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# HYPERSPACES OF LOCALLY CONNECTED CONTINUA OF EUCLIDEAN SPACES

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ABSTRACT. If $X$ is a space then $L(X)$ denotes the subspace of $C(X)$ consisting of all Peano (sub)continua. We announce here that for $n \geq 3$ the space $L(\mathbb{R}^n)$ is topologically homeomorphic to $B^\infty$, where $B$ denotes the pseudo-boundary of the Hilbert cube $Q$.

## 1. INTRODUCTION

For a space $X$, $C(X)$ denote the hyperspace of all nonempty subcontinua of $X$. It is known that for a Peano continuum $X$ without free arcs, $C(X) \approx Q$, where $Q$ denotes the Hilbert cube (Curtis and Schori [5]). $L(X)$ denotes the subspace of $C(X)$ consisting of all nonempty *locally connected* continua.

The spaces $L(X)$ were first studied by Kuratowski in [10]. He proved that $L(X)$ is an $F_{\sigma\delta}$-subset of $C(X)$, i.e., a countable intersection of $\sigma$-compact subsets. A little later, Mazurkiewicz [11] proved that for $n \geq 3$, $L(\mathbb{R}^n)$ belongs to the Borel class $F_{\sigma\delta}\backslash G_{\delta\sigma}$. It is easy to see that $L(\mathbb{R})$ is both $\sigma$-compact and topologically complete.

Our main result is that for $n \geq 3$ the spaces $L(\mathbb{R}^n)$ are homeomorphic to the countable infinite product of copies of the

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pseudo-boundary $B$ of $Q$. Our methods do not apply for the case $n = 2$. We use the theory of absorbing sets in the Hilbert cube and some ideas from Dijkstra, van Mill and Mogilski [7]. In fact, we prove that for $n \geq 3, L([-1,1]^n)$ is an $F_{\sigma\delta}$-absorber in $C([-1,1]^n)$. Our main result then follows easily.

## 2. Terminology

As usual $I$ denotes the interval [0,1] and $Q$ the Hilbert cube $\Pi_{i=1}^{\infty}[-1,1]_i$ with metric $d(x,y) = \sum_{i=1}^{\infty} 2^{-(i+1)}|x_i - y_i|$. In addition, $s$ is the *pseudo-interior* of $Q$, i.e., $s = \{x \in Q; (\forall i \in \mathbb{N})(|x_i| < 1)\}$. The complement $B$ of $s$ in $Q$ is called the *pseudo-boundary* of $Q$. Any space that is homeomorphic to $Q$ is called a *Hilbert cube.*

Let $A$ be a closed subset of a space $X$. We say that $A$ is a $Z$-set provided that every map $f : Q \to X$ can be approximated arbitrarily closely by a map $g : Q \to X \backslash A$. A countable union of $Z$-sets is called a *$\sigma Z$-set.* A *$Z$-embedding* is an embedding the range of which is a $Z$-set.

Let $\mathcal{M}$ be a class of spaces that is topological and closed hereditary.

**2.1. Definition.** Let $X$ be a Hilbert cube. A subset $A \subseteq X$ is called *strongly $\mathcal{M}$-universal* in $X$ if for every $M \in \mathcal{M}$ with $M \subseteq Q$, every embedding $f : Q \to X$ that restricts to a $Z$-embedding on some compact subset $K$ of $Q$, can be approximated arbitrarily closely by a $Z$-embedding $g : Q \to X$ such that $g \mid K = f \mid K$ while moreover $g^{-1}[A] \backslash K = M \backslash K$.

**2.2. Definition.** Let $X$ be a Hilbert cube. A subset $A \subseteq X$ is called an *$\mathcal{M}$-absorber* in $X$ if:

(1) $A \in \mathcal{M}$;
(2) there is a $\sigma Z$-set $S \subseteq X$ with $A \subseteq S$;
(3) $A$ is strongly *$\mathcal{M}$-universal* in $X$.

**2.3 Theorem ( [13,7] ).** Let $X$ be a Hilbert cube and let $A$ and $B$ be a $\mathcal{M}$-absorbers for $X$. Then there is a homeomorphism $h : X \to X$ with $h[A] = B$. Moreover, $h$ can be chosen arbitrarily close to the identity.

Absorbers for the class $F_\sigma$ for all $\sigma$-compact spaces were first constructed by Anderson and Bessaga and Pelczyński. A basic example of such an absorber in $Q$ is $B$. For details, see [2] and [12, Chapter 6]. The space $B^\infty$ in $Q^\infty$ is an absorber for the Borel class $F_{\sigma\delta}$. This was shown in Bestvina and Mogilski [3]; see also [7].

**2.4. Corollary.** Let $X$ be a Hilbert cube and let $A$ be an absorber in $X$ for the Borel class $F_{\sigma\delta}$. Then there is a homeomorphism of pairs $(Q^\infty, B^\infty) \approx (X, A)$. In particular, $A$ is homeomorphism to $B^\infty$.

The space $B^\infty$ has been studied intensively in infinite -dimensional topology during the last years. For more information, see e.g. [3,4,8,7,6,1].

