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Sierpiński's technique and subsets of $\mathbb{R}$

Jan van Mill

Department of Mathematics and Computer Science, Vrije Universiteit, De Boelelaan 1081a, 1081 HV Amsterdam, Netherlands

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Abstract

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Sierpiński invented in 1932 a method for constructing examples, which is now known as the "*technique of killing homeomorphisms*". This method was used by Ohkuma to construct a rigid homogeneous chain, and by van Douwen to construct a compact homogeneous space with a measure that "knows" which sets are homeomorphic, and by Keesling and Wilson to construct an almost uniquely homogeneous subgroup of $\mathbb{R}^n$. The aim of this paper is to derive these results simultaneously.

Keywords: Sierpiński's technique, $\mathbb{R}$, measure.

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1. Introduction

Sierpiński [23] invented a method for constructing examples, which is now known as the "*technique of killing homeomorphisms*". This method was used by various mathematicians for various purposes. For example, Sierpiński [23] used it for the construction of a rigid subset of the real line $\mathbb{R}$, van Douwen [2] used it for the construction of a compact space having a measure that "knows" which sets are homeomorphic, Shelah [21] and van Engelen [3] used it to prove that $\mathbb{R}$ can be partitioned into two homeomorphic rigid sets, Todorčević [24] used it for the construction of various interesting examples concerning cardinal functions, Marciszewski [15] used it for the construction of a compact space K such that the function space $C(K)$ is not weakly homeomorphic to $C(K)\times C(K)$, Keesling and Wilson [12] used it to construct an "almost uniquely homogeneous" subgroup of $\mathbb{R}^n$, and the author used it in [19] for the construction of an infinite-dimensional normed linear space L which is not homeomorphic to $L\times\mathbb{R}$.

The aim of this paper is to construct a new example of a rigid homogeneous chain that can also be used for the construction of a compact space with a measure that "knows" which sets are homeomorphic. Also, its higher dimensional analogues are almost uniquely homogeneous subgroups of $\mathbb{R}^n$. We believe that the fact that we can derive these results simultaneously makes our construction of independent interest.

Let X be a space, and let f be a homeomorphism such that $\mathrm{dom}(f)\subseteq X$ and $\mathrm{range}(f)\subseteq X$. Suppose that we would like to "kill" f. There are several strategies that one could follow. First, one could try to refine the topology of X making sure that f is no longer continuous. Then f is certainly killed. However, by the refinement of the topology, it is possible that some other undesired function that is discontinuous in the old topology, becomes continuous in the new topology. So one has then to continue the process of refining the topology, and it is not impossible that at the end of all the killings, X carries the discrete topology. Then all the dead functions resurrect, and there is deep, deep trouble. One encounters the same difficulties when trying the opposite strategy of making the topology coarser. For then, it is not impossible to end with the indiscrete topology. So it seems that these kinds of strategies should not be considered.

Another possibility is to restrict f to a subspace of X, say A, and hope for the best. If both $\mathrm{dom}(f)$ and $\mathrm{range}(f)$ are subsets of A, then nothing of importance happened. However, if we choose A in such a way that

$$\text{for some } x\in\mathrm{dom}(f)\cap A\colon\quad f(x)\notin A, \tag{1}$$

i.e., $f\restriction A$ is not a function of A into itself, then, by restricting our attention to A, we successfully killed f. Of course, if the killing of f is part of a mass-murder, then we have to prevent f from resurrecting. That is simple. We make sure that the point x in (1) is not removed from A later on, while moreover the point $f(x)$ is never added to A.

We just described Sierpiński's technique of killing homeomorphisms.

2. Definitions

As usual, $\mathbb{R}$ abbreviates the reals, $\mathbb{Q}$ the rationals, $\mathbb{P}$ the irrationals, $\mathbb{N}$ the set of natural numbers, and $\mathbb{Z}$ the set of integers.

The symbol "$X\approx Y$" means that X and Y are homeomorphic spaces. The closure of a set $A\subseteq X$ is denoted by $\bar{A}$. If X is a space, then $\mathscr{H}(X)$ denotes the group of all homeomorphisms of X. The identity function on X will be denoted by 1_X. We say that X is *homogeneous* (or, *topologically homogeneous*) if for all $x, y \in X$ there exists $f \in \mathcal{H}(X)$ such that $f(x) = y$.

As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. The domain and range of a function f will be denoted by dom(f) and range(f), respectively. If A and B are sets, then ${}^A B$ denotes the set of all functions from A to B. If X is a set and κ is a cardinal number, then, as usual,

$$[X]^{<\kappa} = \{A \in \mathcal{P}(X) \colon |A| < \kappa\};$$

$$[X]^{\leqslant\kappa} = \{A \in \mathcal{P}(X) \colon |A| \leqslant \kappa\};$$

$$[X]^{\kappa} = \{A \in \mathcal{P}(X) \colon |A| = \kappa\}.$$

Recall that a *Cantor set* is a space homeomorphic to the Cantor discontinuum ${}^\omega 2$. Observe that ${}^2({}^\omega 2) \approx {}^\omega({}^\omega 2) \approx {}^\omega 2$. Consequently, each Cantor set contains a family of $\mathfrak{c}$ pairwise disjoint Cantor sets. In particular, for every Cantor set K and for every countable set $E \subseteq K$ there exists a Cantor set $L \subseteq K$ which misses E. This fact will be used without explicit reference throughout the remaining part of this paper.

If G is a group and $A \subseteq G$, then $\langle\langle A \rangle\rangle$ denotes the subgroup of G generated by A.

Let G be an Abelian group. A subset A of G is called *algebraically independent* if for all $a_1, \ldots, a_n \in A$ and $m_1, \ldots, m_n \in \mathbb{Z}$ we have

$$\sum_{i=1}^{n} m_i \cdot a_i = 0 \quad \Rightarrow \quad m_1 = \cdots = m_n = 0.$$

It is easy to see that each Cantor set $K \subseteq \mathbb{R}$ contains an algebraically independent Cantor set L [9, p. 477].

For every space X, let $\mathcal{B}(X)$ denote the collection of Borel subsets of X, i.e., $\mathcal{B}(X)$ is the σ-algebra generated by the open subsets of X. If Y is a subspace of X, then for every $B \in \mathcal{B}(Y)$ there clearly exists an element $B' \in \mathcal{B}(X)$ such that $B' \cap Y = B$.

3. Tools

The technique works best with separable metrizable spaces. The following well-known results, the proofs of some of which we will include for the sake of completeness, will be important in our constructions:

3.1. Theorem [14; 5, Theorem 4.3.20]. *Let X be a space and let Y be a completely metrizable space. If $A \subseteq X$ and if $f \colon A \to Y$ is continuous, then there is a G_δ-subset $\tilde{A} \subseteq \bar{A}$ such that f can be extended to a continuous function $\tilde{f} \colon \tilde{A} \to Y$.*

3.2. Theorem [14; 5, Theorem 4.3.21]. *Let X and Y be completely metrizable spaces, let $A \subseteq X$ and $B \subseteq Y$. Then for every homeomorphism $h: A \to B$ there are G_δ-subsets $\tilde{A}$ of X and $\tilde{B}$ of Y with $A \subseteq \tilde{A} \subseteq \bar{A}$ and $B \subseteq \tilde{B} \subseteq \bar{B}$ such that h can be extended to a homeomorphism $\tilde{h}: \tilde{A} \to \tilde{B}$.*

3.3. Corollary. *Let X be a completely metrizable space, and let $Y \subseteq X$. If S and T are Borel sets of Y and if $f: S \to T$ is a homeomorphism, then there exist Borel sets $\hat{S}$ and $\hat{T}$ of X such that*

