

## INCREASING CHAINS OF IDEALS AND ORBIT CLOSURES IN $\beta\mathbb{Z}$

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**ABSTRACT.** Given any discrete semigroup  $(S, +)$ , there is an extension of the operation to  $\beta S$  making  $(\beta S, +)$  a left topological semigroup. The aim of this paper is, among other things, to prove that there exist strictly increasing chains of principle left ideals and of principal closed ideals in  $(\beta\mathbb{Z}, +)$ .

It is a question posed to M. E. Rudin several years ago by some analysts as to whether every point of  $\beta\mathbb{Z} \setminus \mathbb{Z} = \mathbb{Z}^*$  is a member of some maximal orbit closure of the shift function. Here  $\beta\mathbb{Z}$  is the Čech-Stone compactification of the group  $\mathbb{Z}$  of integers and the “shift function”  $\sigma$  refers both to the function from  $\mathbb{Z}$  to  $\mathbb{Z}$  defined by  $\sigma(n) = n + 1$  and to its continuous extension from  $\beta\mathbb{Z}$  to  $\beta\mathbb{Z}$ . The orbit closure of a point  $p$  of  $\mathbb{Z}^*$  is  $\text{cl}\{\sigma^n(p) : n \in \mathbb{Z}\}$ .

A simpler question asks whether there can be any infinite strictly increasing chain of orbit closures. Of course a negative answer to the latter question implies an affirmative question to the former.

It is well known that, given any discrete semigroup  $(S, +)$ , there is an extension of the operation to  $\beta S$  making  $(\beta S, +)$  a left topological semigroup, with  $S$  contained in its topological center. That is to say, for each  $p \in \beta S$  the function  $\lambda_p : \beta S \rightarrow \beta S$  defined by  $\lambda_p(q) = p + q$  is continuous, while for  $s \in S$  the function  $\rho_s : \beta S \rightarrow \beta S$  defined by  $\rho_s(p) = p + s$  is continuous. (See, for example, [4].) In particular, in  $\beta\mathbb{Z}$  we have for each  $n \in \mathbb{Z}$  that  $\sigma^n$  and  $\rho_n$  are continuous functions agreeing on the dense set  $\mathbb{Z}$  so that for all  $p \in \beta\mathbb{Z}$  and all  $n \in \mathbb{Z}$ ,  $p + n = \sigma^n(p)$ . Consequently  $\text{cl}\{\sigma^n(p) : n \in \mathbb{Z}\} = \text{cl}\{p + n : n \in \mathbb{Z}\} = \text{cl}\lambda_p[\mathbb{Z}] = \lambda_p[\text{cl}\mathbb{Z}] = \lambda_p[\beta\mathbb{Z}] = p + \beta\mathbb{Z}$ . That is, the orbit closure of  $p$  is the principal right ideal generated by  $p$ .

Consequently, the question we are addressing is: *is there a strictly increasing chain of principal right ideals in  $\beta\mathbb{Z}$ ?* (Observe that if  $p \in \mathbb{Z}$  then  $p + \beta\mathbb{Z} = \beta\mathbb{Z}$ , so asking for an increasing chain in  $\beta\mathbb{Z}$  or in  $\mathbb{Z}^*$  amounts to the same thing.)

Once the question is phrased in this fashion it immediately suggest two others:

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is there a strictly increasing chain of principal left ideals in  $\beta\mathbb{Z}$ ? Also, by [3] given any  $p$  one has  $\text{cl}(\beta\mathbb{Z} + p)$  is a two-sided ideal which we call the *principal closed ideal generated by  $p$* . One can then ask: *is there a strictly increasing chain of principal closed ideals in  $\beta\mathbb{Z}$ ?*

In §1 we show that there is a sequence  $\langle \widehat{A}_n \rangle_{n=0}^\infty$  of closed  $G_\delta$  subsets of  $\mathbb{Z}^*$  satisfying

- (1) for each  $n \in \omega$ ,  $\widehat{A}_{n+1} + \widehat{A}_{n+1} \subseteq \widehat{A}_n$ .
- (2) for each  $n \in \omega$  and each  $p \in \widehat{A}_n$ ,  $\widehat{A}_{n+1} \cap \text{cl}(\beta\mathbb{Z} + p) = \emptyset$  (and in particular  $\widehat{A}_{n+1} \cap (p + \beta\mathbb{Z}) = \emptyset$ ).

