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CHAPTER 3

# Topological classification of infinite-dimensional spaces with absorbers

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## 1. Introduction

There are two infinite-dimensional generalizations of the euclidean spaces $\mathbb{R}^n$ ($n \in \mathbb{N}$) that are important in analysis: $s = \mathbb{R}^\infty$, the countable infinite product of lines, and $\ell^2$, separable Hilbert space. For an analyst, these two spaces have very little in common: $s$ is locally convex, but not normable, and $\ell^2$ is even a Hilbert space. As a consequence, there cannot be a homeomorphism $f: s \to \ell^2$ that is *linear*. For a topologist $s$ and $\ell^2$ are very similar. For example because they are both topologically complete, infinite-dimensional, connected and locally connected, nowhere locally compact, separable metrizable spaces, to mention a few topological properties that they share. It was asked in [1928] by FRÉCHET and also by BANACH in [1932] whether $s$ and $\ell^2$ are topologically homeomorphic. Observe that homeomorphisms need not be linear in topology.

The question of Fréchet was answered affirmatively by ANDERSON in [1966]. His proof was based on the result earlier obtained by BESSAGA and PEŁCZYŃSKI in [1959] that $\ell^2 \times s \approx \ell^2$, and on the theory of deleting $\sigma$-compact sets from products of the form $X \times s$ in ANDERSON [1967]. A more elementary proof was later found by ANDERSON and BING [1968].

The simplest proof of Anderson's Theorem was found by BESSAGA and PEŁCZYŃSKI in [1969]. Their proof uses so called *absorbing sets* in the Hilbert cube $Q$ as well as KELLER's Theorem [1931] that all infinite-dimensional compact convex subsets of $\ell^2$ are homeomorphic. See also VAN MILL [1989, §6.6].

To explain what an absorbing set is, consider the subspace of rational numbers $\mathbb{Q}$ in $\mathbb{R}$. If one adds $\pi$ to $\mathbb{Q}$, i.e., if one considers the subspace $E = \mathbb{Q} \cup \{\pi\}$ of $\mathbb{R}$, then a straightforward back-and-forth argument yields the existence of an order preserving bijection $f: \mathbb{Q} \to E$, which can, of course, be extended to an order preserving bijection $\bar{f}: \mathbb{R} \to \mathbb{R}$. So $\mathbb{Q}$ and $E$ are homeomorphic, via a homeomorphism that extends to $\mathbb{R}$. In other words, within $\mathbb{R}$ we can add a point of $\mathbb{R}$ to $\mathbb{Q}$ without changing the topology of $\mathbb{Q}$ or, equivalently, $\mathbb{Q}$ can absorb a point of $\mathbb{R}$. It can be shown that a set $C \subseteq \mathbb{R}$ can be absorbed by $\mathbb{Q}$ if and only if $C$ is countable. So sets that can be absorbed are special: uncountable sets can (evidently) not be absorbed. Also, it turns out that "absorbers" for $\mathbb{R}$ are unique: there is only one absorber. If $A$ is a *countable* subset of $\mathbb{R}$ that can absorb every countable subset of $\mathbb{R}$, then there is a homeomorphism $f: \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) = A$. The assumption on countability is essential in this result. It can be shown that if $K \subseteq \mathbb{R}$ is dense, and a countable union of Cantor sets, then $K$ has also the property that it can absorb every countable subset of $\mathbb{R}$; in fact it can absorb every zero-dimensional $\sigma$-compact subset of $\mathbb{R}$. Absorbing sets tend to absorb *small* sets: countable sets in the case of $\mathbb{Q}$ and zero-dimensional sets in the case of $K$.

We now turn our attention to the Hilbert cube $Q = \prod_{i=1}^\infty [-1, 1]_i$. It turns out that $Q$ has also a subset with an absorption property. The set we mean is

$$B = \{x \in Q : |x_i| = 1 \text{ for some } i \in \mathbb{N}\},$$

the so-called *pseudoboundary* of $Q$. It can be shown that $B$ can absorb every countable subset of $Q$. But it can absorb much more. In fact, it can absorb precisely the $\sigma$Z-sets in $Q$, i.e., sets that are a countable union of Z-sets. For details see BESSAGA and PEŁCZYŃSKI [1975] and VAN MILL [1989, Chapter 6]. Recall that a Z-set in $Q$ is a closed set $A \subseteq Q$ such that the identity function $1_Q: Q \to Q$ can be approximated arbitrarily closely by maps $Q \to Q \setminus A$. There also holds a *uniqueness* theorem here. If a $\sigma$Z-set $A$ can absorb every Z-set then there is a homeomorphism $f: Q \to Q$ such that $f(B) = A$. So the property of absorbing Z-sets in a sense characterizes the pseudoboundary $B$ of $Q$.

The proof of Bessaga and Pełczyński of Anderson's Theorem now roughly goes as follows. First consider the subspace

$$K = \left\{ x \in Q : \sum_{i=1}^{\infty} x_i^2 \leq 1 \right\}$$

of $Q$. This space is the so-called *elliptical Hilbert cube*. That it is a Hilbert cube, i.e., a space homeomorphic to $Q$, follows from Keller's Theorem cited above. This can also be shown directly by elementary means, VAN MILL [1989, §6.6]. Then it is shown that the subspace

$$\hat{B} = \left\{ x \in K : \sum_{i=1}^{\infty} x_i^2 < 1 \right\}$$

can absorb arbitrary $\sigma$Z-sets. By the above remark about uniqueness of absorbers, it therefore follows that there is a homeomorphism of pairs $(Q, B) \approx (K, \hat{B})$. This is quite interesting. The points of $B$ are intuitively on the "boundary" of $Q$, while the points of $\hat{B}$ are intuitively in the "interior" of $K$. It follows that the subspace

$$\hat{S} = \left\{ x \in K : \sum_{i=1}^{\infty} x_i^2 = 1 \right\}$$

is homeomorphic to $s \approx (-1,1)^\infty$, the complement of $B$ in $Q$. On the unit sphere $S = \{x \in \ell^2 : \|x\| = 1\}$ of $\ell^2$, the topology of pointwise convergence coincides with the subspace topology that $S$ inherits from $\ell^2$. As a consequence, $S$ and $\hat{S}$ are homeomorphic. It is an easy exercise to prove that $\ell^2$ and $S \setminus \{(-1,0,0,\ldots)\}$ are homeomorphic (Hint: Find an explicit homeomorphism between $S^1 \setminus (-1,0)$ and $\mathbb{R}$). So we arrive at the following situation:

$$s \approx \hat{S} \approx S,$$
$$\ell^2 \approx S \setminus \{\mathrm{pt}\}.$$

It remains to prove that $s \approx s \setminus \{\mathrm{pt}\}$. But this clearly follows from the absorption property of $B$. Since $B$ can absorb every point of $Q$, for every $x \in (-1,1)^\infty$ there is a homeomorphism $f: Q \to Q$ with $f(B) = B \cup \{x\}$. But this means that $f((-1,1)^\infty) = (-1,1)^\infty \setminus \{x\}$, i.e., $s \approx (-1,1)^\infty$ can loose an arbitrary point.