(1) *$\hat{S} \cap Y = S$ and $\hat{T} \cap Y = T$;*
(2) *f can be extended to a homeomorphism $\hat{f}: \hat{S} \to \hat{T}$.*

Proof. Let $\tilde{S}$, $\tilde{T}$ be elements of $\mathscr{B}(X)$ such that $\tilde{S} \cap Y = S$ and $\tilde{T} \cap Y = T$. By Theorem 3.2, there exist G_δ-subsets A, $B \subseteq X$ containing S and T, respectively, such that f can be extended to a homeomorphism $g: A \to B$. Put $\hat{S} = \tilde{S} \cap g^{-1}(B \cap \tilde{T})$. Observe that $\hat{S}$ is a Borel set of X and that $\hat{S} \cap Y = S$. Also, $\hat{T} = g(\hat{S})$ is a Borel set of B such that $\hat{T} \cap Y = T$. Put $\hat{f} = g \restriction \hat{S}$. $\square$

3.4. Theorem [22; 5, Theorem 6.1.27]. *If X is a continuum and if $\mathscr{A}$ is a partition of X into countably many closed sets, then at most one element of $\mathscr{A}$ is nonempty.*

We will also need the following result, basically due to Souslin, cf. Kuratowski and Mostowski [13, p. 437] (see van Douwen [2, 4.2]).

3.5. Theorem. *Let X be a separable completely metrizable space and let $B \in \mathscr{B}(X)$. Let $\mathscr{F}$ be a countable family of continuous functions from B to a space Y such that*

$$\text{for every countable } A \subseteq Y\text{:} \quad \{f^{-1}(A)\text{: } f \in \mathscr{F}\} \text{ does not cover } B.$$

Then there exists a Cantor set $K \subseteq B$ such that $f \restriction K$ is injective for each $f \in \mathscr{F}$.

Proof. There exists a continuous surjection $\xi: \mathbb{P} \to B$ [13, p. 426]. For every $f \in \mathscr{F}$ put $\bar{f} = f \circ \xi$. Let $M \subseteq \mathbb{P}$ be maximal with respect to the property that $\bar{f} \restriction M$ is injective for each $f \in \mathscr{F}$. We claim that M is uncountable. Suppose the contrary, and put

$$A = \bigcup_{f \in \mathscr{F}} \bar{f}(M).$$

Then A is countable, so by assumption there exists $x \in X$ such that

$$x \notin \bigcup_{f \in \mathscr{F}} f^{-1}(A).$$

Then, for $p \in \xi^{-1}(x)$, $p \notin M$ and $\bar{f} \restriction M \cup \{p\}$ is injective for each $f \in \mathscr{F}$. This contradicts the maximality of M. Since each separable metrizable space is the union of a countable set and a set which is dense in itself (this is the so-called Cantor–Bendixson Theorem; for details, see [5, Problem 1.7.11]), the uncountability of M implies that there is a dense-in-itself set $P \subseteq M$. Now enumerate $\mathscr{F}$ as $\langle f_n : n \in \omega \rangle$ such that every $f \in \mathscr{F}$ is listed infinitely often. Let ρ denote an admissible complete metric on $\mathbb{P}$. For each $x \in \mathbb{P}$ and $\varepsilon > 0$, let $B(x, \varepsilon) = \{a \in \mathbb{P}: \rho(a, x) < \varepsilon\}$, and $D(x, \varepsilon) = \{a \in \mathbb{P}: \rho(a, x) \leq \varepsilon\}$, respectively. Using finite disjoint unions of balls about points of P, we may construct a Cantor set $\tilde{K}$ in the complete space $\mathbb{P}$ by the standard procedure; a little extra care will ensure that $\bar{f} \restriction \tilde{K}$ will be injective for each $f \in \mathscr{F}$. It suffices to describe the first two steps in the inductive construction.

Pick two distinct point p_0 and p_1 in P. Since $\bar{f}_0(p_0) \neq \bar{f}_0(p_1)$, there exists $0 < \varepsilon_0 < 1$ such that

$$\bar{f}_0(D(p_0, \varepsilon_0)) \cap \bar{f}_0(D(p_1, \varepsilon_0)) = \emptyset.$$

Put $D_0 = D(p_0, \varepsilon_0) \cup D(p_1, \varepsilon_0)$. Since P is dense-in-itself, there exist two distinct points $p_{0,0}$ and $p_{0,1}$ in $P \cap B(p_0, \varepsilon_0)$. Similarly, pick two distinct points $p_{1,0}$ and $p_{1,1}$ in $P \cap B(p_1, \varepsilon_0)$. Since $\bar{f}_1$ is one-to-one on P, there exists $0 < \varepsilon_1 < \frac{1}{2}$ such that the collection

$$\bar{f}_1(D(p_{0,0}, \varepsilon_1)),\ \bar{f}_1(D(p_{0,1}, \varepsilon_1)),\ \bar{f}_1(D(p_{1,0}, \varepsilon_1)),\ \bar{f}_1(D(p_{1,1}, \varepsilon_1))$$

is pairwise disjoint. Put $D_1 = D(p_{0,0}, \varepsilon_1) \cup D(p_{0,1}, \varepsilon_1) \cup D(p_{1,0}, \varepsilon_1) \cup D(p_{1,1}, \varepsilon_1)$. It is clear that we may choose ε_1 so small that $D_1 \subseteq D_0$.

Continuing with this procedure in the standard manner, we obtain a nested sequence $\langle D_n \rangle_n$ of closed sets in $\mathbb{P}$. Let $\tilde{K} = \bigcap_n D_n$. The requirements of the type $D(p_{0,0}, \varepsilon_1) \cup D(p_{0,1}, \varepsilon_1) \subseteq D_0$ and $D(p_{0,0}, \varepsilon_1) \cap D(p_{0,1}, \varepsilon_1) = \emptyset$, together with the requirement that $\varepsilon_n \to 0$ and the fact that ρ is a complete metric, show that $\tilde{K}$ is a Cantor set. The requirements of type

$$\bar{f}_1(D(p_{0,0}, \varepsilon_1)) \cap \bar{f}_1(D(p_{0,1}, \varepsilon_1)) = \emptyset$$

show that $\bar{f} \restriction \tilde{K}$ is injective for each $f \in \mathscr{F}$: observe that each $f \in \mathscr{F}$ is dealt with infinitely often. Now put $K = \xi(\tilde{K})$. Then K is clearly as required. □

3.6. Corollary [8; 1; 13, p. 427]. *Let S be a Borel set of a separable completely metrizable space. If S is uncountable, then S contains a Cantor set.*

Proof. Apply Theorem 3.5 with $\mathscr{F}$ consisting only of one function, namely the identity function on S. □

Let X be a separable completely metrizable space. A subset A of X is called a *Bernstein set* if A contains no uncountable compact subsets. In addition, A is called a *BB-set* if both A and $X \backslash A$ are Bernstein sets (BB stands for *bi-Bernstein*). Since every uncountable compact subset of X contains a Cantor set (Corollary 3.6), it easily follows that A is a BB-set if and only if, for every Cantor set $K \subseteq X$, $A \cap K \neq \emptyset \neq (X \backslash A) \cap K$.

