Topology and its Applications 50 (1993) 147-182
North-Holland

Absorbing systems in infinite-dimensional manifolds

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Received 29 April 1991
Revised 20 March 1992

Abstract

Baars, J., H. Gladdines and J. van Mill, Absorbing systems in infinite-dimensional manifolds, Topology and its Applications 50 (1993) 147-182.

The aim of this paper is to define absorbing systems in infinite-dimensional manifolds and to derive some basic properties of them along the lines of Chapman. As an application we prove that for a countable nondiscrete Tychonov space X, if $C_p(X)$ is and $F_{\sigma\delta}$ subset of $\mathbb{R}^X$ then it is an $F_{\sigma\delta}$-absorber, and hence homeomorphic to the countable infinite product of copies of l_f^2. This generalizes a result of Dobrowolski, Marciszewski and Mogilski.

Keywords: Hilbert cube, Hilbert space, absorbing system, Z-set, function space, $F_{\sigma\delta}$.

AMS (MOS) Subj. Class.: 57N20, 54C35.

1. Introduction

There are basically two types of absorbing systems in infinite-dimensional topology. Absorbing systems of the first type are generalizations of the capsets of Anderson and the skeletoids of Bessaga and Pełczyński [2]; they are of interest if

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one wishes to prove homeomorphy of *pairs* of spaces. Absorbing systems of the second type are the ones of Bestvina and Mogilski [3]; they have proved to be useful for obtaining internal topological characterizations of various interesting spaces.

Dijkstra, van Mill and Mogilski [17] defined absorbing systems in the Hilbert cube Q; their systems are of the first type. Similar definitions were given earlier by various authors. We refer the reader to Bestvina and Mogilski [3], Cauty and Dobrowolski [8], Dijkstra and Mogilski [16] and Dobrowolski and Mogilski [20] for further details and references. The aim of this paper is to define similar absorbing systems in infinite-dimensional manifolds and to derive some basic properties of them along the lines of Chapman [9]. Some of our results may be known in a different setting.

One of the reasons that we find the absorbing systems that we study interesting, is that we are able to prove the existence of homeomorphisms that preserve the "layers" of the systems under consideration. The following result, due to Dijkstra, van Mill and Mogilski [17], explains this. Let 2^Q denote the hyperspace of Q.

Theorem 1.1. *There exists a homeomorphism* $\alpha : 2^Q \to \prod_{i=1}^{\infty} Q_i$, *where* $Q_i = Q$ *for every* i, *such that for every* $k \in \mathbb{N}$,

$$\alpha(\mathrm{Dim}_{\geq k}) = \underbrace{B \times \cdots \times B}_{k \text{ times}} \times Q \times Q \times \cdots,$$

where $\mathrm{Dim}_{\geq k} = \{A \in 2^Q : \dim A \geq k\}$ *and* $B = \{x \in Q : (\exists i)[|x_i| = 1]\}$.

In Section 12 we will show how our abstract theory can be applied in the concrete situation of function spaces. If X is a Tychonov space then $C_p(X)$ ($C_p^*(X)$) denotes the space of real-valued continuous functions (bounded real-valued continuous functions) endowed with the topology of pointwise convergence, i.e., we regard $C_p(X)$ and $C_p^*(X)$ as subspaces of $\mathbb{R}^X$.

In [23], van Mill showed that for a nonlocally compact countable metric space X, $C_p^*(X)$ is homeomorphic to σ_ω, where

$$\sigma_\omega = (l_f^2)^\infty \quad \text{and} \quad l_f^2 = \{x \in l^2 : x_i = 0 \text{ for all but finitely many } i\}$$

