can assume that $f(A)$ is contained in some finite-dimensional linear subspace of X.

Now pick $\varepsilon > 0$. Lemma 3.10 gives a map $k: Y \to X$ such that

$$d(k, f) < \tfrac{1}{2}\varepsilon,$$

$$k|A = f|A,$$

and

$$k(Y \backslash A) \subset X_0,$$

where X_0 is an $\aleph_0$-dimensional linear subspace of X. The linear subspace $X_1 = \operatorname{lin}(X_0 \cup f(A))$ is $\aleph_0$-dimensional as well, so by [5, Corollary 4.2] it is homeomorphic to the space l^2_f. This subspace contains $k(Y)$, so by [15, Theorem 5] there is an embedding $g: Y \to X_1$ with

$$d(g, k) < \tfrac{1}{2}\varepsilon,$$

$$g|A = k|A.$$

This gives an embedding $g: Y \to X$ such that $g|A = f|A$ and $d(g, f) < \varepsilon$. □

By well-known methods (for example, see [19, § 7.4]) and the help of this proposition we can prove the desired theorem.

Theorem 3.12. *Let X be a linear space with the HEP for $\mathcal{K}_{\mathrm{fd}}$, $V \subset X$ open and $\mathcal{U}$ a collection of open subsets of V; let K be finite dimensional compact and $F: K \times I \to V$ a map such that F_0 and F_1 are embeddings and F is a $\mathcal{U}$-homotopy. Then there is an isotopy $H: X \times I \to X$ such that*

(i) $H_0 = \mathrm{id}$,
(ii) $H_1 \circ F_0 = F_1$,
(iii) $H_t | X \setminus V = \mathrm{id}$ *for* $t \in I$, *and*
(iv) $H | V \times I : V \times I \to V$ *is a* $\mathcal{U}$*-isotopy.*

Corollary 3.13. *Any non-σ-compact linear space has the estimated form, as described in the above theorem, of the HEP for $\mathcal{K}_{\mathrm{fd}}$.*

Proof. Combine Theorems 3.9 and 3.12. □

4. The HEP in two more or less concrete cases

In the first part of this section, let C be a fixed, infinite-dimensional, convex, compact set in some linear space. Remember that metrics on linear spaces are always chosen to be invariant and strictly monotone.

Lemma 4.1. *There exists an $x_0 \in C$ such that*

$$\inf_{z \in C} \mathrm{diam}([0,\infty) \cdot (z - x_0) \cap (C - C)) = 0.$$

Proof. We may assume $0 \in C$. Note that this has the effect that every point of the form

$$\sum_{n=1}^{k} \lambda_n z_n, \quad z_n \in C,\ \lambda_n \geq 0,\ \sum_{n=1}^{k} \lambda_n \leq 1$$

is an element of C. Fix a dense set $\{x_n\}_{n=1}^{\infty}$ with $x_n \neq x_m$ $(n \neq m)$ in C. Let

$$s_p = \sum_{n=1}^{p} 2^{-n} x_n.$$

Since $s_p \in C$, there is a subsequence $(s_{p_r})_r$, convergent to some $x_0 \in C$. Suppose that this x_0 does not satisfy the requirement. Then there is $\varepsilon > 0$ such that

$$\mathrm{diam}([0,\infty) \cdot (z - x_0) \cap (C - C)) > \varepsilon, \quad z \in C.$$

Set $z_{k,l} = x_0 + 2^{-l}(x_k - x_l)$. These points are elements of C for $z_{k,l}$ is limit of the sequence $(s'_{p_r})_r$, where s'_p is given by an expression, analogous to the one for s_p, with the single term $2^{-l}x_l$ replaced by $2^{-l}x_k$. We have

$$\mathrm{diam}([0,\infty) \cdot (x_k - x_l) \cap (C - C)) > \varepsilon.$$

It follows that

$$B = \left(\bigcup_{k,l} [0,\infty) \cdot (x_k - x_l) \right) \cap B_L(0, \varepsilon) \subset C - C.$$

(We denote the "ambient" linear space by L.) Let $E=\mathrm{lin}(C)$. From the convexity of C and $0\in C$ it follows that $E=\bigcup_{n=1}^{\infty} n(C-C)$.