3.7. Lemma. *Let X be a separable, completely metrizable, dense-in-itself space. If $Y \subseteq X$ is a BB-set, then Y is a Baire space.*

Proof. Let S be a dense G_δ-subset of X. Then the Baire Category Theorem implies that S is uncountable. So by Corollary 3.6, S contains a Cantor set, say K. By assumption, $Y \cap S \supseteq Y \cap K \neq \emptyset$. So Y intersects every dense G_δ-subset of X, and therefore is a Baire space. □

3.8. Lemma. *Let X be a separable, completely metrizable, dense-in-itself space. If $Y \subseteq X$ is a BB-set, then $|Y| = \mathfrak{c}$.*

Proof. Since X is dense-in-itself, X is uncountable and hence contains a Cantor set (Corollary 3.6). This Cantor set contains a family consisting of $\mathfrak{c}$-many pairwise disjoint Cantor sets, each member of which has to intersect Y. This proves that $|Y| \geqslant \mathfrak{c}$. That $|Y| \leqslant \mathfrak{c}$ is trivial because Y is a separable metrizable space. □

3.9. Lemma. *Let X be a separable, completely metrizable, dense-in-itself space. If $Y \subseteq X$ is a BB-set and if B is a Borel set in X such that $B \cap Y = \emptyset$, then B is countable.*

Proof. If B were uncountable, then it would contain a Cantor set by Corollary 3.6 and so it would intersect X. □

Recall that the *Sorgenfrey line* $\mathbb{S}$ has $\mathbb{R}$ as underlying set, and topology generated by the collection

$$[x, x+\varepsilon) \quad (x \in \mathbb{R}, \varepsilon > 0).$$

A *network* for a space X is a collection $\mathcal{B}$ of subsets of X such that for every $x \in X$ and for every neighborhood U of x there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. If X is a separable metrizable space, then X has a countable base, so X has a countable network. More generally, if X is a continuous image of a separable metrizable space Y, then X has a countable network. To see this, let $f: Y \to X$ be a continuous surjection, and let $\mathcal{E}$ be a countable base for Y. Then $\mathcal{B} = f(\mathcal{E})$ is a countable network for X.

3.10. Lemma. *If $A \subseteq \mathbb{S}$ has a countable network, then A is countable.*

Proof. Let $\mathcal{B}$ be a countable network for A. Assume that A is uncountable. For every $a \in A$ pick an element $B_a \in \mathcal{B}$ such that

$$a \in B_a \subseteq [a, a+1).$$

Since A is uncountable, there are distinct $a, a' \in A$ such that $B_a = B_{a'}$. Without loss of generality, assume that $a' < a$. Then

$$a' \in B_{a'} = B_a \subseteq [a, a+1),$$

so $a' > a$. This is a contradiction. □

3.11. Corollary. *If $A \subseteq \mathbb{S}$ is metrizable, then A is countable.*

Proof. Consider A to be a subspace of $\mathbb{R}$ and let D be a countable dense subset of A. It is easy to see that D is also dense in the Sorgenfrey topology on A. Consequently, A is a separable metrizable subset of $\mathbb{S}$, and consequently has a countable base, so a countable network. Now apply Lemma 3.10. □

We will now describe a useful technique of compactifying a *dense* subset of the interval $(0, 1)$. So assume that $A \subseteq (0, 1)$ is dense. We split each $x \in (0, 1)\backslash A$ into two points, x^- and x^+. The points of A will not be split. Order the set

$$S(A) = \{0, 1\} \cup A \cup \{x^-, x^+ : x \in (0, 1)\backslash A\}$$

in the natural way, so that x^- always precedes x^+. Endow $S(A)$ with the order topology derived from this order. Then $S(A)$ is easily seen to be a compact ordered, zero-dimensional space. In addition, A as subspace of $\mathbb{R}$ is precisely the same space as A as subspace of $S(A)$. Moreover, the sets

$$\{x^- : x \in (0, 1)\backslash A\},$$

and

$$\{x^+ : x \in (0, 1)\backslash A\},$$

are both homeomorphic to subspaces of $\mathbb{S}$. Finally, observe that $S(A)$ is first countable.

3.12. Lemma. *Let $A \subseteq (0, 1)$ be dense, and let $f : S(A) \to S(A)$ be a homeomorphism. Then there is a countable subset $B \subseteq A$ such that $f(A\backslash B) = A\backslash B$.*

Proof. Put $B = \{a \in A : (\exists n \in \mathbb{Z})(f^n(a) \notin A)\}$. Then by Corollary 3.11, B is countable, and clearly $f(A\backslash B) = A\backslash B$. □

4. A rigid homogeneous chain

An ordered set is sometimes called a *chain*. The aim of this section is to prove that there exists an ordered set $\langle X, \leqslant\rangle$ with the property that for all $x, y \in X$ there exists a *unique* order-isomorphism $f : X \to X$ sending x to y. Of course $\mathbb{Z}$ has this property, so to make the result interesting, we want it also to be densely ordered. Such an ordered set is called a *rigid homogeneous chain*. It is called homogeneous for obvious reasons, and rigid because for every x and y in X there is only one isomorphism that takes x onto y.

It can be shown that every rigid homogeneous chain is isomorphic to a subgroup of $\langle \mathbb{R}, +\rangle$ ([20]; see also [6]). We will construct a rigid homogeneous chain in $\mathbb{R}$ by killing certain functions between Cantor sets. For later use, we will describe a killing process in $\langle \mathbb{R}^k, +\rangle$ for arbitrary k.

So from now on in this section, let $G=\langle\mathbb{R}^k,+\rangle$. Let K be a Cantor set, and consider the collection

$$\mathcal{K}=\{\langle f,g\rangle : f,g:K\to G \text{ are embeddings and the functions } f+g \text{ and } f-g \text{ are one-to-one}\}. \tag{2}$$

Observe that if $\langle f,g\rangle\in\mathcal{K}$, then also $\langle g,f\rangle\in\mathcal{K}$. For every $\langle f,g\rangle$ we would like to kill the homeomorphism $g\circ f^{-1}:f(K)\to g(K)$, or if that is not possible, its inverse, namely the homeomorphism $f\circ g^{-1}:g(K)\to f(K)$.

Observe that if A and B are separable metrizable spaces, then the number of continuous functions from A to B is at most $\mathfrak{c}$. It follows that $|\mathcal{K}|\leq\mathfrak{c}$, so we can enumerate it as

$$\{\langle f_\alpha,g_\alpha\rangle : \alpha<\mathfrak{c}\}$$

(repetitions permitted). Let Z be an arbitrary countable subgroup of G, and let Q be an arbitrary countable subset of $G\backslash\{0\}$ disjoint from Z. These sets will play no role in this section, but they will become important later. By transfinite induction on $\alpha<\mathfrak{c}$, we will pick a point $x_\alpha\in K$ and points $p_\alpha, q_\alpha\in G\backslash\{0\}$ such that

(1) $\{p_\alpha,q_\alpha\}=\{f_\alpha(x_\alpha),g_\alpha(x_\alpha)\}$;

(2) $\langle\langle\{p_\beta:\beta\leq\alpha\}\cup Z\rangle\rangle\cap(\{q_\beta:\beta\leq\alpha\}\cup Q)=\emptyset$.

(Observe that the subgroup of G that we are going to construct, will contain Z but will have empty intersection with Q. This will give us a little freedom later.) So assume that we picked x_β, p_β and q_β for every $\beta<\alpha<\mathfrak{c}$ (possibly, $\alpha=0$). For convenience, put $A=\langle\langle\{p_\beta:\beta<\alpha\}\cup Z\rangle\rangle$, $V=\{q_\beta:\beta<\alpha\}\cup Q$, $f=f_\alpha$ and $g=g_\alpha$, respectively. Observe that $\max\{|A|,|V|\}\leq|\alpha|\cdot\omega<\mathfrak{c}$.