(l^2 denotes separable Hilbert space). Baars, de Groot, van Mill and Pelant obtained in [1] the same result for $C_p(X)$: for a nonlocally compact countable metric space X, $C_p(X)$ is homeomorphic to σ_ω. For a finite space X we have $C_p(X) = C_p^*(X) = \mathbb{R}^X$, and for an infinite countable discrete space X we have $C_p(X) = \mathbb{R}^X$ and $C_p^*(X) = \bigcup_{n=1}^{\infty} [-n, n]^X$. The question remained whether for a nondiscrete countable metric space X, $C_p(X)$ and $C_p^*(X)$ are both homeomorphic to σ_ω. Dobrowolski, Gul'ko and Mogilski [18] and Cauty [6] independently answered this question in the affirmative. Their result was later generalized as follows: if X is a nondiscrete countable Tychonov space, and if $C_p(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$ then $C_p(X)$ is homeomorphic to σ_ω, Dobrowolski, Marciszewski and Mogilski [19]. This result is "best possible" because if X is a nondiscrete Tychonov space then $C_p(X)$ is not a G_δ-, an F_σ-, or a $G_{\delta\sigma}$-subset of the $\mathbb{R}^X$ [13]. There are many examples for which $C_p(X)$ is an $F_{\sigma\delta}$. For example, this is so for any countable metric space X, but there are also nonmetric examples [13, 19].

When we talk about function spaces, it is convenient to represent the Hilbert cube Q by $\hat{\mathbb{R}}^\infty$, where $\hat{\mathbb{R}}$ stands for the compactification $[-\infty, \infty]$. If X is a countable space then $C_p(X) \subseteq \mathbb{R}^\infty$ is a subspace of the Hilbert cube $\hat{\mathbb{R}}^\infty$. In Dijkstra, van Mill and Mogilski [17], it was shown that for countable metric nondiscrete X, $C_p(X)$ is an $F_{\sigma\delta}$-absorber in $\hat{\mathbb{R}}^\infty$. This generalized the result of Dobrowolski, Gul'ko and Mogilski, and Cauty, cited above. It is natural to ask whether the result of Dobrowolski, Marciszewski and Mogilski, can be generalized in a similar way. We will answer this question in the affirmative in Section 12. Independently, the same result was obtained by Dijkstra and Mogilski [15]. Our results imply that for nondiscrete Tychonov spaces X and Y such that both $C_p(X)$ and $C_p(Y)$ are absolute $F_{\sigma\delta}$'s, we have that there exists an arbitrarily close to the identity homeomorphism from the Hilbert cube onto itself that maps $C_p(X)$ onto $C_p(Y)$.

2. Terminology

With the exception of Section 12, *all spaces under discussion are separable and metrizable.*

If X is a set then the identity function on X will be denoted by 1_X or, if confusion is unlikely, simply by 1. If f and g are mappings from X into Y then their distance

$$\sup\{d(f(x), g(x)) \colon x \in X\} \in [0, \infty]$$

will be denoted by $\hat{d}(f, g)$. We say that a map $f: X \to Y$ can be approximated arbitrarily closely by a map $g: X \to Y$ with property α, provided that for every $\varepsilon > 0$ there exists a map $g: X \to Y$ having α such that $\hat{d}(f, g) < \varepsilon$. The phrase "arbitrarily closely" will be used in this paper only if the domain of the map under consideration is compact. It is left as an exercise to the reader to verify that in this situation, the choice of the admissible metric d on Y is irrelevant.

It is well known that if X is an ANR then for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for every space Y, any two $\mathcal{V}$-close maps $f, g: Y \to X$ are $\mathcal{U}$-homotopic. It is also well known that if X is an ANR and if $\mathcal{U}$ is an open cover of X then for every closed subset A of a space Y and for every homotopy $H: A \times I \to X$ which is limited by $\mathcal{U}$, if H_0 can be extended to a continuous function $h_0: Y \to X$ then there exists a homotopy $\bar{H}: Y \times I \to X$ such that

(1) $\bar{H}$ is limited by $\mathcal{U}$.

(2) $\bar{H}_0 = h_0$ and $\bar{H} \mid (A \times I) = H$.

In particular, if $g: Y \to X$ is a map, and $f: A \to X$ is a map such that f and $g \mid A$ are "close", then f can be extended to a map $\bar{f}: Y \to X$ which is "close" to g. For details, see [24, Chapter 5]. These results will be used without explicit reference throughout the remaining part of this paper.

For basic facts on infinite-dimensional topology we refer the reader to Bessaga and Pełczyński [2], Chapman [10] and van Mill [24].