The aim of this paper is to give an overview of the work that has been done in infinite-dimensional topology on absorbing sets and absorbing systems during the last decade. As is clear from the above, the interest in absorbing sets stems from the desire to prove that certain spaces are homeomorphic. It turns out that the theory of absorbing sets is extremely useful in several situations ranging from hyperspaces and function spaces, to finite-dimensional absorbers in euclidean spaces.

## 2. Absorbers and generalized absorbers

We start with some definitions. A closed subset $F$ of a space $X$ is called a *Z-set* if for every open cover $\mathcal{U}$ of $X$ there is a map ( = continuous mapping) $f: X \to X \setminus F$ that is $\mathcal{U}$-close to the identity $1_X$. A closed subset $F$ of a space $X$ is called a *strong Z-set* if for every open cover $\mathcal{U}$ of $X$ there is a map $f: X \to X$ that is $\mathcal{U}$-close to $1_X$ and is such that $\mathrm{Cl}_X(f(X)) \cap F = \varnothing$. A countable union of (strong) Z-sets is called a *(strong)* $\sigma$*Z-set*. A space $X$ that can be written $X = \bigcup_{i=1}^{\infty} X_i$, where each $X_i$ is a (strong) Z-set in $X$, is called a *(strong)* $\sigma$*Z-space*. An embedding $f: X \to Y$ is called a *(strong) Z-embedding* if $f(X)$ is a (strong) Z-set in $Y$.

We now review some topological properties of Hilbert space and the Hilbert cube (see Bessaga and Pełczyński [1975] and van Mill [1989] for additional information). Let $E$ be a topological Hilbert space (respectively, a Hilbert cube). Every Z-set in $E$ is a strong Z-set (in general, Z-sets need not be strong Z-sets, even within the class of topologically complete AR's, cf. Bestvina et al. [1986]). A closed $\sigma$Z-set in $E$ is a Z-set. Every continuous map $f$ from a complete (compact) space $X$ into $E$ that restricts to a Z-embedding on some closed set $K$ can be approximated by a Z-embedding $g: X \to E$ with $g \mid K = f \mid K$. Every homeomorphism between Z-sets in $E$ can be extended (with control) to an autohomeomorphism of $E$.

We now turn to absorbers in $E$. The first attempts to axiomatise this concept can be found in Anderson [19??] and Bessaga and Pełczyński [1970]. Their notions were generalized by Toruńczyk [19??] and West [1970]. Because it is relatively easy to use, West's concept of an absorptive set (and its derivatives) is currently the classification method of choice. The definition that follows is inspired by West and can in essence be found in Bestvina and Mogilski [1986].

Let $\mathcal{M}$ be a class of topological spaces that is topological (spaces homeomorphic to elements of $\mathcal{M}$ are in $\mathcal{M}$) and closed hereditary (closed subspaces of elements of $\mathcal{M}$ are also in $\mathcal{M}$). A subset $X$ of $E$ is called *strongly $\mathcal{M}$-universal in $E$* if for every map $f: E \to E$ that restricts to a Z-embedding on a closed set $K \subseteq E$ and for every subset $A$ of $E$ that is in the class $\mathcal{M}$ there exists a Z-embedding $g: E \to E$ that can be chosen arbitrarily close to $f$ with the properties $g \restriction K = f \restriction K$ and $g^{-1}(X) \setminus K = A \setminus K$. The set $X$ is called an *$\mathcal{M}$-absorber* in $E$ if

(1) $X$ is contained in a $\sigma$Z-set of $E$,
(2) $X \in \mathcal{M}_\sigma = \{\bigcup_{i=1}^{\infty} A_i : A_i \in \mathcal{M}\}$,
(3) $X$ is strongly $\mathcal{M}$-universal.

The pseudoboundary $B$ of the Hilbert cube is the standard example of an absorber for the class of compacta. The most important property of absorbers is the uniqueness theorem (Anderson [19??], Bessaga and Pełczyński [1970], Toruńczyk [19??], West [1970] and Bestvina and Mogilski [1986]).

**2.1.** Theorem. *If $X$ and $Y$ are two $\mathcal{M}$-absorbers in $E$ then there exists a homeomorphism $h: E \to E$, arbitrarily close to the identity, that maps $X$ onto $Y$.*

It is also possible to study absorbers without an ambient Hilbert space or Hilbert cube as was shown by Bestvina and Mogilski [1986]. Let $\mathcal{M}$ be a topological class that is closed hereditary. In addition, assume that $\mathcal{M}$ is additive: $A \in \mathcal{M}$

whenever $A$ can be written as a union of two closed subsets that are in $\mathcal{M}$. An ANR $X$ is called *strongly $\mathcal{M}$-universal* if for every $A \in \mathcal{M}$ and every map $f: A \to X$ that restricts to a Z-embedding on a closed set $K \subseteq A$ there exists a Z-embedding $g: A \to X$ that can be chosen arbitrarily close to $f$ with the properties $g \mid K = f \mid K$. The ANR $X$ is called a *generalized $\mathcal{M}$-absorber* if

(1) $X$ is a strong $\sigma$Z-space,
(2) $X \in \mathcal{M}_\sigma$,
(3) $X$ is strongly $\mathcal{M}$-universal.

Bestvina and Mogilski prove the following characterisation theorem for generalized absorbers in [1986, Theorem 5.3]):

**2.2.** Theorem. *If $X$ is an $\mathcal{M}$-absorber in $E$ and an AR $Y$ is a generalized $\mathcal{M}$-absorber then $X$ and $Y$ are homeomorphic.*

Bestvina and Mogilski also show that there exists a "standard absorber" in Hilbert space for every Borel class.

## 3. Absorbing systems

Where as the generalized absorber method that is discussed in the preceding section applies to the widest range of classification problems, we find that in practice virtually every space that merits consideration has a canonical embedding in a topological Hilbert space or Hilbert cube. In order to deal with these spaces in a more efficient way absorbing systems were introduced in the papers Dijkstra et al. [1992] and Dijkstra and Mogilski [1991]. The material in this section has been taken from these publications. Throughout this section $E$ stands for either a topological Hilbert space or Hilbert cube.

Let $\Gamma$ be a fixed index set. A collection $\mathfrak{X} = (X_\gamma)_{\gamma\in\Gamma}$ of subsets of the space $E$ (formally the pair $(E, \mathfrak{X})$) is called a *Z-system* if $\bigcup\{X_\gamma : \gamma \in \Gamma\}$ is contained in a $\sigma$Z-set of $E$. Let $\Delta$ be a subset of $\Gamma$. We say that a Z-system $(E, \mathcal{X})$ is *$\Delta$-embeddable in* (*$\Delta$-homeomorphic to*) a Z-system $(E', \mathcal{Y})$ if there exists a closed embedding (homeomorphism) $f: E \to E'$ such that $f^{-1}(Y_\gamma) = X_\gamma$ for each $\gamma \in \Delta$. The map $f$ is called a *$\Delta$-embedding ($\Delta$-homeomorphism).* If $\Delta = \Gamma$ then we simply say that $\mathcal{X}$ is embeddable in (homeomorphic to) $\mathcal{Y}$.