Claim. $B_E(0,\frac{1}{2}\varepsilon)\subset C-C$.

Proof. Let $z\in E$, $|z|<\frac{1}{2}\varepsilon$. Then $z=n(x-y)$ for some $x,y\in C$ and $n\in\mathbb{N}$. There are strictly increasing integer sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ such that $n(x_{n_k}-x_{m_k})\to z$. Since for large k we have $|n(x_{n_k}-x_{m_k})|<\varepsilon$, we find $z\in \bar{B}\subset C-C$.

By infinite dimensionality of E and compactness of $C-C$ this gives a contradiction, [16]. □

Remark. We could take $x_0=\sum_{n=1}^{\infty} t_n x_n$, where $t_n\geqslant 0$, $\sum_{n=1}^{\infty} t_n\leqslant 1$ and $\sum t_n x_n$ is convergent.

We recall a few standard definitions, cf. [3; 4, V.2].

(i) A subset A of C is a *Z-set* in C iff A is closed and for every $n\geqslant 0$, every map $f:I^n\to C$ is uniformly approximable by maps $g:I^n\to C\backslash A$.

(ii) The subset A is a *Z_σ-set* in C iff A is the union of countably many Z-sets in C; equivalently (by completeness of C): iff A is a union of countably many closed sets and for every $n\geqslant 0$, every map $f:I^n\to C$ is uniformly approximable by maps $g:I^n\to C\backslash A$.

(iii) The point x in C is called a *central point* for C if the set $x+[0,1)\cdot(C-x)$ is a Z_σ-set in C.

In [3] it is proved that every Keller cube (see a few lines below) contains a central point. We start with strengthening this result.

Proposition 4.2. *There exists a central point for C.*

Proof (cf. [3, Proposition 2.6]). Take x_0 from Lemma 4.1. We may assume that $x_0=0$ and we shall show that 0 is central. By compactness, the set $[0,1)\cdot C$ is a countable union of closed sets. We check the approximation property. Take $\varepsilon>0$ and a map $f:I^n\to C$. By Lemma 2.4 there are linearly independent $x_1,\ldots,x_r$ in C and there is a map $f_1:I^n\to C'$, where $C'=C\cap L$, $L=\mathrm{lin}\{x_1,\ldots,x_r\}$, such that $d(f_1,f)<\frac{1}{2}\varepsilon$. By replacing f_1 by $p+\lambda\cdot(f_1-p)$ for some $\lambda\in(0,1)$, $p\in\mathrm{int}_L C'$, we have $A:=f_1(I^n)\subset\mathrm{int}_L C'$. Since

$$\inf_{z\in C'}\mathrm{diam}([0,\infty)\cdot z\cap(C-C))\geqslant\inf_{z\in C'}\mathrm{diam}([0,\infty)\cdot z\cap(C'-C'))>0,$$

there is $q\in C\backslash L$ such that

$$\mathrm{diam}([0,\infty)\cdot q\cap(C-C))<\tfrac{1}{2}\varepsilon.$$

Write $E=\mathrm{lin}(L,q)$ and $\{f_2(x)\}=(x+[0,\infty)\cdot q)\cap\mathrm{Bd}_E(C\cap E)$ for $x\in A$. Note that this set indeed contains precisely one point for every $x\in A$ and that $f_2:A\to C$ is continuous. It is easy to check that $f_2(A)\subset C\backslash[0,1)\cdot C$ and

$$d(f_2,\mathrm{id})\leqslant\mathrm{diam}([0,\infty)\cdot q\cap(C-C)).$$

The map $g=f_2\circ f_1$ approximates f and satisfies $g(I^n)\subset C\backslash[0,1)\cdot C$. □

Remark. By almost the same proof we see that every point $x_0 \in C$, satisfying the conclusion of Lemma 4.1, is a central point for C.