As usual $Q=\prod_{i=1}^{\infty}[-1,1]_i$ and l^2 denote the Hilbert cube and separable Hilbert space, respectively. In addition, s denotes the pseudo-interior of Q, i.e., $s=\{x\in Q\colon (\forall i\in\mathbb{N})[|x_i|<1]\}$. The complement B of s in Q is called the pseudo-boundary of Q. Finally, let Σ denote the linear span in l^2 of $\{x\in l^2\colon (\forall i\in\mathbb{N})[|x_i|\leq 1/i]\}$.

We sometimes identify s and $\mathbb{R}^{\infty}$ as well as Q and $\hat{\mathbb{R}}^{\infty}$, where $\hat{\mathbb{R}}$ stands for the compactification $[-\infty,\infty]$. If we talk about a linear subspace of s, we mean a linear subspace of $\mathbb{R}^{\infty}$.

Let A be a closed subset of a space X. We say that A is a *Z-set* provided that every map $f\colon Q\to X$ can be approximated arbitrarily closely by a map $g\colon Q\to X\backslash A$. The collection of all Z-sets in X will be denoted by $\mathscr{Z}(X)$. A countable union of Z-sets is called a σZ-set: the collection of all σZ-sets of X will be denoted by $\mathscr{Z}_\sigma(X)$. It is known that a compact subset of an s-manifold M is a Z-set in M, see Bessaga and Pełczyński [2] for details.

We will use the Z-set Unknotting Theorem: If E is a Q- or s-manifold and if $\alpha\colon A\to B$ is a homeomorphism between Z-sets A and B in E such that for some open cover $\mathscr{U}$ of E, α and 1_A are $\mathscr{U}$-homotopic, then α can be extended to a homeomorphism $\bar{\alpha}\colon E\to E$ which is $\mathscr{U}$-close to the identity. A space X has the *Z-approximation property* if for every compact space B, every map $f\colon B\to X$ that restricts to a Z-embedding on some compact subset K of B, can be approximated arbitrarily closely by a Z-embedding $g\colon B\to X$ such that $g|K=f|K$. Moreover X is said to have the *disjoint-cells property* if for every $n\in\mathbb{N}$, every map $f\colon I^n\times\{0,1\}\to X$ can be approximated arbitrarily closely by a map $g\colon I^n\times\{0,1\}\to X$ such that $g(I^n\times\{0\})\cap g(I^n\times\{1\})=\emptyset$. Observe that every infinite product of nondegenerate spaces has the disjoint-cells property. It is known that every topologically complete ANR with the disjoint-cells property has the Z-approximation property [27, Theorem 7.3.5]. It is well known that every Q- or s-manifold M has the disjoint-cells property. This follows for example easily from the fact that $M\times Q\approx M$. See Bessaga and Pełczyński [2] and Chapman [10] for details.

The following result is probably well known: its proof is included for the sake of completeness.

Lemma 2.1. *Let X be an* ANR *and let Z be an* ANR *with the Z-approximation property. Then $Y=Z\times X$ has the Z-approximation property.*

Proof. Let B be a compact space, and let $f=(f_1,f_2)\colon B\to Y$ be a map that restricts to a Z-embedding on some compact subset $K\subseteq B$. Write $B\backslash K$ as $\bigcup_{i=0}^{\infty}C_i$, where $C_0=\emptyset$, each C_i is compact, and $C_0\subseteq C_1\subseteq\cdots$. Let $\varepsilon>0$. Put $f_0=f$, and suppose that we constructed a map $f_i=(f_1^i,f_2^i)\colon B\to Y$ such that

(1) $f_i|K\cup C_{i-1}=f_{i-1}|K\cup C_{i-1}$,

(2) $f_i(C_i)\cap f(K)=\emptyset$,

(3) $f_1^i|C_i$ is a Z-embedding,

(4) $\hat{d}(f_i, f_{i-1}) < 2^{-i} \cdot \varepsilon$.

Since Y is an ANR, we can approximate $f_i|C_{i+1}$ arbitrarily closely by a map $\xi = (\xi_1, \xi_2): C_{i+1} \to Y \backslash f(K)$ such that $\xi|C_i = f_i|C_i$ [24, Lemma 7.2.1. and Corollary 7.2.6]. Since Z has the Z-approximation property, we can approximate ξ_1 arbitrarily closely by a Z-embedding η such that $\eta|C_i = \xi_1|C_i$. Observe that $(\eta, \xi_2)(C_{i+1})$ is a Z-set in Y because it projects onto a Z-set in the Z-coordinate direction. We can moreover assume that the range of (η, ξ_2) misses $f(K)$, and that (η, ξ_2) is also close to $f_i|C_{i+1}$ that it can be extended to a map $f_{i+1}: B \to Y$ satisfying our inductive hypotheses.