A Z-system $\mathcal{X}$ is called *reflexively universal* if for every map $f: E \to E$ that restricts to a Z-embedding on some *closed* set $K \subseteq E$, there exists a Z-embedding $g: E \to E$ that can be chosen arbitrarily close to $f$ with the properties: $g \mid K = f \mid K$ and $g^{-1}(X_\gamma) \setminus K = X_\gamma \setminus K$ for every $\gamma \in \Gamma$.

These notions come together in the following:

**3.1.** Theorem. *Let $\mathcal{X}$ and $\mathcal{Y}$ be reflexively universal Z-systems in $E$ respectively $E'$. If $\mathcal{X}$ is $\Delta$-embeddable in $\mathcal{Y}$ and $\mathcal{Y}$ is $\Delta$-embeddable in $\mathcal{X}$ then $\mathcal{X}$ is $\Delta$-homeomorphic to $\mathcal{Y}$.*

$\Box$ This is a standard back and forth argument. Obviously, we may assume that $E = E'$. Let $\bigcup_\gamma X_\gamma \subseteq \bigcup_i A_i$ and let $\bigcup_\gamma Y_\gamma \subseteq \bigcup_i B_i$, where $\varnothing = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ and $\varnothing = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ are sequences of Z-sets in $E$. By induction we shall construct sequences of homeomorphisms $f_i\colon E \to E$ and $g_i = f_i \circ \cdots \circ f_0$ with the properties (for each $\gamma \in \Delta$):

$$A_i \cap X_\gamma = A_i \cap g_i^{-1}(Y_\gamma),$$
$$B_i \cap g_i(X_\gamma) = B_i \cap Y_\gamma,$$
$$f_i \mid (g_{i-1}(A_{i-1}) \cup B_{i-1}) = 1,$$

where 1 denotes the identity map. Put $f_0 = 1$.

Assume that $f_i$ has been constructed. Put $K = g_i(A_i) \cup B_i$ and observe that $g_i(X_\gamma) \cap K = Y_\gamma \cap K$. Let $p\colon E \to E$ be a $\Delta$-embedding of the system $\mathcal{X}$ into $\mathcal{Y}$. Then the inverse of $p \circ g_i^{-1}$ is defined on a closed subset of $E$ and can therefore be extended to a map $r\colon E \to E$. Since $\mathcal{Y}$ is reflexively universal we can approximate $r$ by a Z-embedding $\tilde{r}\colon E \to E$ with the properties $\tilde{r}^{-1}(Y_\gamma) = Y_\gamma$ for each $\gamma \in \Delta$ and $\tilde{r}$ coincides with $r$ on $p \circ g_i^{-1}(K)$. Let $\alpha$ be the Z-embedding $\tilde{r} \circ p \circ g_i^{-1}$ and note that $\alpha$ fixes $K$ and that it has the property $\alpha^{-1}(Y_\gamma) = g_i(X_\gamma)$ for each $\gamma \in \Delta$. Observe that $\alpha \mid g_i(A_{i+1}) \cup B_i$ is a homeomorphism between Z-sets in $E$ and hence it can be extended to a homeomorphism $\tilde{\alpha}$ of $E$. This homeomorphism satisfies

$$\tilde{\alpha}^{-1}(Y_\gamma) \cap g_i(A_{i+1}) = g_i(X_\gamma \cap A_{i+1}).$$

By a similar argument we can find a homeomorphism $\tilde{\beta}$ of $E$ that fixes $K' = \tilde{\alpha} \circ g_i(A_{i+1}) \cup B_i$ and that has the property

$$\tilde{\beta}^{-1}(\tilde{\alpha} \circ g_i(X_\gamma)) \cap B_{i+1} = Y_\gamma \cap B_{i+1}.$$

If we put $f_{i+1} = \tilde{\beta}^{-1} \circ \tilde{\alpha}$ then one can easily verify the induction hypothesis for $i+1$. Since $\tilde{\alpha}$ and $\tilde{\beta}$ and hence $f_{i+1}$ can be chosen arbitrarily close to the identity we may assume that $h = \lim_{i\to\infty} g_i$ is a homeomorphism of $E$. The function $h$ maps $X_\gamma$ onto $Y_\gamma$ for each $\gamma \in \Delta$. □

The reflexive universality of a system can often be obtained without much effort. We mention the most common method for recognizing reflexive universality.

A subset $A$ is *locally homotopy negligible in* $X$ if for every map $f\colon M \to X$ from an absolute neighbourhood retract $M$ and for every open cover $\mathcal{U}$ of $X$ there exists a homotopy $h\colon M \times [0,1] \to X$ such that $\{h(\{x\} \times [0,1])\}_{x\in M}$ refines $\mathcal{U}$, $h(x,0) = f(x)$ and $h(M \times (0,1]) \subseteq X \setminus A$. For a space $X$ and $* \in X$ we define the weak cartesian product

$$W(X,*) = \{x \in X^{\mathbb{N}} : x_i = * \text{ for all but finitely many } i\}.$$

**3.2.** Lemma. *Let $\mathcal{X} = (X_\gamma)_{\gamma\in\Gamma}$ be a system in $E$ such that $E \setminus \bigcap_{\gamma\in\Gamma} X_\gamma$ is locally homotopy negligible in $E$ and let $* \in \bigcap_{\gamma\in\Gamma} X_\gamma$. Assume that there exists a homeomorphism $\Phi\colon E \to E^{\mathbb{N}}$ satisfying*

$$W(X_\gamma,*) \subseteq \Phi(X_\gamma) \subseteq X_\gamma^{\mathbb{N}}$$

*for all $\gamma \in \Gamma$. Then $\mathcal{X}$ is reflexively universal.*

All of this is closely related to absorbing systems which we introduce presently. Let $\Gamma$ be an ordered set and let $\mathcal{M}_\gamma$ be a collection of spaces for each $\gamma \in \Gamma$. Each $\mathcal{M}_\gamma$ is assumed to be *topological* and *closed hereditary*. Let $\mathcal{M}$ stand for the whole system $(\mathcal{M}_\gamma)_{\gamma\in\Gamma}$. Let $\mathcal{X} = (X_\gamma)_{\gamma\in\Gamma}$ be an order preserving indexed collection of subsets of a topological Hilbert cube (Hilbert space) $E$, i.e., $X_\gamma \subseteq X_{\gamma'}$ if and only if $\gamma \leq \gamma'$.

The system $\mathcal{X}$ is called $\mathcal{M}$-*universal* if for every order preserving system $(A_\gamma)_\gamma$ in $E$ such that $A_\gamma \in \mathcal{M}_\gamma$ for every $\gamma \in \Gamma$, there is a closed embedding $f\colon E \to E$ with $f^{-1}(X_\gamma) = A_\gamma$. The system $\mathcal{X}$ is called *strongly* $\mathcal{M}$-*universal* if for every order preserving system $(A_\gamma)_\gamma$ in $E$ such that $A_\gamma \in \mathcal{M}_\gamma$ for every $\gamma \in \Gamma$, and for every map $f\colon E \to E$ that restricts to a Z-embedding on some compact set $K$, there exists a Z-embedding $g\colon E \to E$ that can be chosen arbitrarily close to $f$ with the properties: $g \mid K = f \mid K$ and $g^{-1}(X_\gamma) \setminus K = A_\gamma \setminus K$ for every $\gamma$. Observe that $\mathcal{X}$ is strongly $\mathcal{M}$-universal whenever $\mathcal{X}$ is $\mathcal{M}$-universal and reflexively universal. If $X_\gamma \in \mathcal{M}_\gamma$ then the converse is also true.