4.1. Lemma. *$E_f=\{x\in K:\langle\langle f(x)\cup A\rangle\rangle\cap V\neq\emptyset\}$ has cardinality less than $\mathfrak{c}$.*

Proof. For every $x\in E_f$ there exists $n_x\in\mathbb{Z}$ such that $n_x\cdot f(x)\in V-A$. Since $A\cap V=\emptyset$, $V-A\subseteq G\backslash\{0\}$, so always $n_x\neq 0$. Then for every $x\in E_f$,

$$f(x)\in\frac{1}{n_x}\cdot(V-A)\subseteq\mathbb{Q}\cdot(V-A).$$

Since $|\mathbb{Q}\cdot(V-A)|<\mathfrak{c}$, and f is one-to-one, we are done. □

By precisely the same argumentation one obtains:

4.2. Lemma. *$E_g=\{x\in K:\langle\langle g(x)\cup A\rangle\rangle\cap V\neq\emptyset\}$ has cardinality less than $\mathfrak{c}$.*

We now come to the crucial step in our argumentation.

4.3. Lemma. *If $F\subseteq K$ has cardinality $\mathfrak{c}$, then there exists $x\in F$ such that $f(x)\notin\langle\langle\{g(x)\}\cup A\rangle\rangle$ or $g(x)\notin\langle\langle\{f(x)\}\cup A\rangle\rangle$.*

Proof. Fix an arbitrary $F \subseteq K$ such that $|F| = \mathfrak{c}$ and assume that for every $x \in F$, $f(x) \in \langle\langle \{g(x)\} \cup A \rangle\rangle$. Let $\kappa = |A| \cdot \omega$. Then $|\mathbb{Z} \times A| = \kappa < \mathfrak{c}$, so there are an $n \in \mathbb{Z}$, an element $a \in A$, and a subset $\hat{F} \subseteq F$ of cardinality greater than κ such that for every $x \in \hat{F}$, $f(x) = n \cdot g(x) + a$. Since the functions $f + g$ and $f - g$ are both one-to-one, $|A| \leq \kappa$ and $|\hat{F}| > \kappa$, $n \neq 1$ and $n \neq -1$. We claim that there exists an element $x \in \hat{F}$ such that $g(x) \notin \langle\langle \{f(x)\} \cup A \rangle\rangle$. Assume the contrary. Since $|\hat{F}| > \kappa$ and $|\mathbb{Z} \times A| = \kappa$, there are a subset $\tilde{F}$ of $\hat{F}$ of cardinality greater than κ, an $m \in \mathbb{Z}$, and an element $\bar{a} \in A$, such that for every $x \in \tilde{F}$, $g(x) = m \cdot f(x) + \bar{a}$. Now pick an arbitrary element $x \in \tilde{F}$. Then

$$f(x) = n \cdot g(x) + a = n \cdot m \cdot f(x) + n \cdot \bar{a} + a.$$

Put $\tilde{a} = n \cdot \bar{a} + a$. Then for every $x \in \tilde{F}$, $(nm - 1) f(x) = \tilde{a}$. If $nm - 1 = 0$, then $|n| = 1$, which is impossible by the above. So $nm - 1 \neq 0$. This clearly contradicts the fact that f is one-to-one. □

Let E_f and E_g be as in Lemma 4.1 and Lemma 4.2, respectively. Let $F = K \setminus (E_f \cup E_g)$. Without loss of generality we assume that there exists $x \in F$ such that $f(x) \notin \langle\langle \{g(x)\} \cup A \rangle\rangle$ (Lemma 4.3). Now put $x_\alpha = x$, $p_\alpha = g(x_\alpha)$, and $q_\alpha = f(x_\alpha)$, respectively. It is clear that our choice of x_α is as required. This completes the transfinite construction. Put $X = \langle\langle \{p_\alpha : \alpha < \mathfrak{c}\} \cup Z \rangle\rangle$.

We now formulate and prove a curious property of X.

4.4. Theorem. *Let $S, T \subseteq X$ be Borel sets and let $f : S \to T$ be a homeomorphism. Then there is a countable subgroup A of X such that for every $x \in S$ there exists $a \in A$ such that $f(x) = x + a$ or $f(x) = -x + a$.*

Proof. By Corollary 3.3, there exist Borel sets $\hat{S}$ and $\hat{T}$ in G such that $\hat{S} \cap X = S$ and $\hat{T} \cap X = T$, while moreover f can be extended to a homeomorphism $\hat{f} : \hat{S} \to \hat{T}$.

Let $E \subseteq \hat{S}$ be maximal with respect to the properties that the functions $\xi, \eta : E \to G$, defined by

$$\xi(x) = x + \hat{f}(x);$$
$$\eta(x) = x - \hat{f}(x),$$

are one-to-one.

Claim. *E is countable.*

To the contrary, assume that E is uncountable. Let $i : \hat{S} \to \hat{S}$ denote the identity. Since the functions i, ξ and η are one-to-one on E, and E is uncountable, by Theorem 3.5 there exists a Cantor set $L \subseteq \hat{S}$ such that i, ξ and η are one-to-one on L. By considering an arbitrary homeomorphism between K and L we can now define in the obvious way an element of $\mathcal{K}$. Consequently, by construction, there exists $x \in L$ such that either $x \in X$ and $\hat{f}(x) \notin X$, or $\hat{f}(x) \in X$ and $x \notin X$. Suppose first that $x \in X$ and $\hat{f}(x) \notin X$. Then $x \in \mathrm{dom}(\hat{f}) \cap X = \hat{S} \cap X = S$. But now we contradict the fact that $\hat{f}$ extends f so that $\hat{f}(x) = f(x) \in X$. Next, suppose that $\hat{f}(x) \in X$

and $x \notin X$. Then $\hat{f}(x) \in \hat{T} \cap X = T$. Consequently, there exists $y \in S$ such that $f(y) = \hat{f}(x)$. Since $\hat{f}$ extends f, $\hat{f}(y) = \hat{f}(x)$. But this contradicts the fact that $\hat{f}$ is one-to-one.

So we conclude that E is countable. Now let $F = \langle\!\langle E \cup \hat{f}(E) \rangle\!\rangle$. Take an arbitrary element $x \in \hat{S} \setminus E$. Then by maximality of E, $\xi \restriction E \cup \{x\}$ or $\eta \restriction E \cup \{x\}$ is not one-to-one. So assume e.g. that there exists $e \in E$ such that $\xi(e) = \xi(x)$. Then

$$x + \hat{f}(x) = e + \hat{f}(e),$$

which implies that $x + \hat{f}(x) \in F$. If $\eta \restriction E \cup \{x\}$ is not one-to-one, then similarly, $x - \hat{f}(x) \in F$. Moreover, if $x \in E$, then $\hat{f}(x) \in F$ so in that case we also have $x - \hat{f}(x) \in F$. It is now clear that for the required countable subgroup of X we may take $F \cap X$. Simply observe that for every $x \in S$ we have $\hat{f}(x) = f(x)$ and both $f(x) + x$ and $f(x) - x$ belong to X. $\square$

We proceed by deriving another important property of X.

4.5. Proposition. *X is a BB-set.*

Proof. Let $C \subseteq G$ be a Cantor set. Pick an algebraically independent Cantor set $L \subseteq C$. Now take disjoint Cantor sets $A, B \subseteq L$. It easily follows that if $f: K \to A$ and $g: K \to B$ are homeomorphisms, then $\langle f, g \rangle \in \mathscr{K}$. By construction it therefore follows that $A \cup B$ intersects both X and $G \setminus X$. $\square$

We are now prepared to derive the main result in this section:

4.6. Theorem [20]. *$\mathbb{R}$ contains a dense rigid homogeneous chain.*

Proof. In this proof we assume X to be constructed in $\mathbb{R}$. We claim that X is a rigid homogeneous chain. Take arbitrary $x, y \in X$. Since X is a subgroup, the translation

$$p \mapsto p + (y - x) \quad (p \in X)$$

is an order-isomorphism of X that takes x to y. This proves that X is homogeneous. It remains to verify that X is rigid.