Now put $g = \lim_{i\to\infty} f_i$. Then g is one-to-one, and hence an embedding, and its range is contained in a σZ-set, hence it is a Z-embedding, Since $\hat{d}(f, g) < \varepsilon$ and $g|K = f|K$, we are done. $\square$

For a space X, $\mathscr{H}(X)$ denotes the group of all homeomorphisms of X. If X is complete, and if $(f_n)_n$ is an inductively constructed sequence of homeomorphisms of X such that each f_n can be chosen as close to the identity as we please then we can choose them in such a way that $L\prod_{i=1}^{\infty} f_i = \lim_{i\to\infty} f_i \circ \cdots \circ f_1$ is a homeomorphism of X (see e.g. Dijkstra [14, Lemma 1.1.2]). We call $L\prod_{i=1}^{\infty} f_i$ the *infinite left-product* of the sequence $(f_i)_i$.

Let X be a space, and let $A \subseteq X$ be closed. Then there exists an open cover $\mathscr{U}$ of $X \backslash A$ such that for any homeomorphism $h: X\backslash A \to X\backslash A$ that is $\mathscr{U}$-close to the identity the function $\bar{h} = h \cup 1_A: X \to X$ is a homeomorphism. Indeed, it is easily seen that a so-called Dugundji cover $\mathscr{U}$ of $X\backslash A$ is as desired. For details, see [24, Lemma 1.4.12]. We will use this often without explicit reference throughout the remaining part of this paper.

By a *Borel class of spaces* we mean a class of the form F_σ, G_δ, $F_{\sigma\delta}$, $G_{\delta\sigma}$, etc. If $\mathscr{M}$ is a class of spaces then $\mathscr{M}_\delta$ denotes the class of spaces that are homeomorphic to a countable intersection of elements of $\mathscr{M}$ that are contained in Q.

If A_n $(n \in \mathbb{N})$ is a decreasing sequence of subsets of a set X, then $\bigcap_{n=1}^{\infty} A_n$ will sometimes be denoted by A_∞.

Let X be a space and let $x \in X^\infty$. Define the *weak product* $W(X, x)$ of X with basepoint x to be the subspace

$$W(X, x) = \{p \in X^\infty\colon\ p_n = x_n \text{ for all but finitely many } n\}$$

of X^∞.

If $A \subseteq X$ then A° denotes the interior of A in X. The Cantor set $\{0, 1\}^{\mathbb{N}}$ will be denoted by C.

3. Definitions

For every ordered set Γ, $\vec{\mathscr{M}}_\Gamma$ stands for a system $(\mathscr{M}_\gamma)_{\gamma\in\Gamma}$ of classes of spaces which are all topological and closed hereditary. An *$\mathscr{M}_\Gamma$-system* in a space X is an order preserving (with respect to inclusion) indexed collection $(A_\gamma)_{\gamma\in\Gamma}$ of subsets of X such that $A_\gamma \in \mathcal{M}_\gamma$ for every γ.

Let $\mathscr{X}=(X_\gamma)_{\gamma\in\Gamma}$ be an order preserving indexed collection of subsets of a space E.

Definition 3.1. The system $\mathscr{X}$ is called *strongly $\mathcal{M}_\Gamma$-universal* in E if for every $\mathcal{M}_\Gamma$-system $(A_\gamma)_{\gamma\in\Gamma}$ in Q, every map $f:Q\to E$ that restricts to a Z-embedding on some compact subset K of Q, can be approximated arbitrarily closely by a Z-embeddding $g:Q\to E$ such that $g|K=f|K$ while moreover for every $\gamma\in\Gamma$ we have $g^{-1}(X_\gamma)\backslash K=A_\gamma\backslash K$.

This definition is not essentially different from the corresponding one in Dijkstra, van Mill and Mogilski [17].

In case E is an ANR, Q can be "replaced" by any compact space B. In the process of proving this, the following result will be helpful.