The system $\mathcal{X}$ is called $\mathcal{M}$-*absorbing* if

(1) $X_\gamma \in \mathcal{M}_\gamma$ for every $\gamma \in \Gamma$,
(2) $\{X_\gamma : \gamma \in \Gamma\}$ is a Z-system of $E$, and
(3) $X$ is strongly $\mathcal{M}$-universal.

This notion appears to be a successful synthesis of the $Q$-matrices technique of VAN MILL [1987] and the generalized absorbers of BESTVINA and MOGILSKI [1986]. The power of the method we introduce here comes mainly from the relative ease of application.

The following uniqueness result follows immediately from 3.1. It contains 2.1 as a special case.

**3.3.** COROLLARY. *If $\mathcal{X}$ and $\mathcal{Y}$ are both $\mathcal{M}$-absorbing systems in $E$ respectively $E'$ then $(E, \mathcal{X})$ and $(E', \mathcal{Y})$ are homeomorphic, i.e., there is a homeomorphism $h\colon E \to E'$ such that $h(X_\gamma) = Y_\gamma$ for all $\gamma \in \Gamma$. If $E = E'$ then the map $h$ can be found arbitrarily close to the identity.*

We mention a special case that is particularly useful. A Z-system $(X_i)_{i\in\mathbb{N}}$ is called a $\delta$-sequence if $X_i \supseteq X_{i+1}$ for every $i$. The following situation is very common. One needs to show that a certain set is an absorber for the class of absolute $F_{\sigma\delta}$-sets. It would suffice to show that the set is the intersection of a $\delta$-sequence that is $\mathcal{F}_\sigma$-absorbing. Absorbers for $\sigma$-compacta are copies of the pseudointerior $B$ and are well understood and easily recognized.

## 4. Infinite-dimensional applications

As we said in the introduction, interest in absorbers comes from the desire to prove that certain spaces are homeomorphic. The aim of this section is to present evidence that absorbers are a very useful tool for obtaining that goal in the infinite-dimensional setting. In the next section we will show that the interplay between finite-dimensional and infinite-dimensional absorbers sometimes works very well in the realm of finite-dimensional spaces. That this is so, is not very surprising.

Consider for example the elegant proof via infinite-dimensional topology that every compact ANR $X$ (which may be finite-dimensional) has the homotopy type of a compact polyhedron: all one needs to remark is that $X \times Q$ is a compact $Q$-manifold, and that every compact $Q$-manifold $M$ is homeomorphic to $P \times Q$ for some compact polyhedron $P$. For details, see CHAPMAN [1975].

The first topic that we discuss is that of function spaces with the topology of point-wise convergence. Let $X$ be a Tychonov space and let $C_p(X)$ stand for the space of all continuous real-valued functions on $X$ endowed with the topology of pointwise convergence. In other words, we regard $C_p(X)$ to be a subspace of the product $\mathbb{R}^X$. In the last decade, these function spaces have been studied intensively, primarily in the Soviet Union. For more information, see ARHANGEL′SKIĬ [1987].

It is known that $C_p(X)$ is always a dense linear subspace of $\mathbb{R}^X$, so it follows that $C_p(X)$ is metrizable iff $X$ is countable iff $C_p(X)$ can be thought of as a linear subspace of $s$. It therefore comes as no surprise that in infinite-dimensional topology one is mostly interested in function spaces of *countable* spaces.

It is known by DIJKSTRA ET AL. [1985] that for non-discrete $X$, $C_p(X)$ cannot be an $F_\sigma$ and a $G_{\delta\sigma}$-subset of $\mathbb{R}^X$. In contrast, it is easily seen that $C_p(X)$ can be an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$. For example, let $X$ be $\omega + 1$, a convergent sequence. Then $C_p(X)$ can be described as follows:

$$\begin{aligned} C_p(X) &= \left\{ f \in \mathbb{R}^{\omega+1} : (\forall m)\,(\exists N)\,(\forall n \geq N)\,(|f(n) - f(\omega)| \leq \frac{1}{m} \right\} \\ &= \bigcap_{m\in\mathbb{N}} \bigcup_{N\in\omega} \bigcap_{n\geq N} \left\{ f \in \mathbb{R}^{\omega+1} : |f(n) - f(\omega)| \leq \frac{1}{m} \right\}, \end{aligned}$$

which is clearly an $F_{\sigma\delta}$-subset of $\mathbb{R}^{\omega+1}$.

It is easy to see that the above observation can be generalized for arbitrary countable metrizable spaces: for such spaces $C_p(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X \approx s$. Notice that since $s$ is topologically complete, it follows that if $C_p(X)$ is a Borel subset of $\mathbb{R}^X$ then it is absolutely Borel. It can be shown that for countable $X$, $C_p(X)$ can in fact be of arbitrarily large Borel complexity. See LUTZER ET AL. [1985] and CALBRIX [1985, 1988] for more information. The following interesting result was recently obtained by CAUTY ET AL. [19??]:

**4.1.** THEOREM. *If $X$ is countable and non-discrete and if $C_p(X)$ is a Borel subset of $\mathbb{R}^X$ then for some $\alpha < \omega_1$, $C_p(X)$ is of multiplicative class $\alpha$ in $\mathbb{R}^X$.*

This result had been obtained earlier for spaces with only one non-isolated point by CALBRIX [1988].

ARHANGEL′SKIĬ [1982] proved that if $X$ is compact, and $C_p(X)$ is *linearly* homeomorphic to $C_p(Y)$, then $Y$ is compact. This result has motivated several questions, among them whether the linearity of the homeomorphism involved is essential. This question was answered by GUL′KO and KHMYLEVA [1986], who proved that $C_p(\mathbb{R})$ and $C_p([0,1])$ are homeomorphic. More recently, it has been shown by DOBROWOLSKI ET AL. [1990], and independently CAUTY [1991], that if $X$ is any countable metrizable non-discrete space then $C_p(X)$ is homeomorphic to $\sigma_\omega$, the countable infinite product of copies of the space $\ell^2_f = \{x \in \ell^2 : x_n = 0 \text{ for almost all } n\}$. In particular, it follows that the compact space $\omega + 1$ and the non-compact space $\mathbb{Q}$ have homeomorphic function spaces. This result was generalized even further by Dobrowolski et al. [1991], who proved that for a countable non-discrete space $X$, if $C_p(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$ then $C_p(X)$ is homeomorphic to $\sigma_\omega$. Their result provides a very satisfactory classification result for function spaces of the lowest possible Borel complexity, i.e., the class $F_{\sigma\delta}$ of all absolute $F_{\sigma\delta}$-spaces.