(Recall that  $\omega = \{0, 1, 2, \dots\}$ .)

The motivation for this construction comes from the proof [2, Corollary 2.10] that any compact Hausdorff left topological semigroup has an idempotent. This proof begins with any compact nonempty set  $A$  with  $A + A \subseteq A$  and, in effect, shrinks  $A$  to a point  $x$  with  $\{x\} + \{x\} \subseteq \{x\}$ , i.e., with  $x + x = x$ . In a converse fashion it has recently been shown [1] that given any countable subsemigroup  $T$  of  $\beta\mathbb{Z}$ , each point  $t \in T$  can be expanded to a compact  $G_\delta$   $G(t)$  with  $G(t) + G(s) \subseteq G(t + s)$  for all  $t, s \in T$ . Taken together these facts indicate that one might expect to be able to treat compact subsets of  $\beta\mathbb{Z}$  as “big points.” If we could then collapse each  $\widehat{A}_n$  to a point  $p_n$  we would get for each  $n$  that  $p_{n+1} + p_{n+1} = p_n$  (so that  $p_n \in p_{n+1} + \beta\mathbb{Z}$  and hence  $p_n + \beta\mathbb{Z} \subseteq p_{n+1} + \beta\mathbb{Z}$ ). The second part of the construction would then show that  $p_{n+1} \notin p_n + \beta\mathbb{Z}$  so that  $p_n + \beta\mathbb{Z}$  is properly contained in  $p_{n+1} + \beta\mathbb{Z}$ .

Of course one does not need this much; it is enough to get points  $p_n$  in  $\widehat{A}_n$  and  $q_n \in \beta\mathbb{Z}$  with for all  $n \in \omega$ ,  $p_{n+1} + q_{n+1} = p_n$ . We conclude §1 by showing that we can accomplish this in reverse order. That is, we can get  $p_n$  and  $q_n$  in  $\widehat{A}_n$  with  $q_{n+1} + p_{n+1} = p_n$  for all  $n \in \omega$ . As a consequence one obtains strictly increasing chains of principal left ideals and of principal closed ideals.

The construction of the sequences  $\langle p_n \rangle_{n=0}^\infty$  and  $\langle q_n \rangle_{n=0}^\infty$  is based solely on the topological-algebraic properties of the sets  $\widehat{A}_n$ . We show in §2 that one cannot hope to appeal only to such properties to obtain a strictly increasing chain of principal right ideals in  $\beta\mathbb{Z}$ .

We take the points of  $\beta\mathbb{Z}$  to be the ultrafilters on  $\mathbb{Z}$ , the principal ultrafilters being identified with the points of  $\mathbb{Z}$ . Given  $p$  and  $q$  in  $\beta\mathbb{Z}$  the operation  $+$  is characterized by

$$A \in p + q \quad \text{if and only if} \quad \{n \in \mathbb{Z} : A - n \in p\} \in q.$$

The topology of  $\beta\mathbb{Z}$  is characterized by the fact that for  $A \subseteq \mathbb{Z}$ ,  $p \in \text{cl} A$  if and only if  $A \in p$ .

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### 1. INCREASING CHAINS OF PRINCIPAL LEFT IDEALS AND PRINCIPAL CLOSED IDEALS

For  $A \subseteq \mathbb{Z}$ , by  $\bigoplus_{n < \omega} A$  we mean the set of all sequences in  $A$  with only finitely many nonzero terms.