## 3. Results

For a continuum $X$ and $n \in \mathbb{N}$ define

$$\mathcal{B}(X)^m_n = \{C \in C(X) : C \text{ can be covered by at most } m \text{ subcontinua of diameter} \le \tfrac{1}{n} \cdot \operatorname{diam}(C)\}.$$

A routine verification shows that each $\mathcal{B}(X)^m_n$ is compact, and that

$$L(X) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{B}(X)^m_n.$$

We show that for $n \geq 2$, $L(\mathbb{R}^n)$ belongs to the Borel class $F_{\sigma\delta} \backslash G_{\delta\sigma}$, generalizing the result of Mazurkiewicz mentioned in the introduction. Let $\hat{c}_0 = \{x \in Q : \lim_{n\to\infty} x_n = 0\}$. It follows from Dijkstra, van Mill and Mogilski [7] that $\hat{c}_0$ is an $F_{\sigma\delta}$-absorber in $Q$, and hence that it belongs to the Borel class $F_{\sigma\delta} \backslash G_{\delta\sigma}$. For every $x \in Q$ define $S(x) \subseteq [-1,1]^2$ by

$$S(x) = (\{0\}\times[-1,1])\cup([0,1]\times\{0\})\cup\bigcup_{n=1}^{\infty}\{\frac{1}{n}\}\times\begin{cases}\Big[0, x_n\Big](x_n \geq 0),\\ \\ \Big[x_n, 0\Big](x_n \leq 0).\end{cases}$$

It is clear that the function $S : Q \to C([-1,1]^2) \subseteq C(\mathbb{R}^2)$ defined by $x \mapsto S(x)$ is an embedding. Moreover, $S(x)$ is locally connected if and only if $x \in \hat{c}_0$. As a consequence,

$$S[Q] \cap L([-1,1]^2) = S[\hat{c}_0],$$

and so $L([-1,1]^2)$ belongs to the Borel class $F_{\sigma\delta} \backslash G_{\delta\sigma}$. The result for all $n \geq 2$ now follows easily because for these $n, L([-1,1]^n)$ contains a closed copy of $L([-1,1]^2)$.

**3.1 Theorem.** If $n \geq 3$ then $L([-1,1]^n)$ is contained in a $\sigma Z$-set in $C([-1,1]^n)$.

The strategy of the proof is roughly speaking the following. First we push $C([-1,1]^n)$ by a small movement into $C(\Gamma)$ for a certain finite connected graph $\Gamma \subseteq [-1,1]^n$. Then we carefully "blow up" each subcontinuum of $\Gamma$ to a close subcontinuum of $[-1,1]^n$ that has more or less the following shape:

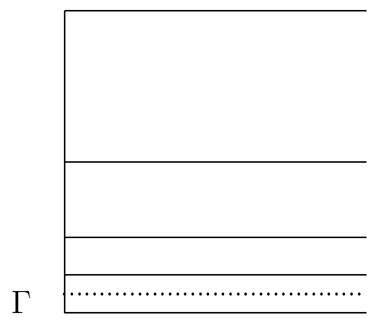


FIGURE 1

We next consider the collection

$$\mathcal{B} = \{C \in C([-1,1]^n) : C \text{ can be covered by finitely many subcontinua of diameter} \leq \tfrac{1}{3} \,.\, \mathrm{diam}(C)\}$$

and observe that $L([-1,1]^n) \subseteq \mathcal{B}$ and that $\mathcal{B}$ is $\sigma$-compact. We then prove that $\mathcal{B}$ is a $\sigma Z$-set by observing that continua $C$

of the type as shown in Figure 1 cannot be covered by finitely many subcontinua of diameter $\leq \frac{1}{3} \cdot \mathrm{diam}(C)$.

**3.2 Theorem.** If $n \geq 2$ then $L([-1,1]^n)$ is strongly $F_{\sigma\delta}$-universal in $C([-1,1]^n)$.

The strategy of the proof is roughly speaking the following. First we approzimate a continuum $C \subseteq [-1,1]^n$ arbitrarily closely by a finite set $F$. Then we add straight-line intervals to $F$ to make it connected. Moreover, to each point of $F$ we add small sets of the form that were used in the proof that $L([-1,1]^2)$ belongs to the Borel class $F_{\sigma\delta} \backslash G_{\delta\sigma}$. These sets are needed to make sure that some but not all of the approzimations that we construct are locally connected. Then we add to each point of $F$ a half-closed ball. This ball is added for technical reasons: it allows us later to establish rather easily that our approzimation is an embedding.

So we arrive at the conclusion that for $n \geq 3, L([-1,1]^n)$ is an $F_{\sigma\delta}$-absorber in $C([-1,1]^n)$. Fix $n \geq 3$. It is clear that $\{A \in C([-1,1]^n) : A \cap \partial([-1,1]^n) \neq \emptyset\}$ is a $Z$-set in $C([-1,1]^n)$.

Since an $F_{\sigma\delta}$-absorber in $Q$ minus a $Z$-set in $Q$ is an $F_{\sigma\delta}$-absorber (Baars, Gladdines and van Mill [1, Theorem 9.3]), it follows that the set of all Peano continua in $[-1,1]^n$ that miss the boundary also forms an $F_{\sigma\delta}$-sbsorber in $C([-1,1]^n)$. So an application of Corollary 2.4 now yields our main result.

**3.3. Theorem.** If $n \geq 3$ then $L(\mathbb{R}^n)$ is homeomorphic to $B^\infty$.

For details, see Gladdines and van Mill [9].

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