Claim. *If f is an order-isomorphism of X, then f is a translation.*

By Theorem 4.4 there exists a countable subgroup A of X such that for every $x \in X$ there exists $a \in A$ such that $f(x) = x + a$ or $f(x) = -x + a$. There also exists an order-isomorphism φ of $\mathbb{R}$ that extends f. For every $a \in A$ and $\varepsilon \in \{-1, 1\}$ put

$$S_a^\varepsilon = \{x \in \mathbb{R}: \varphi(x) = \varepsilon \cdot x + a\}.$$

It is clear that for every $a \in A$ and $\varepsilon \in \{1, -1\}$, S_a^ε is closed in $\mathbb{R}$. Also, the collection of all S_a^ε covers X. Consequently,

$$Y = \mathbb{R} \setminus \bigcup \{S_a^\varepsilon: \varepsilon \in \{-1, 1\}, a \in A\}$$

is a G_δ-subset of $\mathbb{R}$ that misses X. Since X is a BB-set (Proposition 4.5), we conclude that Y is countable (Lemma 3.9).

Fix $a \in A$ and assume that there exist two distinct elements $x, y \in S_a^{-1}$. Without loss of generality, $x < y$. Then

$$\varphi(y) = -y + a < -x + a = \varphi(x)$$

contradicts the fact that φ is an order-isomorphism. So each S_a^{-1} contains at most one point. Now put

$$B = A \cup Y \cup \bigcup_{a \in A} S_a^{-1}.$$

Then B is countable, and we claim that there exists a countable subgroup $\hat{B}$ of $\mathbb{R}$ which contains B such that $\varphi(\hat{B}) = \hat{B}$. Indeed, inductively define subsets $B_n \subseteq \mathbb{R}$ $(n \in \omega)$ as follows:

$$B_0 = B \quad \text{and} \quad B_{n+1} = \left\langle\!\!\left\langle \bigcup_{m \in \mathbb{Z}} \varphi^m(B_n) \right\rangle\!\!\right\rangle.$$

Then $\hat{B} = \bigcup_{n \in \omega} B_n$ is clearly as required.

We claim that for *every* $x \in \mathbb{R}$ we have $\varphi(x) - x \in \hat{B}$. This is clear for the elements in $\hat{B}$. So take an $x \notin \hat{B}$. Then there exists $a \in A$ such that $x \in S_a^1$. Consequently, $\varphi(x) - x = a \in A \subseteq B \subseteq \hat{B}$.

The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \varphi(x) - x$ is continuous and has countable range. Consequently, g is constant, i.e., f is a translation. □

The following question still seems to be open, cf. [6].

4.7. Question. Is there a rigid homogeneous chain $X \subseteq \mathbb{R}$ such that X is Lebesgue measurable?

5. A subgroup of $\mathbb{R}$ with a measure that "knows" which Borel sets are homeomorphic

Let $X \subseteq \mathbb{R}$ be a BB-set. Our aim is to define a natural Borel measure on X. To this end, let μ denote Lebesgue measure on $\mathbb{R}$, and let μ_* denote the inner measure induced by μ, i.e., for every subset $E \subseteq \mathbb{R}$,

$$\mu_*(E) = \sup\{\mu(B) : B \in \mathscr{B}(\mathbb{R}) \text{ and } B \subseteq E\}.$$

We claim that $\mu_*(\mathbb{R} \setminus X) = 0$. To this end, pick $B \in \mathscr{B}(\mathbb{R})$ such that $B \subseteq \mathbb{R} \setminus X$. Then B is countable by Lemma 3.9, so $\mu(B) = 0$. This proves our claim. It now follows from Halmos [7, p. 75] that the following defines unambiguously a Borel measure $\bar{\mu}$ on X:

$$\text{if } B \in \mathscr{B}(X),\ B' \in \mathscr{B}(\mathbb{R}) \text{ and } B' \cap X = B, \quad \text{then } \bar{\mu}(B) = \mu(B'). \tag{3}$$

From now on, for a BB-set X, $\bar{\mu}$ denotes the Borel measure on X defined in (3).

5.1. Theorem [2]. *There exists a subgroup X of $\mathbb{R}$ with the following property*:

$$\textit{if } S,\ T\in\mathscr{B}(X) \textit{ and if } S\approx T, \quad \textit{then } \bar{\mu}(S)=\bar{\mu}(T).$$

Proof. Let X be the subgroup of $\mathbb{R}$ constructed in the previous section. Then X is the required example. To prove this, let S, $T\in\mathscr{B}(X)$ and let $f: S\to T$ be a homeomorphism. By Theorem 4.4 there is a countable subgroup A of X such that for every $x\in S$ there exists $a\in A$ such that $f(x)=x+a$ or $f(x)=-x+a$. For every $a\in A$ and $\varepsilon\in\{-1,1\}$ put

$$S_a^\varepsilon=\{x\in S: f(x)=\varepsilon\cdot x+a\}.$$

Observe that each S_a^ε is closed in S, and hence is a Borel subset of X. Since f is a homeomorphism, $f(S_a^\varepsilon)$ is a Borel subset of T, so $f(S_a^\varepsilon)$ is also a Borel subset of X. Finally, observe that f restricted to S_a^ε is equal to the identity or the function $x\mapsto -x$, followed by a translation. Both these functions preserve measure, i.e.,

$$\bar{\mu}(S_a^\varepsilon)=\bar{\mu}(f(S_a^\varepsilon)) \quad (a\in A,\ \varepsilon\in\{-1,1\}). \tag{$*$}$$

Claim. *If a, $a'\in A$ are distinct and ε, $\eta\in\{-1,1\}$, then $S_a^\varepsilon\cap S_{a'}^\eta$ and $f(S_a^\varepsilon)\cap f(S_{a'}^\eta)$ are countable.*

Take an arbitrary $x\in S_a^\varepsilon\cap S_{a'}^\eta$. Then

$$\varepsilon\cdot x+a=f(x)=\eta\cdot x+a'.$$

So $\varepsilon\neq\eta$ for otherwise $a=a'$. But this now easily implies that $x\in\mathbb{Q}\cdot A$, which is countable. The second part of the claim is a triviality because f is one-to-one.

Since countable subsets of X have measure 0, the Claim now implies that

$$\bar{\mu}(S)=\sum_{\substack{a\in A\\ \varepsilon\in\{-1,1\}}}\bar{\mu}(S_a^\varepsilon), \quad\text{and}\quad \bar{\mu}(T)=\sum_{\substack{a\in A\\ \varepsilon\in\{-1,1\}}}\bar{\mu}(f(S_a^\varepsilon)).$$

By $(*)$ we therefore obtain $\bar{\mu}(S)=\bar{\mu}(T)$, as required. □

Observe that this is a very curious result. For example, let $A=(0,1)\cap X$ and $B=(0,2)\cap X$. Then clearly $\bar{\mu}(A)=1$ and $\bar{\mu}(B)=2$. One would expect A and B to be homeomorphic, via a homeomorphism of type $x\mapsto 2\cdot x$. Theorem 5.1 disproves this. It is illustrative to find out precisely where in the construction we killed each candidate for a homeomorphism between A and B, such as $x\mapsto 2\cdot x$.