The proofs of the above results all use the technique of absorbing sets as proposed by Bestvina and Mogilski [1986]. A drawback of this method is that one does not obtain homeomorphisms of pairs of spaces. Since for countable $X$, $C_p(X)$ can be thought of as being a subspace of the Hilbert cube $Q = [-\infty, \infty]^X$, it is natural to try to find homeomorphisms of pairs $\big(Q, C_p(X)\big) \approx \big(Q, C_p(Y)\big)$. This can be done with the help of our technique of absorbing sets as explained in section 2. For countable, metric, non-discrete spaces it was shown in Dijkstra et al. [1992] that $C_p(X)$ is an $F_{\sigma\delta}$-absorber in $Q$; there consequently is a homeomorphism of pairs $\big(Q, C_p(X)\big) \approx (Q^\infty, B^\infty)$, see section 2. This result was later generalized by Baars et al. [19??] and independently by Dijkstra and Mogilski [19??] who proved that for countable non-discrete $X$, if $C_p(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$ then it is an $F_{\sigma\delta}$-absorber in $Q$; in this case there consequently also exists a homeomorphism of pairs $\big(Q, C_p(X)\big) \approx (Q^\infty, B^\infty)$. A slightly different approach is to classify $C_p(X)$ with $s = \mathbb{R}^\infty$ as ambient space. This is done in Dijkstra and Mogilski [19??]. Cauty et al. [19??] recently obtained similar results about equivalence of triples of the form $(Q, s, C_p(X))$ and quadruples of the form $\big(Q, s, C_p^*(X), C_p(X)\big)$, where $C_p^*(X)$ denotes the subspace of $C_p(X)$ consisting of all *bounded* functions.

This completely solves the problem of the classification problem for function spaces of the lowest possible Borel complexity.

We now give an application of the absorbing systems method (in particular $\delta$-sequences). Define the following subspaces of $s$:

$$c_0 = \{x \in \mathbb{R}^{\mathbb{N}} : \lim_{i\to\infty} x_i = 0\}$$

and for $n \in \mathbb{N}$

$$\Sigma_n = \{x \in \mathbb{R}^{\mathbb{N}} : |x_i| \le 2^{-n} \text{ for all but finitely many } i\}.$$

Observe that $\Sigma = (\Sigma_n)_n$ is a $\delta$-sequence of $\sigma$Z-sets in $Q$ with the property that its intersection is $c_0$.

**4.2.** Proposition. *The system* $\Sigma$ *is* $\mathcal{F}_\sigma$*-absorbing (and hence* $c_0$ *is an* $\mathcal{F}_{\sigma\delta}$*-absorber) in* $Q = [-\infty, \infty]^{\mathbb{N}}$.

This is Theorem 6.3 of Dijkstra et al. [1992]. The fact that $C_p(X)$ is an $\mathcal{F}_{\sigma\delta}$-absorber in $Q$ if $X$ is countable and metric follows easily from this result.

□ It is obvious that we only need to prove that the $\delta$-sequence is strongly $\mathcal{F}_\sigma$-universal, i.e., $\mathcal{F}_\sigma$-universal and reflexively universal. Now let $\Phi\colon [-\infty, \infty]^{\mathbb{N}} \to ([-\infty, \infty]^{\mathbb{N}})^{\mathbb{N}}$ be any map that simply rearranges coordinates. It is easily seen that with this map the system $\Sigma$ satisfies the conditions of 3.2. So the system is reflexively universal.

To prove $\mathcal{F}_\sigma$-universality we shall use the following fact: if $A$ is an $\mathcal{F}_\sigma$-absorber in $Q$ and $A'$ is a $\sigma$Z-set then for every $\sigma$-compactum $C$ in $Q$ there is an embedding

$f : Q \to Q$ such that $f^{-1}(A) = C$ and $f(Q \setminus C) \cap A' = \varnothing$ (see DIJKSTRA ET AL. [1992, Lemma 6.1]).

Let $A_1 \supset A_2 \supset \cdots$ be a sequence of $\sigma$-compacta in $Q$. Let $\alpha$ be a bijection from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ and define $N_i = \{\alpha(i,j) : j \in \mathbb{N}\}$. For every $i \in \mathbb{N}$ define the Hilbert cube $Q_i = [-2^{-i+1}, 2^{-i+1}]^{N_i}$. It is easily verified with the capset characterization theorem in CURTIS [1985] that

$$C_i = \{x \in Q_i : |x_{\alpha(i,j)}| \le 2^{k-j} \text{ for some } k\}$$

is an $\mathcal{F}_\sigma$-absorber in $Q_i$. Observe that for every $x \in C_i$ we have $\lim_{j\to\infty} x_{\alpha(i,j)} = 0$. Define in $Q_i$ the $\sigma$Z-set

$$D_i = \{x \in Q_i : |x_{\alpha(i,j)}| \le 2^{-i} \text{ for all but finitely many } j\}.$$

Let $f_i : Q \to Q_i$ be an embedding such that $f_i^{-1}(C_i) = A_i$ and $f_i(Q \setminus A_i)$ does not meet $D_i$. Consider the embedding $f = (f_i)_{i\in\mathbb{N}} : Q \to \prod_{i=1}^\infty Q_i \subseteq Q$. Let $x \in A_n$. If $i > n$ then we have $f_i(x) \in Q_i$ and hence all components of $f_i(x)$ are in $[-2^{-n}, 2^{-n}]$. If $i \le n$ then we have $x \in A_i$ and hence $f_i(x) \in C_i$. Note that only finitely many components of $f_i(x)$ are outside $[-2^{-n}, 2^{-n}]$ and hence only finitely many components of $f(x)$ are outside this interval. This means that $f(x)$ is an element of $\Sigma_n$. If $x \notin A_n$ then we have $f_n(x) \notin D_n$. This means that infinitely many components of $f_n(x)$ have absolute value greater than $2^{-n}$ and hence $f(x) \notin \Sigma_n$. So we may conclude that $f^{-1}(\Sigma_n) = A_n$, proving that $\Sigma$ is $\mathcal{F}_\sigma$-universal. $\square$

The question naturally arises what can be said of function spaces of higher Borel complexity. The following was conjectured in DOBROWOLSKI ET AL. [1991].

**4.3.** CONJECTURE. *If $X$ is countable and non-discrete and if $C_p(X)$ is an absolute Borel subset of $\mathbb{R}^X$ then $C_p(X)$ is an absorber for the Borel class to which it belongs. In other words, we conjecture that the topological type of $C_p(X)$ is for absolutely Borel $C_p(X)$ completely determined by its Borel type.*

There is quite a lot of interesting evidence supporting this conjecture. Cauty, Dobrowolski, and Marciszewski have shown in CAUTY ET AL. [19??] that for countable spaces $X$ and $Y$, if $C_p(X)$ and $C_p(Y)$ are Borel then they are homeomorphic if and only $C_p(X)$ contains a closed homeomorph of $C_p(Y)$ and vice versa. They also proved that for each countable ordinal number $\alpha \ge 2$ (natural number $n$) there exists a countable space $X$ for which $C_p(X)$ is an absorbing set for the class of absolute Borel sets of the multiplicative class $\alpha$ (the class of projective sets of the class $n$).

It is known that for countable $X$, $C_p(X)$ need not be a Borel subset of $\mathbb{R}^X$. MARCISZEWSKI [19??] recently showed that in Gödel's constructable universe $L$ there exist countable spaces $X$ and $Y$ such that $C_p(X)$ and $C_p(Y)$ are analytic, non-Borel, and not homeomorphic. So it is essential that in 4.3 we restrict our attention to Borel sets.