1.1. **Definition.** (a) For  $m, n \in \omega$ ,

$$\theta_n^m = \left\{ \vec{a} \in \bigoplus_{n < \omega} \omega : \sum_{i=0}^{\infty} \frac{a_i}{2^i} = \frac{1}{2^n} \text{ and } \min\{i \in \omega : a_i \neq 0\} \geq m \right\}.$$

(b)  $\langle \xi_n \rangle_{n < \omega}$  is some sequence in  $\omega \setminus \{0\}$  such that for all  $n \in \omega$ ,

$$\xi_{n+1} > \sum_{i=0}^n 2^{i+1} \cdot \xi_i.$$

(c) For  $n, m \in \omega$ ,

$$A_n^m = \left\{ \sum_{i=0}^{\infty} a_i \xi_i : \vec{a} \in \theta_n^{n+m} \right\}.$$

Observe that each  $A_n^m$  is infinite and that  $A_n^m \subseteq A_n^{m'}$  if  $m' \leq m$ .

1.2. **Lemma.** For each  $m$  and  $n$  in  $\omega$ ,  $A_{n+1}^m + A_{n+1}^m \subseteq A_{n+1}^{m+1}$ .

*Proof.* Let  $x, y \in A_{n+1}^m$  and pick  $\vec{a}, \vec{b} \in \theta_{n+1}^{n+m+1}$  with  $x = \sum_{i=0}^{\infty} a_i \xi_i$  and  $y = \sum_{i=0}^{\infty} b_i \xi_i$ . For each  $i < \omega$  let  $c_i = a_i + b_i$ . Then  $x + y = \sum_{i=0}^{\infty} c_i \xi_i$  so it suffices to show that  $\vec{c} \in \theta_{n+1}^{n+m+1}$ . Certainly  $\min\{i \in \omega : c_i \neq 0\} = \min(\{i \in \omega : a_i \neq 0\} \cup \{i \in \omega : b_i \neq 0\}) \geq n + m + 1$ . Further  $\sum_{i=0}^{\infty} c_i / 2^i = \sum_{i=0}^{\infty} a_i / 2^i + \sum_{i=0}^{\infty} b_i / 2^i = 1/2^{n+1} + 1/2^{n+1} = 1/2^n$  as required.  $\square$

1.3. **Lemma.** Let  $\vec{a}, \vec{b} \in \bigoplus_{n < \omega} \mathbb{Z}$  such that for each  $i < \omega$ ,  $|a_i| \leq 2^i$  and  $|b_i| \leq 2^i$ . If  $\sum_{i=0}^{\infty} a_i \xi_i = \sum_{i=0}^{\infty} b_i \xi_i$ , then  $\vec{a} = \vec{b}$ .

*Proof.* Suppose  $\vec{a} \neq \vec{b}$ . Since  $\vec{a}$  and  $\vec{b}$  have each only finitely many nonzero values we may pick the largest  $n$  such that  $a_n \neq b_n$ . Then  $\sum_{i=0}^n a_n \xi_n = \sum_{i=0}^n b_n \xi_n$ . Without loss of generality, we have  $a_n > b_n$ . Then  $\xi_n \leq (a_n - b_n) \xi_n = \sum_{i=0}^{n-1} (b_i - a_i) \xi_i \leq \sum_{i=0}^{n-1} 2^{i+1} \xi_i < \xi_n$ , a contradiction.  $\square$

1.4. **Definition.** For each  $n < \omega$ ,  $\hat{A}_n = \bigcap_{m=0}^{\infty} \text{cl } A_n^m$ .

We now show that the sets  $\hat{A}_n$  are as promised.

1.5. **Theorem.** For each  $n < \omega$ ,

(a)  $\hat{A}_{n+1} + \hat{A}_{n+1} \subseteq \hat{A}_n$ , and

(b) For each  $p \in \hat{A}_n$ ,  $\text{cl}(\beta\mathbb{Z} + p) \cap \hat{A}_{n+1} = \emptyset$ .

*Proof.* (a) Let  $p, q \in \hat{A}_{n+1}$ . To see that  $p + q \in \hat{A}_n$ , let  $m \in \omega$  be given. We must show  $A_n^m \in p + q$ . But by Lemma 1.2 we have immediately that  $A_{n+1}^m \subseteq \{x \in \mathbb{Z} : A_{n+1}^{m+1} - x \in p\}$  and hence that  $A_{n+1}^{m+1} \in p + q$ . Since  $A_{n+1}^{m+1} \subseteq A_n^m$ , this suffices.