5.2. Corollary. *There exists a subgroup X of $\mathbb{R}$ which is a BB-set such that if $Y=\mathbb{R}\setminus X$, then Y has the following property*:

$$\textit{If } S,\ T\in\mathscr{B}(Y) \textit{ and if } S\approx T, \quad \textit{then } \bar{\mu}(S)=\bar{\mu}(T).$$

Proof. Let X be such as in the proof of Theorem 5.1. By Corollary 3.3 there exist Borel sets $\hat{S}$ and $\hat{T}$ in $\mathbb{R}$ such that $\hat{S} \cap Y = S$ and $\hat{T} \cap Y = T$, while moreover f can be extended to a homeomorphism $\hat{f}: \hat{S} \to \hat{T}$. Observe that $\hat{f}(\hat{S} \setminus S) = \hat{T} \setminus T$. By Theorem 5.1 we therefore obtain the following:

$$\bar{\mu}(S) = \mu(\hat{S}) = \bar{\mu}(\hat{S} \cap X) = \bar{\mu}(\hat{T} \cap X) = \bar{\mu}(T).$$

We are done. $\square$

6. The Homogeneity Lemma and applications

In this section we discuss a criterion for proving homogeneity of certain spaces.

Let X be a zero-dimensional space. We say that two points x and y have *arbitrarily small homeomorphic clopen neighborhoods* if for all neighborhoods U and V of x and y, respectively, there are clopen neighborhoods U' of x and V' of y with $U' \subseteq U$ and $V' \subseteq V$ such that $U' \approx V'$. Observe that we do *not* require the homeomorphism between U' and V' to map x onto y.

6.1. Lemma [16]. *Let X be a zero-dimensional first countable space. If $x, y \in X$, then the following statements are equivalent*:

(1) *there is a homeomorphism $h: X \to X$ with $h(x) = y$;*

(2) *x and y have arbitrarily small homeomorphic clopen neighborhoods.*

Observe that the first countability in this lemma is essential. If $X = \beta\omega \setminus \omega$, then all clopen subsets of X are homeomorphic, so X satisfies condition (2) of Lemma 6.1 for all x and y, but X is not homogeneous. For details, see [18].

From now on we refer to Lemma 6.1 as the "Homogeneity Lemma".

Let G be a dense subgroup of $\mathbb{R}$ and let X be a proper subset of $\mathbb{R}$ such that $X + G = X$. Then clearly $\mathbb{R} \setminus X$ is dense. We now construct a locally compact extension of $\mathbb{R} \setminus X$, similar to the spaces $S(A)$ defined in Section 3. We split each $x \in X$ into two points, x^- and x^+. The points of $\mathbb{R} \setminus X$ with not be split. Order the set

$$T(X) = (\mathbb{R} \setminus X) \cup \{x^-, x^+ : x \in X\}$$

in the natural way, so that x^- always precedes x^+. Endow $T(X)$ with the order topology derived from this order. Then $T(X)$ is easily seen to be a locally compact, zero-dimensional first countable space. In addition, $\mathbb{R} \setminus X$ as subspace of $\mathbb{R}$ is precisely the same space as $\mathbb{R} \setminus X$ as subspace of $T(X)$. Moreover, the sets

$$\{x^+ : x \in X\},$$

and

$$\{x^- : x \in X\},$$

are both homeomorphic to subspaces of $\mathbb{S}$. Our aim is to prove that $T(X)$ is homogeneous.

It will be convenient to introduce some notation. A basic clopen subset of $T(X)$ is an interval of the form $[a^+, b^-]$, where $a, b \in X$ and $a < b$. We call such clopen sets *clopen arcs*. The *length* of a clopen arc $[a^+, b^-]$ is $b - a$.

6.2. Lemma. *Let $[a^+, b^-]$ and $[c^+, d^-]$ be clopen arcs such that $c - a = d - b \in G$. Then $[a^+, b^-]$ and $[c^+, d^-]$ are order-isomorphic.*

Proof. Put $g = c - a$. Define $h : [a^+, b^-] \to [c^+, d^-]$ as follows:

$$\text{if } a^+ < y < b^-,\ y \in \mathbb{R}\backslash X, \quad \text{then } h(y) = y + g;$$

$$\text{if } a^+ \leq x^+ < b^-,\ x \in X, \quad \text{then } h(x^+) = (x+g)^+;$$

$$\text{if } a^+ < x^- \leq b^-,\ x \in X, \quad \text{then } h(x^-) = (x+g)^-.$$

Since $X + G = X$, it is clear that h is a well-defined order-isomorphism. □

We now come to the main result in this section, which is implicit in [2].

6.3. Theorem. *Let G be a dense subgroup of $\mathbb{R}$ and let X be a proper subset of $\mathbb{R}$ such that $X + G = X$. Then $T(X)$ is homogeneous.*

Proof. Take $p, q \in T(X)$. In view of the Homogeneity Lemma 6.1, all we need to prove is that these points have arbitrarily small homeomorphic clopen neighborhoods. To this end, let U and V be arbitrary neighborhoods of p and q, respectively. We have to distinguish several cases:

Case 1: $p, q \in \mathbb{R}\backslash X$. There clearly exist $x, y \in X$ and $g \in G$ such that

$$x < p < y \quad \text{and} \quad x + g < q < y + g,$$

while moreover

$$[x^+, y^-] \subseteq U \quad \text{and} \quad [(x+g)^+, (y+g)^-] \subseteq V.$$

So we are done by Lemma 6.2.

Case 2: *$p \in \mathbb{R}\backslash X$ and $q = a^-$ for certain $a \in X$.* There clearly exist $h, g \in G\backslash\{0\}$ such that

$$[(a-g)^+, a^-] \subseteq V, \qquad a - g - h < x < a - h \quad \text{and}$$

$$[(a-g-h)^+, (a-h)^-] \subseteq U.$$

So we are again done by Lemma 6.2.

The remaining cases are left as exercises to the reader.

Case 3: *$p = a^-$ and $q = b^-$ for certain $a, b \in X$.*

Case 4: *$p = a^-$ and $q = b^+$ for certain $a, b \in X$.*

Case 5: *$p = a^+$ and $q = b^+$ for certain $a, b \in X$.* □

6.4. Lemma. *Let X be zero-dimensional and homogeneous. Then every clopen subspace of X is homogeneous.*

Proof. This is easy. Let $C \subseteq X$ be clopen and pick distinct $x, y \in C$. Let $f: X \to X$ be a homeomorphism such that $f(x) = y$. There exists a clopen neighborhood U of x such that $U \cap f(U) = \emptyset$ and $U \cup f(U) \subseteq C$. Now define $\xi: C \to C$ as follows:

$$\xi(x) = \begin{cases} x & (x \notin U \cup f(U)); \\ f(x) & (x \in U); \\ f^{-1}(x) & (x \in f(U)). \end{cases}$$

It is clear that ξ is a homeomorphism such that $\xi(x) = y$. □

So by Theorem 6.3 and Lemma 6.4 we obtain:

6.5. Corollary. *Let G be a dense subgroup of $\mathbb{R}$ and let X be a proper subset of $\mathbb{R}$ such that $X + G = X$. Then every clopen arc of $T(X)$ is homogeneous.*

6.6. Question (Aarts). Does there exist a homogeneous space X containing a clopen subspace Y such that Y is not homogeneous?

7. A homogeneous compact space with a measure that "knows" which Borel sets are homeomorphic

In Section 5 we constructed a subgroup of $\mathbb{R}$ having a measure that "knows" which Borel sets are homeomorphic. This space is not compact of course. We would like to get a compact example. There exist of course infinite compact spaces which have a nonzero Borel measure invariant under all homeomorphisms: simply take one which is rigid [11, 10]. So to make the result nontrivial, we want it to be homogeneous. The aim of this section is to construct an infinite compact homogeneous space with a Borel measure $\bar{\mu}$ invariant under all homeomorphisms. It turns out that $\bar{\mu}$ really knows which *clopen* subsets are homeomorphic: two clopen subsets are homeomorphic if and only if they have the same measure.