It seems that our technique of absorbing systems is the right framework for obtaining results on function spaces as mentioned above.

The second topic is that of hyperspaces. If $X$ is a compact metric space, then $2^X$ denotes the hyperspace consisting of all non-empty closed subsets of $X$, endowed with the Hausdorff metric. It is known that $2^X \approx Q$ if and only if $X$ is a non-degenerate Peano continuum (CURTIS and SCHORI [1978]; see also VAN MILL [1989, Chapter 8]). For a space $X$ we let $\dim X$ denote its covering dimension. In addition, for $k \in \{0, 1, 2, \ldots, \infty\}$ we let $\dim_{\geq k}(X)$ denote the subspace consisting of all $\geq k$-dimensional elements of $2^X$. We define $\dim k(X)$ and $\dim_{\leq k}(X)$ in the same way. Let $\overline{\mathrm{Dim}}_{\geq k}(X)$ stand for all uniformly $\geq k$-dimensional compacta in $2^X$, i.e., spaces such that every nonempty open subset is at least $k$-dimensional. Define, for $n \in \mathbb{N}$, the set $\mathcal{G}_n$ to be the set of elements $A$ of $2^X$ for which is in $X$ a finite open cover of $A$ with mesh $\leq 1/n$ and order $\leq k$ Obviously, $\mathcal{G}_n$ is an open subset of $2^X$. Note that $\dim_{\geq k}(X) = 2^X \setminus \bigcap_{n=1}^{\infty} \mathcal{G}_n$ is therefore an $F_\sigma$-set.

It is natural to consider the sequence $\big(\dim_{\geq k}(Q)\big)_{n=1}^{\infty}$ in $2^Q$. By the above, this sequence consists of $F_\sigma$-subsets of $2^Q$. As a consequence,

$$\dim \infty(Q) = \bigcap_{n=1}^{\infty} \dim_{\geq k}(Q),$$

"the space of all infinite-dimensional compacta" is an $F_{\sigma\delta}$-subset of $2^Q$. The main result in DIJKSTRA ET AL. [1992] is the following:

**4.4.** THEOREM.

(a) *There exists a homeomorphism $\alpha$ from $2^Q$ onto $Q^\infty = \prod_{i=1}^{\infty} Q$ such that for every $k \in \{0, 1, 2, \ldots\}$,*

$$\alpha(\dim_{\geq k}) = \underbrace{B \times \cdots \times B}_{k \text{ times}} \times Q \times Q \times \cdots .$$

*This implies that $\alpha(\dim \infty) = B^\infty$.*

(b) *There exists a homeomorphism $\beta$ from $2^Q$ onto $Q^\infty$ such that for every $k \in \{0, 1, 2, \ldots\}$,*

$$\beta(\dim_{\leq k}) = \underbrace{Q \times \cdots \times Q}_{k \text{ times}} \times s \times s \times \cdots . \qquad \square$$

The proof of 4.4 is based in an essential way on the "convex" structure of $Q$ as well as on the technique of absorbing systems. Since for every non-degenerate Peano continuum $X$ we have $2^X \approx Q$, it is natural to ask for which Peano continua $X$ there is a homeomorphism of pairs $(2^X, \dim \infty(X)) \approx (Q^\infty, B^\infty)$. Since $\dim \infty(X) \neq \varnothing$ iff $\dim X = \infty$, this is of interest only when $X$ is infinite-dimensional. This question was considered in GLADDINES and VAN MILL [19??] who proved that if $X$ is an infinite product of non-degenerate Peano continua then there is a homeomorphism of pairs $(2^X, \dim \infty(X)) \approx (Q^\infty, B^\infty)$. This partly generalizes 4.4 above. The proof of their result differs from the proof of 4.4: a "convex" structure is not available on an arbitrary infinite product of Peano continua. They also present an example of an everywhere infinite-dimensional Peano continuum $X$ such that for every $n \in \mathbb{N}$,

$\dim \infty(X^n) \not\approx B^\infty$. So their result is in a sense "best possible". GLADDINES [19??] has recenly improved this result by showing that 4.4 holds with $Q$ replaced by an arbitrary countable infinite product of Peano continua. She has also shown that the sequence $\bigl(\dim_{\geq k}(Q) \cap C(Q)\bigr)_{k=2}^{\infty}$ is $F_\sigma$-absorbing in the subspace $C(Q)$ of $2^Q$ consisting of all subcontinua of $Q$.

The third and last topic of this section concerns the classification of dense subspaces of Banach spaces. We restrict our attention to the standard examples $l^p$ and $L^p$, which are homeomorphic to $\mathbb{R}^\infty$, according to the Anderson-Kadec Theorem (ANDERSON [1966], KADEC [1967]).

Consider the Banach space consisting of all sequences of real numbers that converge to 0, equipped with the norm

$$|x|_\infty = \max\{|x_n| : n \in \mathbb{N}\}.$$

We shall call this space $l^\infty$. Let $p$ be an arbitrary element of the interval $(0, \infty]$. For every $p < \infty$ define the following function from $l^\infty$ into $[0, \infty]$:

$$|x|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

The Banach space $(l^p, |\cdot|_p)$ consists of all $x \in l^\infty$ with $|x|_p < \infty$ — actually $l^p$ is a quasi-Banach space if $p < 1$. If $q \in (0, p)$ then

$$l^p_q = \{x \in l^p : |x|_q < \infty\}$$

is considered a subspace of $l^p$. Since the expression $|x|_q$ is nonincreasing as a function of $q$ we have $l^p_q \subseteq l^p_{q'}$ whenever $q < q'$. We are also interested in the spaces

$$\tilde{l}^p_q = \bigcap_{q<q'<p} l^p_{q'} \subseteq l^p.$$

The topological classification of these spaces was carried out by CAUTY and DOBROWOLSKI [19??] and DIJKSTRA and MOGILSKI [1991]. The method described here is the one in DIJKSTRA and MOGILSKI [1991]. We note that $(l^p_q)_q$ is an ordered system like the ones in the previous section.

The central idea is to compare this system with systems in $\mathbb{R}^\infty$ that are linked to the product structure of that topological Hilbert space.

We need some definitions. If $A$ is a countable infinite set then we define the following subspaces of the topological Hilbert space $\mathbb{R}^A$: the capset

$$\Sigma(A) = \{x \in \mathbb{R}^A : x = (x_a)_{a\in A} \text{ is bounded}\}$$

and the fd-capset

$$\sigma(A) = \{x \in \mathbb{R}^A : x_a = 0 \text{ for all but finitely many } a \in A\}.$$

In the standard model $\mathbb{R}^\mathbb{N}$ we have $\Sigma = \Sigma(\mathbb{N})$ and $\sigma = \sigma(\mathbb{N})$. It is easily seen that every $l^p_q$ is a so-called $\sigma$Z-set in $l^p$ and one may expect that $l^p_q$ is a Z-absorber, i.e., the pair $(l^p, l^p_q)$ is homeomorphic to the pairs $(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}} \times \Sigma)$ and $(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}} \times \sigma)$. This leads to the following definitions. Let $p$ be some fixed number in $(0, \infty]$ and let $A$ be a countable dense subset of the interval $(0, p)$. If $q \in (0, p)$ then we have

$$Z^p_q = Z^p_q(A) = \mathbb{R}^{(0,q]\cap A} \times \Sigma((q, p) \cap A) \subseteq \mathbb{R}^A$$

and

$$\zeta^p_q = \zeta^p_q(A) = \mathbb{R}^{(0,q]\cap A} \times \sigma((q, p) \cap A) \subseteq \mathbb{R}^A.$$

The main result of DIJKSTRA and MOGILSKI [1991] can now be formulated.