Theorem 4.3. *If C has the HEP for Z-sets, then C is an AR.*

Proof. By [6, Corollary 1] it is enough to prove admissibility of C. To that end take a central point x_0 for C, Proposition 4.2. We may assume $x_0 = 0$. Since the sequence, formed by the maps

$$x \to \left(1 - \frac{1}{n}\right) \cdot x, \quad x \in C,$$

converges uniformly to id_C, it suffices to show that every Z-set $A \subset C$ is admissible in C. By [5, Proposition 3.5] there is a copy of Q contained in $\frac{1}{2}C$. Since this Q is a Z-set in C, there is a homeomorphism $h \in \mathscr{H}(C)$ with $h(A) \subset Q$. Lemma 3.2 finishes the argument. □

Corollary 4.4. *C has the HEP for Z-sets iff C is homeomorphic to Q.*

Proof. See [8]. □

We point out that usually for non-ANR's, Z-sets are defined in a different way, namely as what we call for the moment Z*-sets (while our Z-sets are commonly named *locally homotopy negligible*):

(i′) The closed subset A of the space X is said to be a *Z*-set* iff the identity map $\mathrm{id}: X \to X$ can be approximated uniformly by maps $f: X \to X \setminus A$.

It is obvious that every Z*-set is a Z-set and we are led to the following question:

Question. If C has the HEP for Z*-sets, can we conclude that it is an AR?

Furthermore, we would like to have an answer to:

Question. Can we weaken "compactness" in Theorem 4.3 to "local compactness"?

In the light of Theorem 3.9 we are interested in:

Question. Does every compact convex infinite-dimensional subset of a linear space have the HEP for finite-dimensional Z-sets?

This question is somewhat premature, for singletons are Z-sets, and we even do not know the answer to:

Question. Is every compact convex infinite-dimensional subset of every linear space homogeneous?

The second part of this section is devoted to the HEP in linear spaces, spanned by a Cantor set.

Recall that a *Keller cube* in some linear space is a convex subset which is affinely homeomorphic to a compact, convex, infinite-dimensional subset of l_2, cf. [4]. Let Σ denote the linear space

$$\Sigma = \{x \in l_2 \mid \sup_i |ix_i| < \infty\}.$$

Proposition 4.5. *The following assertions are equivalent*:

(i) *every linear space, which is spanned by a Cantor set and contains a Keller cube, has the HEP,*

(ii) *every σ-compact linear space, which contains a Keller cube, is homeomorphic to Σ,*

(iii) *every linear space is admissible.*

Proof. (iii)$\Rightarrow$(ii): a linear space as in (ii) is admissible, so by σ-compactness it is an AR, [6]. By [5, Corollary 4.2] we have homeomorphy to Σ.

(ii)$\Rightarrow$(i): a linear space as in (i) is σ-compact, so homeomorphic to Σ, which has the HEP by [4].

(i)$\Rightarrow$(iii): by Theorem 2.9 we may take a complete linear space L. Pick $K \subset L$ compact. By [5, Proposition 3.5] there exists a Keller cube Q in L. This cube contains a homeomorph K' of K. Using Proposition 2.10 we find a Cantor set C in L such that $K \cup Q \subset \mathrm{lin}(C)$. The linear space $L' = \mathrm{lin}(C)$ has the HEP, so

$$\exists h \in \mathscr{H}(L'): \quad h(K) = K'.$$

Now Lemma 3.2 implies the admissibility of K' in L', so certainly in L. □

Remark [5, after Proposition 3.5]. For a linear space the condition that it contains a Keller cube may be (apparently) weakened by the condition that it contains an infinite-dimensional convex subset, which is somewhere locally complete.

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