Lemma 3.2. *Let E be an* ANR, *$Z\in\mathscr{Z}(E)$, and let $\mathscr{X}$ be strongly $\mathcal{M}_\Gamma$-universal in E. Then for every M_Γ-system $(A_\gamma)_{\gamma\in\Gamma}$ in Q, every map $f:Q\to E$ that restricts to a Z-embedding on some compact subset K of Q such that $f(K)\subseteq Z$, can be approximated arbitrarily closely by a Z-embedding $g:Q\to E$ such that $g|K=f|K$, $g(Q\backslash K)\cap Z=\emptyset$, and for every $\gamma\in\Gamma$, $g^{-1}(X_\gamma)\backslash K=A_\gamma\backslash K$.*

Proof. Write $Q\backslash K$ as $\bigcup_{i=0}^{\infty} C_i$, where $C_0=\emptyset$, each C_i is compact, and $C_0\subseteq C_1\subseteq\cdots$. Let $\varepsilon>0$. Put $f_0=f$, and suppose that we constructed a Z-embedding $f_i:Q\to E$ such that

(1) $f_i|K\cup C_{i-1}=f_{i-1}|K\cup C_{i-1}$,

(2) $f_i(C_i)\cap Z=\emptyset$,

(3) $(\forall\gamma\in\Gamma)[f_i^{-1}(X_\gamma)\cap C_i=A_\gamma\cap C_i]$,

(4) $\hat{d}(f_i,f_{i-1})<2^{-i}\cdot\varepsilon$.

Since E is an ANR, we can approximate $f_i|C_{i+1}$ arbitrarily closely by a map $\xi:C_{i+1}\to E\backslash Z$ such that $\xi|C_i=f_i|C_i$ [2, 4, Lemma 7.2.1 and Corollary 7.2.6]. Moreover, again since E is an ANR, we can assume that ξ and $f_i|C_{i+1}$ are so close that ξ can be extended to a map $\eta:Q\to E$ that restricts to f_i on K, and is as close to f_i as we please. Now observe that $\eta(C_{i+1})$ and Z have positive distance. By the strong $\mathcal{M}_\Gamma$-universality of $\mathscr{X}$, we can consequently approximate η by a Z-embedding $f_{i+1}:Q\to E$ such that

- $f_{i+1}|K\cup C_i=\eta|K\cup C_i=f_i|K\cup C_i$,
- $f_{i+1}(C_{i+1})\cap Z=\emptyset$,
- $(\forall\gamma\in\Gamma)[f_{i+1}^{-1}(X_\gamma)\backslash(K\cup C_i)=A_\gamma\backslash(K\cup C_i)]$, so $f_{i+1}^{-1}(X_\gamma)\cap C_{i+1}=A_\gamma\cap C_{i+1}$,
- η and f_{i+1} are as close as we please.

Now put $g=\lim_{i\to\infty}f_i$. Then g is one-to-one, and is therefore an embedding. Moreover, $g(Q)$ is a countable union of Z-sets, and therefore a Z-set itself. Clearly,

$\hat{d}(f, g) < \varepsilon$ and $g(Q \backslash K) \subseteq E \backslash Z$. Since for every $\gamma \in \Gamma$,

$$
\begin{aligned}
g^{-1}(X_\gamma) \backslash K &= \bigcup_{i=1}^{\infty} g^{-1}(X_\gamma) \cap C_i \\
&= \bigcup_{i=1}^{\infty} f_i^{-1}(X_\gamma) \cap C_i \\
&= \bigcup_{i=1}^{\infty} A_\gamma \cap C_i \\
&= A_\gamma \backslash K,
\end{aligned}
$$

we are done. □

Proposition 3.3. *Let E be an* ANR. *The following statements are equivalent*:

(1) *$\mathscr{X}$ is strongly $\mathscr{M}_\Gamma$-universal in E*;

(2) *for every compact space B and for every $\mathscr{M}_\Gamma$-system $(A_\gamma)_{\gamma \in \Gamma}$ in B, every map $f: B \to E$ that restricts to a Z-embedding on some compact subset K of B, can be approximated arbitrarily closely by a Z-embedding $g: B \to E$ such that $g \mid K = f \mid K$ while moreover for every $\gamma \in \Gamma$ we have $g^{-1}(X_\gamma) \backslash K = A_\gamma \backslash K$.*