**4.5.** THEOREM. *If $\Delta$ is an arbitrary countable dense subset of $(0, p)$ then the systems $l^p_q$, $Z^p_q$, and $\zeta^p_q$ are $\Delta$-homeomorphic, i.e., there exist homeomorphisms $\alpha$ and $\beta$ from $\mathbb{R}^A$ onto $l^p$ such that for every $q \in \Delta$*

$$\alpha(Z^p_q) = l^p_q \quad \textit{and} \quad \beta(\zeta^p_q) = l^p_q.$$

Consequently, every $l^p_q$ is a Z-absorber and homeomorphic to $\mathbb{R}^{\mathbb{N}} \times \sigma$. Theorem 4.5 is obtained as an application of 3.1. It is obvious that the systems are Z-systems and their reflexive universality follows from lemmas similar to 3.2. Most of the work is in proving that the three systems $l^p_q$, $Z^p_q$, and $\zeta^p_q$ are embeddable in each other.

If $q \in [0, p)$ then we define the subspaces

$$\tilde{Z}^p_q = \bigcap_{q<q'<p} Z^p_{q'} \subseteq \mathbb{R}^A$$
and
$$\tilde{\zeta}^p_q = \bigcap_{q<q'<p} \zeta^p_{q'} \subseteq \mathbb{R}^A.$$

Since the spaces originate from the product structure it is easily seen that they are homeomorphic to $\Sigma^\infty$ and $\sigma^\infty$ and hence to $B^\infty$.

It follows from 4.5 that we have for every $q \in [0, p)$,

$$\alpha(\tilde{Z}^p_q) = \tilde{l}^p_q \quad \text{and} \quad \beta(\tilde{\zeta}^p_q) = \tilde{l}^p_q.$$

This implies that $(l^p, \tilde{l}^p_q)$ is homeomorphic to the pairs $(s^\infty, \Sigma^\infty)$ and $(s^\infty, \sigma^\infty)$. Consequently, $\tilde{l}^p_q$ is an $\mathcal{F}_{\sigma\delta}$-absorber in $l^p$.

Similar results are obtained for the function spaces $L^p$.

A number of other pre-Banach spaces have been classified, most notably by Cauty. We mention one of the more interesting results (see CAUTY [19??]). Let $L^1$ stand for the space of Lebesque integrable real functions on the interval $I$, equipped with the usual norm $\|f\| = \int_0^1 |f(t)|\, dt$.

**4.6.** THEOREM. *Both the space of continuous functions and the space of Riemann integrable functions in $L^1$ are homeomorphic to $B^\infty$.*

We finish this section with a classification problem that is related to the $l^p_q$'s. Let $l_p$ stand for the subspace of $p$-summable sequences in $s$. So instead of a norm topology we have the topology of coordinate-wise convergence. Put $\tilde{l}_p = \bigcap_{q>p} l_q$. These spaces are less natural than the normed spaces above but important because they were classified first. Dobrowolski and Mogilski [19∞] used generalized absorbers to show that every $\tilde{l}_p$ is homeomorphic to $B^\infty$. This result was later strengthened (using absorbing systems) by Dijkstra and Mogilski [19??] to: $(s, \tilde{l}_p) \approx (s, c_0)$ and $(Q, \tilde{l}_p) \approx (Q^\infty, B^\infty)$.

## 5. Finite-dimensional applications

In this section we consider absorbers in $\mathbb{R}^n$ and their complements in $\mathbb{R}^n$. Let $k$ and $n$ be two fixed integers that satisfy $0 \le k < n$. The $k$-dimensional Nöbeling space $N_k^n$ in $\mathbb{R}^n$ consists of all points of $\mathbb{R}^n$ that have at most $k$ rational coordinates. These spaces were introduced by Nöbeling in [1931], who essentially proved the following:

**5.1.** Theorem. *If $n \ge 2k+1$ then every $k$-dimensional space can be embedded in $N_k^n$.*

The Nöbeling spaces were the first examples of universal spaces in Dimension Theory. Let $P_k^n$ stand for the space $\mathbb{R}^n \setminus N_{n-k-1}^n$ and note that this space consists of a countable union of $k$-dimensional hyperplanes in $\mathbb{R}^n$.

The notion of an absorber of a class of spaces as presented in sections 2 and 3 is particularly useful when we are dealing with infinite-dimensional manifolds but not appropriate in the finite-dimensional setting. In this section we shall use West's original definition of absorbers.

Let $\mathfrak{M}$ be a collection of closed subsets of a complete space $X$ that is invariant under autohomeomorphisms of $X$. An element $A$ of $\mathfrak{M}_\sigma$ is called an $\mathfrak{M}$-*absorber in* $X$ if for every $S \in \mathfrak{M}$ and every collection $\mathcal{U}$ of open subsets in $X$ there is a homeomorphism $h: X \to X$ such that $h$ is $\mathcal{U}$-close to $1_X$ and $h(S \cap \bigcup \mathcal{U}) \subseteq A$. If $X$ is complete then the Uniqueness Theorem holds (West [1970]).

Toruńczyk [19??] showed that $P_k^n$ is an absorber for the collection of $\le k$-dimensional tame polyhedra in $\mathbb{R}^n$. In [1974], Geoghegan and Summerhill introduced the following similar spaces. They consider the $k$-skeleton of the barycentric subdivisions of some triangulation of $\mathbb{R}^n$, called the $k$-dimensional polyhedral pseudoboundary. Since this object is just as $P_k^n$ an absorber for the $\le k$-dimensional tame polyhedra we have with the Uniqueness Theorem that it is homeomorphic to $P_k^n$; its complement, the $(n-k-1)$-dimensional polyhedral pseudointerior, is homeomorphic to $N_{n-k-1}^n$.

Define the collection

$$\mathfrak{M}_k^n = \{h(S) : h \text{ a homeomorphism of } \mathbb{R}^n \text{ and} S \text{ a compact subset of } N_k^n\}.$$

Geoghegan and Summerhill construct in [1974] an $\mathfrak{M}_k^n$-absorber. This is the *$k$-dimensional universal pseudoboundary in* $\mathbb{R}^n$ and we denote it by $B_k^n$.

We now sketch the construction of $B_k^n$. Define for $i = 0, 1, 2, \cdots$,

$$K_i = \{(m + 1/2)3^{-i} : m \text{ an integer}\}$$

and let $\mathcal{P}$ be the group of homeomorphisms of $\mathbb{R}^n$ that correspond to coordinate permutations. We denote the open $\varepsilon$-ball with respect to the max metric in $\mathbb{R}^n$ by $U_\varepsilon$. Define for $m \in \mathbb{N}$ the following closed subset of $\mathbb{R}^n$:

$$A_m = \mathbb{R}^n \setminus \bigcup_{i=0}^{\infty} \bigcup_{\alpha \in \mathcal{P}} \alpha(U_{\frac{1}{2}3^{-i-m}}(K_i^{k+1} \times \mathbb{R}^{n-k-1})).$$

Then the set $B_k^n = \bigcup_{m=1}^{\infty} A_m$ is an $\mathfrak{M}_k^n$-absorber. For details, see GEOGHEGAN and SUMMERHILL [1974] and DIJKSTRA [1985].