(b) Suppose we have some  $r \in \hat{A}_{n+1} \cap \text{cl}(\beta\mathbb{Z} + p)$ . Then  $A_{n+1}^0 \in r$  so  $\text{cl } A_{n+1}^0$  is a neighborhood of  $r$  so  $(\text{cl } A_{n+1}^0) \cap (\beta\mathbb{Z} + p) \neq \emptyset$ . Pick  $q \in \beta\mathbb{Z}$  with  $q + p \in \text{cl } A_{n+1}^0$ , i.e.,  $A_{n+1}^0 \in q + p$ . Let  $B = \{x \in \mathbb{Z} : A_{n+1}^0 - x \in q\}$ . Then  $B \in p$  so pick  $x \in B \cap A_n^0$ . Pick  $\vec{a} \in \theta_n^n$  with  $x = \sum_{i=0}^{\infty} a_i \xi_i$  and let  $m = 1 + \max\{i \in \omega : a_i \neq 0\}$ . Since  $A_n^m \in p$ , pick  $y \in B \cap A_n^m$  and pick  $\vec{b} \in \theta_n^{n+m}$  with  $y = \sum_{i=0}^{\infty} b_i \xi_i$ . Since  $x$  and  $y$  are in  $B$ , pick  $z \in (A_{n+1}^0 - x) \cap (A_{n+1}^0 - y)$ . Since  $z + x$  and  $z + y$  are in  $A_{n+1}^0$ , pick  $\vec{c}$  and  $\vec{d} \in \theta_{n+1}^{n+1}$  with  $z + x = \sum_{i=0}^{\infty} c_i \xi_i$

and  $z + y = \sum_{i=0}^{\infty} d_i \xi_i$ . Note that for each  $i$  each of  $a_i, b_i, c_i$  and  $d_i$  is in  $\{0, 1, \dots, 2^i\}$ . Now  $y - x = \sum_{i=0}^{\infty} (b_i - a_i) \xi_i = \sum_{i=0}^{\infty} (d_i - c_i) \xi_i$ , so by Lemma 1.3 we have for each  $i$  that  $b_i - a_i = d_i - c_i$ . Now  $\vec{b} \in \theta_n^{n+m}$  so  $b_i = 0$  for  $i < n + m$ . Thus  $1/2^n = \sum_{i=0}^{\infty} b_i/2^i = \sum_{i=n+m}^{\infty} b_i/2^i$ . Since  $a_i = 0$  for  $i \geq m$  we have for each  $i \geq n + m$  that  $d_i - c_i = b_i - a_i = b_i$  and hence  $d_i = b_i + c_i$ . But then  $1/2^{n+1} = \sum_{i=0}^{\infty} d_i/2^i \geq \sum_{i=n+m}^{\infty} d_i/2^i = \sum_{i=n+m}^{\infty} (b_i + c_i)/2^i \geq \sum_{i=n+m}^{\infty} b_i/2^i = 1/2^n$ , a contradiction.  $\square$

1.6. **Corollary.** For each  $n \in \omega$  and each  $p \in \widehat{A}_n, (p + \beta\mathbb{Z}) \cap \widehat{A}_{n+1} = \emptyset$ .

*Proof.* We have  $p + \beta\mathbb{Z} = \text{cl}(p + \mathbb{Z}) = \text{cl}(\mathbb{Z} + p) \subseteq \text{cl}(\beta\mathbb{Z} + p)$ .  $\square$

We are now prepared to deduce from the topological and algebraic facts contained in Theorem 1.5 that there are strictly increasing chains of principal left ideals and of principal closed ideals. We utilize the notion of the limit along an ultrafilter. Recall that if  $\xi$  is an ultrafilter on a set  $I$  and  $\langle x_i \rangle_{i \in I}$  is an  $I$ -sequence in a topological space the statement  $\xi - \lim_{i \in I} x_i = y$  means that for every neighborhood  $U$  of  $y, \{i \in I : x_i \in U\} \in \xi$ . Recall also (or do the easy exercise which is the proof) that if  $f$  is a continuous function and  $\xi - \lim_{i \in I} x_i$  exists, then  $f(\xi - \lim_{i \in I} x_i) = \xi - \lim_{i \in I} f(x_i)$ .