Proof. (2)⇒(1) is trivial. For (1)⇒(2), let B be an arbitrary compact space, $(A_\gamma)_{\gamma \in \Gamma}$ an $\mathscr{M}_\Gamma$-system in B, and $f: B \to E$ a map that restricts to a Z-embedding on some compact subset K of B. We may assume that B is a closed subset of Q. Let $\varepsilon > 0$. Since E is an ANR, we can extend f to a map $\bar{f}: U \to E$, where U is an open neighborhood of B in Q. By compactness of B there exist finitely many infinite-dimensional subcubes $Q_1, \ldots, Q_n$ of Q, such that $B \subseteq \hat{Q} = \bigcup_{i=1}^n Q_i \subseteq U$. Put $h = \bar{f} \mid \hat{Q}$, $Q_0 = \emptyset$ and $h_0 = h$. For every $0 \leq i \leq n$, put $K_i = K \cup Q_0 \cup \cdots \cup Q_i$. Then $\hat{Q} = K_n$. Assume that we constructed $h_i: \hat{Q} \to E$ $(0 \leq i < n)$ such that

(1) $h_i \mid K_{i-1} = h_{i-1} \mid K_{i-1}$,

(2) $h_i \mid K_i$ is a Z-embedding,

(3) $(\forall \gamma \in \Gamma)[(h_i^{-1}(X_\gamma) \cap \bigcup_{t=0}^i Q_t) \backslash K = (A_\gamma \cap \bigcup_{t=0}^i Q_t) \backslash K]$,

(4) $\hat{d}(h_{i-1}, h_i) < \varepsilon / n$.

By Lemma 3.2, we can approximate $h_i \mid Q_{i+1}$ arbitrarily closely by a Z-embedding $\eta: Q_{i+1} \to E$ such that

- $\eta \mid K_i \cap Q_{i+1} = h_i \mid K_i \cap Q_{i+1}$,
- $\eta(Q_{i+1} \backslash K_i) \cap h_i(K_i) = \emptyset$,
- $(\forall \gamma \in \Gamma)[\eta^{-1}(X_\gamma) \backslash K_i = (Q_{i+1} \cap A_\gamma) \backslash K_i]$.

(Observe that all classes under consideration are closed hereditary.) Now since E is an ANR we may assume that $\eta \cup (h_i \mid K_i): K_{i+1} \to E$ is so close to $h_i \mid K_{i+1}$ that it can be extended to a map $h_{i+1}: \hat{Q} \to E$ satisfying our inductive requirements.

Now put $g = h_n$. Then it easily follows that $g \mid B$ is as required. □

Definition 3.4. The system $\mathscr{X}$ is called *$\mathcal{M}_\Gamma$-absorbing* in E if:

(1) $\mathscr{X}$ is an $\mathcal{M}_\Gamma$-system;

(2) $\bigcup_{\gamma\in\Gamma} X_\gamma \subseteq \bigcup_{i=1}^{\infty} A_i$, where each A_i is a compact Z-set in E;

(3) $\mathscr{X}$ is strongly $\mathcal{M}_\Gamma$-universal in E.

In Example 10.6 we will make clear why we require the A_i to be compact.

If Γ contains one element only then it is confusing to talk about systems. In that case there is only one class of spaces under consideration, say $\mathcal{M}$, and $\mathscr{X}$ contains only one set. In this situation we use the terms strongly $\mathcal{M}$-universal *set* and *$\mathcal{M}$-absorber.*

Let $\mathcal{M}$ be a Borel class of spaces, and let M be an strongly $\mathcal{M}$-universal in a space X. It follows straight from the definition that $X\backslash M$ is a strongly $\mathcal{N}$-universal in X, where $\mathcal{N}$ is the dual class of $\mathcal{M}$, i.e., if e.g. $\mathcal{M} = G_\delta$ then $\mathcal{N} = F_\sigma$, etc. This triviality will be used without explicit reference several times in the forthcoming.