The *k-dimensional universal pseudointerior* $s_k^n$ is the complement of $B_{n-k-1}^n$ in $\mathbb{R}^n$. The space $B_k^n$ is a countable union of topological copies of $\mu_k^n$, the $k$-dimensional universal Menger compactum in $\mathbb{R}$. The space $\mu_k^n$ was introduced by MENGER in [1926] — its universality was proved by LEFSCHETZ [1931] (see also BOTHE [1963] and ŠTAN'KO [1971]). In addition, there exists an absorber $B_k^\omega$ for the $\le k$-dimensional compacta in the topological Hilbert space $\mathbb{R}^\omega$, see DIJKSTRA [1985]. If $n \ge 2k+1$, then a slight modification of the construction presented in GEOGHEGAN and SUMMERHILL [1974] or DIJKSTRA [1985] gives an absorber $\beta_k^n$ for the $\le k$-dimensional Z-sets in $\mu_k^n$. Let $\nu_k^n = \mu_k^n \setminus \beta_k^n$ be the corresponding pseudointerior.

The following classification result was obtained by DIJKSTRA ET AL. [1990, 19∞].

**5.2.** THEOREM. *If $m, n \ge 2k+1$ then*

(a) *$B_k^\omega$ is homeomorphic to $B_k^n$ and*
(b) *$s_k^n$ is homeomorphic to $s_k^m$.*

□ *Idea of the proof.* First we note that (b) follows from (a). The pseudointeriors $s_k^m$ and $s_k^n$ contain embedded copies of $B_k^m$ and $B_k^n$, respectively. By a classic Theorem of Lavrentiev the homeomorphism between $B_k^m$ and $B_k^n$ can be extended to a homeomorphism between $G_\delta$-subsets $X$ and $Y$ of $s_k^m$ and $s_k^n$, respectively. Since it can be shown that the complements of $X$ and $Y$ are negligible we have a homeomorphism between $s_k^m$ and $s_k^n$.

We now give a rough sketch of the most important part of the proof. Let $\pi: \varrho \to \mathbb{R}^n$ stand for the projection onto the first $n$ coordinates. Using the fact that $B_k^\omega$ and $B_k^n$ are absorbers and that homeomorphisms between compacta can be extended with control to global homeomorphisms of $\varrho$, we can reposition $B_k^\omega$ in $\varrho$ so that $\pi(B_k^\omega) = B_k^n$. Put $f = \pi \mid B_k^\omega: B_k^\omega \to B_k^n$. We then use a version of Bing's shrinking criterion that was developed by TORUŃCZYK [1985] for incomplete spaces to show that $f$ is a near homeomorphism. Since we use shrinking all the "work" is done again in the Hilbert space $\varrho$ making the proof "infinite-dimensional" in spirit. □

From 5.2 one may derive:

(c) $P_k^n$ is homeomorphic to $P_k^m$ if $m, n \ge 2k+1$.

If $n$ and $m$ are distinct and not both $n$ and $m$ are greater than $2k$ then $B_k^n$, $s_k^n$ and $P_k^n$ are not homeomorphic to $B_k^m$, $s_k^m$ and $P_k^m$, respectively. Hence we have obtained a complete classification of these spaces. Observe that a similar classification of $\beta_k^n$ and $\nu_k^n$ follows immediately from BESTVINA's characterisation [1988] of $\mu_k^n$. The following conjecture can essentially be found in GEOGHEGAN and SUMMERHILL [1974].

**5.3.** CONJECTURE. *If $n \geq 2k+1$ then $N_k^n$ and $s_k^n$ are homeomorphic.*

There are also strong indications for the validity of the following conjectures.

**5.4.** CONJECTURE. *If $n \geq 2k+1$ then $\beta_k^n$ is homeomorphic to $B_k^n$.*

**5.5.** CONJECTURE. *If $n \geq 2k+1$ then $s_k^n$ and $\nu_k^n$ are homeomorphic.*

**5.6.** CONJECTURE. *$N_k^n$ is homeomorphic to $N_k^m$ if and only if $n = m$ or $n, m \geq 2k+1$.*

Observe that 5.6 follows from 5.2 and 5.3. Conjecture 5.5 follows from 5.4 and DIJKSTRA ET AL. [19∞].

It was observed by R. D. Anderson that $N_k^{k+1}$ and $s_k^{k+1}$ are not homeomorphic if $k > 0$. This suggests the following:

**5.7.** CONJECTURE. *$N_k^n$ and $s_k^n$ are not homeomorphic if $n \leq 2k$.*

The spaces introduced above are considered $k$-dimensional analogues of the pseudointerior and the pseudoboundary of the Hilbert cube. Let us recall that according to TORUŃCZYK [1981, 1985] the topological Hilbert space $s$ is characterised by the following:

(1) topological completeness,
(2) the absolute retract property,
(3) the strong discrete approximation property (SDAP).

If $n \geq 2k+1$ then the spaces $N_k^n$, $s_k^n$ and $\nu_k^n$ satisfy property (1) and the following $k$-dimensional versions of (2) and (3):

$(2_k)$ they are $k$-dimensional and have the absolute extension property for $\leq k$-dimensional spaces,
$(3_k)$ the discrete $k$-cells property.

A space $X$ has the *discrete $k$-cells property* if every sequence $(f_i\colon I^k \to X)_{i=1}^{\infty}$ can be approximated by a sequence $(g_i\colon I^k \to X)_{i=1}^{\infty}$ such that the images of the $g_i$'s form a discrete collection in $X$. The main unsolved problem in this area is expressed by the following:

**5.8.** CONJECTURE. *There is only one space (up to topological equivalence) that satisfies the properties* (1), $(2_k)$ *and* $(3_k)$.

Note that this conjecture implies 5.2 and 5.3 through 5.6. Other spaces that satisfy these three conditions have been constructed – for instance the spaces $\nu^k(\mathbb{R}^\omega)$ obtained by CHIGOGIDZE [19??] and CHIGOGIDZE and VALOV [1990] in analogy to DRANIŠNIKOV'S [1986] construction of certain Menger compacta.

Let us now turn to the pseudoboundaries. According to MOGILSKI [1984] the space $B$ is characterised by the following properties:

$(1')$ $\sigma$-compactness,
(2) the absolute retract property,

(3) the SDAP,

(4) strong universality for compacta.

If $n \geq 2k+1$ then the spaces $B_k^\omega$, $B_k^n$ and $\beta_k^n$ satisfy the properties $(1')$, $(2_k)$, $(3_k)$ and

$(4_k)$ strong universality for $\leq k$-dimensional compacta.

We have the following:

**5.9.** CONJECTURE. *There is only one space (up to topological equivalence) that satisfies the properties* $(1')$, $(2_k)$, $(3_k)$ *and* $(4_k)$.

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