1.7. **Theorem.** Let  $(S, +)$  be a compact Hausdorff left-topological semigroup and for each  $n \in \omega$  let  $A_n$  be a nonempty closed subset of  $S$  with  $A_{n+1} + A_{n+1} \subseteq A_n$ . Given any sequence  $\langle q_n \rangle_{n=0}^{\infty}$  with each  $q_n \in A_n$ , there is a sequence  $\langle p_n \rangle_{n=0}^{\infty}$  with each  $p_n \in A_n$  such that for each  $n \in \omega, q_{n+1} + p_{n+1} = p_n$ .

*Proof.* Choose any nonprincipal ultrafilter  $\xi$  on  $\omega$ . For  $n$  and  $m \in \omega$  with  $n \geq m$  let  $r_{n,m} = q_n$ . By downward induction for  $n < m$  let  $r_{n,m} = q_{n+1} + r_{n+1,m}$ . For each  $n \in \omega$  let  $p_n = \xi - \lim_{m \in \omega} r_{n,m}$ . Observe that since for each  $n$  and  $m$  in  $\omega$  we have  $r_{n,m} \in A_n$  we have  $p_n \in A_n$ . Note also that for  $n \in \omega, \{m \in \omega : r_{n,m} = q_{n+1} + r_{n+1,m}\}$  is cofinite and is hence in  $\xi$ . Thus for  $n \in \omega$  we have  $q_{n+1} + p_{n+1} = \lambda_{q_{n+1}}(\xi - \lim_{m \in \omega} r_{n+1,m}) = \xi - \lim_{m \in \omega} \lambda_{q_{n+1}}(r_{n+1,m}) = \xi - \lim_{m \in \omega} (q_{n+1} + r_{n+1,m}) = p_n$ .  $\square$

1.8. **Corollary.** There exist strictly increasing chains of principal left ideals and of principal closed ideals in  $\beta\mathbb{Z}$ .

*Proof.* Pick by Theorems 1.5(a) and 1.7 sequences  $\langle p_n \rangle_{n=0}^{\infty}$  and  $\langle q_n \rangle_{n=0}^{\infty}$  with each  $p_n$  and  $q_n$  in  $\widehat{A}_n$  and each  $q_{n+1} + p_{n+1} = p_n$ . Then for each  $n, \beta\mathbb{Z} + p_n \subseteq \beta\mathbb{Z} + p_{n+1}$  and hence  $\text{cl}(\beta\mathbb{Z} + p_n) \subseteq \text{cl}(\beta\mathbb{Z} + p_{n+1})$ . By Theorem 1.5(b) we have  $p_{n+1} \in (\beta\mathbb{Z} + p_{n+1}) \setminus \text{cl}(\beta\mathbb{Z} + p_n)$ .  $\square$

## 2. COMPACT SUBSETS OF LEFT TOPOLOGICAL SEMIGROUPS NEED NOT BE JUST BIG POINTS

We show here that Theorem 1.5 and the fact that  $\beta\mathbb{Z}$  is a compact left topological semigroup are not sufficient to conclude that there exists a strictly increasing chain of principal right ideals (i.e., orbit closures) in  $\beta\mathbb{Z}$ . Specifically, we produce a compact left topological semigroup  $(S, +)$  and a sequence  $\langle A_n \rangle_{n=0}^{\infty}$  of compact subsets of  $S$  satisfying

- (1) for each  $n \in \omega, A_{n+1} + A_{n+1} \subseteq A_n$ ;
- (2) for each  $n \in \omega$  and each  $p \in A_n, (p + S) \cap A_{n+1} = \emptyset$  and  $\text{cl}(S + p) \cap A_{n+1} = \emptyset$ ; and

- (3) there does not exist a sequence  $\langle p_n \rangle_{n=0}^\infty$  with each  $p_n \in A_n$  and each  $p_n \in p_{n+1} + S$ .