Let X be the rigid homogeneous chain constructed in Section 4. Recall that we constructed X in such a way that X contained a pre-given countable subgroup Z of $\mathbb{R}$, and missed a pre-given countable subset Q of $\mathbb{R}\setminus\{0\}$ which is disjoint from Z. We now specify Z and Q by taking $Z = \mathbb{Z}$ and $Q = \mathbb{Q}\setminus\mathbb{Z}$. Observe that this implies that

$$X \cap (0, 1) \cap \mathbb{Q} = \emptyset. \tag{4}$$

Also recall that X is a subgroup of $\mathbb{R}$, and also that X is a BB-set (Lemma 4.5).

The space $T(X)$ is homogeneous by Theorem 6.3. In addition, since $0, 1 \in X$, the clopen arc $[0^+, 1^-]$ of $T(X)$ is also homogeneous by Lemma 6.4. Put $D = (\mathbb{R}\setminus X) \cap (0, 1)$. Observe that the arc $[0^+, 1^-]$ is equal to $S(D)$, the compactification of D constructed in Section 3. We already observed the following important:

7.1. Theorem. *$S(D)$ is homogeneous.*

Since X is a BB-set, so is $\mathbb{R}\backslash X$, so in Section 5 we defined a natural Borel measure $\bar{\mu}$ on $\mathbb{R}\backslash X$. The restriction of $\bar{\mu}$ to D will also be denoted by $\bar{\mu}$. We will use $\bar{\mu}$ to define a natural Borel measure $\bar{\bar{\mu}}$ on $S(D)$. Indeed, define

$$\bar{\bar{\mu}}(B)=\bar{\mu}(B\cap D) \quad (B\subseteq S(D) \text{ Borel}).$$

7.2. Theorem. *Let S and T be homeomorphic Borel sets in $S(D)$. Then $\bar{\bar{\mu}}(S)=\bar{\bar{\mu}}(T)$.*

Proof. Let $f: S\to T$ be a homeomorphism. Put $B=\{d\in D\cap S: f(d)\notin D\}$. Then by Corollary 3.11, B is countable. In addition, let $C=\{d\in D\cap T: f^{-1}(d)\notin D\}$. Then again by Corollary 3.11, C is countable. Since $f((D\cap S)\backslash B)=(D\cap T)\backslash C$, by Corollary 5.2 we obtain

$$\bar{\mu}(D\cap S)=\bar{\mu}((D\cap S)\backslash B)=\bar{\mu}((D\cap T)\backslash C)=\bar{\mu}(D\cap T).$$

Consequently, $\bar{\bar{\mu}}(S)=\bar{\bar{\mu}}(T)$, as required. □

We now aim at a partial converse to Theorem 7.2. Before we can formulate what we mean, we need to derive the following:

7.3. Proposition. *Let $[a^+, b^-]$ and $[c^+, d^-]$ be clopen arcs in $T(Y)$ such that $b-a=d-c$. Then $[a^+, b^-]\approx[c^+, d^-]$.*

Proof. Since X is dense in $\mathbb{R}$, there is $z\in X$ such that

$$d-(b+z)<\tfrac{1}{2}(d-c). \qquad (*)$$

Observe that $b, c\in X$ implies $c-z, b+z\in X$. So we can consider the clopen arcs

$$[a^+, (c-z)^-] \quad \text{and} \quad [(b+z)^+, d^-]. \qquad (**)$$

These arcs have the same length, while moreover the clopen arcs

$$[(c-z)^+, b^-] \quad \text{and} \quad [c^+, (b+z)^-]$$

are order-isomorphic by Lemma 6.2. By $(*)$, the clopen arcs in $(**)$ have length less than $\frac{1}{2}$ times the length of the arcs we started with. So by repeating the same construction infinitely often, we can easily construct a partition $\langle A_n\rangle_n$ of $(a^+, b^-]$ into clopen arcs, and a partition $\langle B_n\rangle_n$ of $[c^+, d^-)$ into clopen arcs, such that for every n, $A_n\approx B_n$. From this we conclude that $(a^+, b^-]\approx[c^+, d^-)$. Since $[a^+, b^-]$ is the one-point compactification of $(a^+, b^-]$, and similarly since $[c^+, d^-]$ is the one-point compactification of $[c^+, d^-)$, we conclude that $[a^+, b^-]\approx[c^+, d^-]$. (Recall that each homeomorphism between locally compact spaces extends to a homeomorphism between their respective one-point compactifications.) □

7.4. Corollary. *Let* $E \subseteq S(D)$ *be clopen and nonempty. Then* $\bar{\mu}(E) \in X$ *and* $E \approx [0^+, \bar{\mu}(E)^-]$.

Proof. It is clear that E is the union of a finite family of pairwise disjoint clopen arcs, say

$$[a_0^+, a_1^-], [a_2^+, a_3^-], \ldots, [a_{n-1}^+, a_n^-].$$

Without loss of generality we may assume that

$$a_0^+ < a_1^- < a_2^+ < a_3^- < \cdots < a_{n-1}^+ < a_n^-.$$

For every $1 \leqslant i \leqslant n$ put

$$b_i = a_i - a_{i-1}$$

and let

$$b = \sum_{i=1}^{n} b_i.$$

Then $b \leqslant 1$ and $\bar{\mu}(E) = b$. Since X is a subgroup of $\mathbb{R}$, for every i, $b_i \in X$ and also $b \in X$, i.e., the clopen arcs

$$[0^+, b_1^-], [b_1^+, (b_1 + b_2)^-], \ldots, \left[\left(\sum_{i=1}^{n-1} b_i\right)^-, b^+\right]$$

exist. By Proposition 7.3, it now follows that E is homeomorphic to $[0^+, b^-]$. □

This result is the key in the proof of the following interesting:

7.5. Theorem. *If* E *and* F *are clopen subspaces of* $S(D)$, *then*

$$E \approx F \quad \textit{iff} \quad \bar{\mu}(E) = \bar{\mu}(F).$$

Proof. By Theorem 7.2, it suffices to prove sufficiency. Without loss of generality assume that $E \neq \emptyset \neq F$. So $\bar{\mu}(E) = \bar{\mu}(F) > 0$. By Corollary 7.4, E is homeomorphic to $[0^+, \bar{\mu}(E)^-]$ and F is homeomorphic to $[0^+, \bar{\mu}(F)^-]$. So we are done. □

We summarize the results obtained so far in this section:

7.6. Theorem [2]. *There exists a compact zero-dimensional homogeneous space* $S(D)$ *having a nonzero Borel measure* $\bar{\mu}$ *with the following properties*:

(1) *if* S *and* T *are homeomorphic Borel sets of* $S(D)$, *then* $\bar{\mu}(S) = \bar{\mu}(T)$;

(2) *if* $E, F \subseteq S(D)$ *are clopen, then* $E \approx F$ *iff* $\bar{\mu}(E) = \bar{\mu}(F)$.

We finish this section by establishing a few other curious properties of $S(D)$.