We begin by defining an operation  $*$  on the finite nonempty subsets of  $\mathbb{Z}$ .

2.1. **Definition.** (a)  $\mathcal{F} = \{F : F \subseteq \mathbb{Z}, F \neq \emptyset, \text{ and } F \text{ is finite}\}$ .

(b) Define  $\varphi : \mathcal{F} \rightarrow \mathbb{R}$  by  $\varphi(F) = \sum_{n \in F} 2^{-n}$ .

(c) For  $F, G \in \mathcal{F}$ ,  $F * G = \varphi^{-1}(\varphi(F) + \varphi(G))$ .

An example will make the operation clear. Consider  $F = \{-4, -2, 1, 4\}$  and  $G = \{-4, -3, 0, 1, 3\}$ . Write numbers in binary labelling the columns in the reverse of the usual order and add:

|           |    |    |    |    |    |   |   |   |   |   |
|-----------|----|----|----|----|----|---|---|---|---|---|
|           | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
| $F$ :     |    | 1  | 0  | 1  | 0  | 0 | 1 | 0 | 0 | 1 |
| $G$ :     |    | 1  | 1  | 0  | 0  | 1 | 1 | 0 | 1 | 0 |
| $F * G$ : | 1  | 0  | 1  | 1  | 1  | 0 | 0 | 0 | 1 | 1 |

We read off from the addition that  $F * G = \{-5, -3, -2, -1, 3, 4\}$ . It is then clear (since we are copying ordinary addition) that  $*$  is associative and cancellative. Note also that always  $\min(F * G) \leq \min F$ .

2.2. **Definition.** (1)  $T = \{(m, F) : m \in \omega \text{ and } F \in \mathcal{F}\}$ .

(2) Given  $(m, F)$  and  $(s, G)$  in  $T$ ,

$$(m, F) + (s, G) = (m + \min F - \min(F * G), F * G).$$

Since always  $\min(F * G) \leq \min F$  we have for  $(m, F)$  and  $(s, G)$  in  $T$  that the sum  $(m, F) + (s, G)$  is again in  $T$ . It is an easy exercise to verify that  $+$  is an associative operation on  $T$ .

We topologize  $T$  as follows. Given  $F \in \mathcal{F}$ , we make  $\{(m, F) : m \in \omega\}$  an open and closed copy of the one point compactification of the positive integers with  $(0, F)$  as its point at infinity. That is, if  $m > 0$ ,  $\{(m, F)\}$  is clopen and  $\{(0, F)\} \cup \{(m, F) : m > n\} : n \in \omega$  is a basic neighborhood system for  $(0, F)$ . Observe that as a topological space  $T$  is homeomorphic to the topological sum of countably many convergent sequences. The following lemma is therefore a triviality.

2.3. **Lemma.** *With the topology described above  $T$  is a locally compact Hausdorff space. Given  $B \subseteq T$ ,  $B$  is compact if and only if  $B$  is closed and there is a finite  $\mathcal{H} \subseteq \mathcal{F}$  with  $B \subseteq \{(m, F) : m \in \omega \text{ and } F \in \mathcal{H}\}$ .  $\square$*

2.4. **Definition.**  $S = T \cup \{\infty\}$  where topologically  $S$  is the one point compactification of  $T$  and algebraically  $\infty = \infty + \infty = (m, F) + \infty = \infty + (m, F)$  for any  $(m, F) \in T$ .

2.5. **Lemma.**  *$S$  is a compact Hausdorff left topological semigroup.*

*Proof.* One immediately concludes that  $S$  is compact and Hausdorff. Further  $\lambda_\infty$  is constant. Let  $(m, F) \in T$ . Given  $(k, G) \in T$  we have  $\{(n, G) : n \in \omega\}$  is a neighborhood of  $(k, G)$  on which  $\lambda_{(m, F)}$  is constantly equal to  $(m + \min F - \min(F * G), F * G)$ . Finally we show that  $\lambda_{(m, F)}$  is continuous at  $\infty$ .

Let  $U$  be an open neighborhood of  $\infty$  and pick by Lemma 2.3 finite  $\mathcal{H} \subseteq \mathcal{F}$  with  $S \setminus U \subseteq \{(n, H) : n \in \omega \text{ and } H \in \mathcal{H}\}$ . Let  $\mathcal{G} = \{G \in \mathcal{F} : F * G \in \mathcal{H}\}$ . Since  $*$  is cancellative,  $\mathcal{G}$  is finite (possibly empty). Let  $V = S \setminus \{(k, G) : G \in \mathcal{G}\}$ . Then  $V$  is a neighborhood of  $\infty$  and  $\lambda_{(m, F)}[V] \subseteq U$ .  $\square$

**2.6. Definition.** For  $n \in \omega$ , let  $A_n = \{(m, \{n\}) : m \in \omega\}$ .

Unlike the situation in  $\beta\mathbb{Z}$  we need a separate verification of conclusions 2(a) and 2(b) in the following theorem. The reason is that while the center of  $\beta\mathbb{Z}$  (namely,  $\mathbb{Z}$ ) is dense, the center of  $S$  is  $\{\infty\}$ .

**2.7. Theorem.** *The sequence  $\langle A_n \rangle_{n=0}^\infty$  in the compact left topological semigroup  $(S, +)$  satisfies*

- (1) For each  $n \in \omega$ ,  $A_n$  is compact and  $A_{n+1} + A_{n+1} \subseteq A_n$ ,
- (2) For each  $n \in \omega$  and each  $p \in A_n$ ,
  - (a)  $(p + S) \cap A_{n+1} = \emptyset$  and
  - (b)  $\text{cl}(S + p) \cap A_{n+1} = \emptyset$ .
- (3) There is no sequence  $\langle p_n \rangle_{n=0}^\infty$  with each  $p_n \in A_n$  and each  $p_n \in p_{n+1} + S$ .

*Proof.* For (1) we have immediately that each  $A_n$  is compact. Since  $\{n+1\} * \{n+1\} = \{n\}$  we have  $A_{n+1} + A_{n+1} \subseteq A_n$ .

For (2) let  $p = (m, \{n\})$ . Then  $p + \infty = \infty$  while for  $(k, F) \in T$  we have  $(m, \{n\}) + (k, F) = (m + n - \min(\{n\} * F), \{n\} * F)$  and  $\min(\{n\} * F) \leq n$  so 2(a) holds.

For 2(b) since  $A_{n+1}$  is open in  $S$  it suffices to show that  $(S + p) \cap A_{n+1} = \emptyset$ . Again  $\infty + p = \infty$  and  $(k, F) + (m, \{n\}) = (k + \min F - \min(F * \{n\}), F * \{n\})$  and  $\min(F * \{n\}) \leq n$ .

To verify (3) suppose we have a sequence  $\langle p_n \rangle_{n=0}^\infty$  with each  $p_n \in A_n$  and each  $p_n \in p_{n+1} + S$ . For each  $n \in \omega$  pick  $q_{n+1} \in S$  with  $p_n = p_{n+1} + q_{n+1}$ . For each  $n$  pick  $m_n \in \omega$  with  $p_n = (m_n, \{n\})$  and pick  $r_{n+1} \in \omega$  and  $F_{n+1} \in \mathcal{F}$  with  $q_{n+1} = (r_{n+1}, F_{n+1})$ . Since  $p_n = p_{n+1} + q_{n+1}$  we have  $\{n+1\} * F_{n+1} = \{n\}$  and hence  $F_{n+1} = \{n+1\}$ . But now for each  $n$  we have

$$(m_n, \{n\}) = (m_{n+1}, \{n+1\}) + (r_{n+1}, \{n+1\}) = (m_{n+1} + (n+1) - n, \{n\})$$

so that  $m_{n+1} = m_n - 1$ . But one cannot have an infinite decreasing sequence in  $\omega$ .  $\square$

Observe that by the left-right switch of Theorem 1.7 we have that  $S$  is not a right topological semigroup. (This is also easy to verify directly.)

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