7.7. Theorem. *If $E \subseteq S(D)$ is clopen and $\emptyset \neq E \neq S(D)$, then $\bar{\mu}(E)$ is irrational.*

Proof. By Corollary 7.4, $\bar{\mu}(E) \in X$. Also, $0 \neq \bar{\mu}(E) \neq 1$. So we are done by (4). □

So $S(D)$ has the amusing property that since it has no clopen subspace of measure $\frac{1}{2}$, no clopen subspace of it is homeomorphic to its complement. In [2], van Douwen claimed to be able to construct a space cG with this property, but did not give details; presumably because his space cG is not perfectly normal. Our construction of $S(D)$ differs essentially from van Douwen's spaces bH and cG: in our construction it is a triviality to build in that all proper nonempty clopen sets have irrational measure, while moreover $S(D)$ is clearly perfectly normal (it is a separable ordered space). Another reason why we find $S(D)$ more interesting than bH and cG is the following. It is still unknown whether there exists a compact homogeneous zero-dimensional space with the fixed-point property for homeomorphisms. Although $S(D)$ does not seem to answer this problem, we can prove the following:

7.8. Theorem. *Let $h: S(D) \to S(D)$ be a homeomorphism such that $h \circ h = 1_{S(D)}$. Then h has a fixed point.*

Proof. To the contrary, assume that h has no fixed point. Let $U = \{x \in S(D): x < h(x)\}$, and $V = \{x \in S(D): h(x) < x\}$, respectively. Then U and V are complementary clopen sets, and $h(U) = V$ because $h \circ h = 1_{S(D)}$. Consequently, $\bar{\mu}(U) = \bar{\mu}(V) = \frac{1}{2}$ (Theorem 7.2). This contradicts Theorem 7.7. (Observe that such a simple argument does not work if the space under consideration is not ordered.) □

7.9. Question (van Douwen and van Mill). Is there a zero-dimensional homogeneous compact space with the fixed-point property for homeomorphisms?

Below we will present an example of a locally compact zero-dimensional homogeneous space with the fixed-point property for homeomorphisms.

A space X is said to be *halvable* if it contains a subset homeomorphic to its complement. A space with precisely one isolated point, or with precisely one nonisolated point, is of course not halvable. There also exist spaces without isolated points that are not halvable. For example, consider the one-point compactification of the topological sum of the spaces $I, I^2, \ldots, I^n, \ldots$. (Use a dimension argument.) The following question is still open:

7.10. Question [4]. Is there a homogeneous metrizable (preferably: separable) space that cannot be halved?

The reason that this question seems only of interest for metrizable spaces, is because of the following:

7.11. Example. There exists a locally compact, locally metrizable homogeneous zero-dimensional space X that cannot be halved. Moreover, X has the fixed-point property for homeomorphisms.

Consider the ordinal space ω_1, and replace each isolated point by a copy of the standard Cantor set in $[0, 1]$. Order the resulting set X in the natural way, and give it the order topology. Then X is locally homeomorphic to the Cantor set, and is therefore locally metrizable, zero-dimensional and homogeneous (apply Lemma 6.1, or use a direct argument). Observe that each initial segment of X is separable and metrizable. We will proceed to prove that X cannot be halved. Suppose that $A \subseteq X$ and its homeomorphic image $h(A) = B = X \backslash A$ are two "halves". Either $a < h(a)$ for all a in some unbounded subset of A, or the analogue holds for B and h^{-1}, so we may assume the former. Pick a sequence $a_1, a_2, \ldots$ in A such that

$$a_1 < h(a_1) < a_2 < h(a_2) < a_3 < \cdots,$$

let $p = \sup_i a_i$ and observe that $p = \sup_i h(a_i) = h(p)$, a contradiction.

That X also has the fixed-point property for homeomorphisms follows by a similar argumentation.

7.12. Question (van Douwen and van Mill). Is there a zero-dimensional separable metrizable space with the fixed-point property for homeomorphisms?

Haar measure on a compact group has the property of being invariant under left and right translations, as well as under all topological isomorphisms, i.e., is invariant under all algebraically significant homeomorphisms. In view of Theorem 7.6 it is therefore natural to ask:

7.13. Question (van Douwen). Does there exist an infinite compact connected topological group G such that every homeomorphism of G preserves Haar measure?

8. A subgroup of $\mathbb{R}^n$ with few homeomorphisms

The author constructed in [17] an example of a topological group having no homeomorphisms other than translations. The aim of this section is to present a proof of the following related result:

8.1. Theorem [12]. *For every $k > 1$ there exists a subgroup $G \subseteq \mathbb{R}^k$ such that, for each homeomorphism $f: G \to G$, there exists $a \in G$ such that*

$$\text{either } f(x) = x + a \quad \text{for every } x \in G,$$

$$\text{or } f(x) = -x + a \quad \text{for every } x \in G.$$

Proof. Let G be the subgroup of $\mathbb{R}^k$ constructed in Section 4 and let $f: G \to G$ be a homeomorphism. By Theorem 3.1 there exists a G_δ-subset S of $\mathbb{R}^k$ such that f can be extended to a continuous function $\hat{f}: S \to \mathbb{R}^k$. In addition, by Theorem 4.4, there exists a countable subgroup A of G such that for every $x \in G$ there exists $a \in A$ such that $f(x) = x + a$ or $f(x) = -x + a$. For every $a \in A$ and $\varepsilon \in \{-1, 1\}$ put

$$S_a^\varepsilon = \{x \in S: \hat{f}(x) = \varepsilon \cdot x + a\}.$$

It is clear that every S_a^ε is closed in S.

Claim 1. *If $\langle a, \varepsilon\rangle \neq \langle a', \varepsilon'\rangle$, then $S_a^\varepsilon \cap S_{a'}^{\varepsilon'}$ is countable.*

Take an arbitrary $x \in S_a^\varepsilon \cap S_{a'}^{\varepsilon'}$. Then

$$\varepsilon \cdot x + a = f(x) = \varepsilon' \cdot x + a'.$$

So if $\varepsilon = \varepsilon'$, then $a = a'$. We may therefore assume that $\varepsilon \neq \varepsilon'$. Consequently, $x \in \mathbb{Q} \cdot A$, which is countable.

Put $V = \bigcup \{S_a^\varepsilon \cap S_{a'}^{\varepsilon'}: \langle a, \varepsilon\rangle \neq \langle a', \varepsilon'\rangle\}$ and $\hat{S} = \bigcup \{S_a^\varepsilon: a \in A, \varepsilon \in \{-1, 1\}\}$, respectively.

Claim 2. *$T = \mathbb{R}^k \setminus \hat{S}$ is countable.*

First observe that T is a Borel subset of $\mathbb{R}^k$, being the union of an F_σ- and a G_δ-subset of $\mathbb{R}^k$. So T is countable by Proposition 4.5 and Lemma 3.9.

Claim 3. *At most one of the collection $\{S_a^\varepsilon \setminus V: a \in A, \varepsilon \in \{-1, 1\}\}$ is nonempty.*

Suppose that there exist distinct $\langle a, \varepsilon\rangle$ and $\langle a', \varepsilon'\rangle$ such that there exist $x \in S_a^\varepsilon \setminus V$ and $y \in S_{a'}^{\varepsilon'} \setminus V$. Since $x, y \notin V$ and since by Claim 2 the set T is countable, there is an arc J in $\mathbb{R}^k$ connecting x and y and contained in $\hat{S} \setminus V$. (Here we use the fact that $k > 1$.) Consequently, J is partitioned by the collection

$$\{J \cap S_a^\varepsilon: a \in A, \varepsilon \in \{-1, 1\}\}.$$

Since this collection contains at least two distinct members, and consists of closed subsets of J, we contradict Theorem 3.4.

So there exists a unique $\langle a, \varepsilon\rangle$ for which $S_a^\varepsilon \setminus V$ is nonempty. Observe that S_a^ε is a closed subset of S having countable complement. By another application of Proposition 4.5, every nonempty open subset of G is uncountable. So we conclude that $S_a^\varepsilon = S$, which is obviously as required. □

Notes

The construction in Section 4 is new. It has the advantage over the constructions in [20] and [6] that it simultaneously gives a rigid homogeneous chain as well as the examples in [2] and [12]. Also, for technical reasons it turned out to be simpler to split the points of $X \cap [0, 1]$ instead of splitting the points of its complement.

This is rather unusual, but it simplified the argumentation. The idea of splitting the points of a subset of $\mathbb{R}$ is well known of course.

I am indebted to the referee for simplifying the proof of Example 7.11.

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