Let Γ_1 and Γ_2 be two disjoint ordered sets. Denote by Γ the ordered set $\Gamma_1 \cup \Gamma_2$, where both Γ_1 and Γ_2 are endowed with their initial ordering and each element of Γ_1 precedes every element of Γ_2. In this situation we use $(\mathcal{M}_{\Gamma_1}, \mathcal{M}_{\Gamma_2})$ as an abbreviation for $\mathcal{M}_\Gamma$. In addition, if $\mathscr{X}_i = (X_\gamma)_{\gamma\in\Gamma_i}$ is an indexed collection of subsets of a space E for $i = 1, 2$, then $(\mathscr{X}_1, \mathscr{X}_2)$ will sometimes be used as an abbreviation for the system $(X_\gamma)_{\gamma\in\Gamma_1} \cup (X_\gamma)_{\gamma\in\Gamma_2}$.

Definition 3.5. A subset A of a space X is *locally homotopy negligible in X* if for every map $f: M \to X$, where M is any ANR, and for every open cover $\mathcal{U}$ of X there exists a homotopy $H: M \times I \to X$ such that $\{H(\{x\}\times I)\}_{x\in M}$ refines $\mathcal{U}$, $H_0 = f$ and $H(M \times (0, 1]) \subseteq X\backslash A$.

By a result of Toruńczyk [22], if X is an ANR, for a set $A \subseteq X$ to be locally homotopy negligible it suffices to consider maps $f: M \to X$, where $M \in \{\{pt\}, I, I^2, \ldots, I^n, \ldots\}$. Moreover, if X is an ANR and if $A \subseteq X$ is locally homotopy negligible, then for every map $f: B \to X$, where B is *any* space, and for every open cover $\mathcal{U}$ of X there exists a homotopy $H: B \times I \to X$ such that $\{H(\{x\}\times I)\}_{x\in B}$ refines $\mathcal{U}$, $H_0 = f$ and $H(B \times (0, 1]) \subseteq X\backslash A$. To see this, first observe that B can be thought of as being a closed subset of an AR E [24, Lemma 1.2.3(1) and Theorem 1.5.1]. Since X is an ANR there is a neighborhood U of B in E such that f can be extended to a continuous function $\bar{f}: U \to X$. Since U is an ANR [24, Theorem 5.4.1], we can find a homotopy for U of which the restriction to B is as desired.

Observe that if X is an ANR and $A \subseteq X$ then A is locally homotopy negligible in X iff there exist small homotopies $H: X \times I \to X$ such that H_0 is the identity and $H(X \times (0, 1]) \subseteq X\backslash A$.

It is easy to see that if X is an ANR and A is a σZ-set in X then A is locally homotopy negligible (Toruńczyk [22, Corollary 3.3]).

We let $\mathscr{C}$ denote the class of all compact spaces. Observe that all concepts that were defined in this section are *topological* in the sense that if the system $\mathscr{A}$ in the space E has property $\mathfrak{P}$ in E then for every homeomorphism $h: E \to F$, the image system $h(\mathscr{A})$ has property $\mathfrak{P}$ in F.

4. Complements of strongly $\mathscr{M}$-universal sets

As stated in the introduction, our aim is to study $\mathscr{M}$-absorbing sets in infinite-dimensional manifolds. Since such spaces are ANR's, in this section we will study $\mathscr{M}$-absorbing sets in ANR's first.

We begin by deriving the somewhat surprising result that the complement of a strongly $\mathscr{C}$-universal set is again strongly $\mathscr{C}$-universal.

Theorem 4.1. *Let E be an* ANR *and $\mathscr{M}$ a class of spaces that contains the collection of all compact spaces. Then for every strongly $\mathscr{M}$-universal set A in E, $E \backslash A$ is strongly $\mathscr{C}$-universal in E.*

Proof. Let B be compact, and let $f: B \to E$ be a continuous function that restricts to a Z-embedding on some compact subset K. In addition, let $C \subseteq B$ be compact. Since A is strongly $\mathscr{C}$-universal, we can approximate f arbitrarily closely by a Z-embedding $g: B \to E$ such that $g \mid K = f \mid K$ while moreover $g(B \backslash K) \subseteq E \backslash A$ (apply the definition of strong $\mathscr{C}$-universality with the closed subset of B that has to be "absorbed" by A equal to the empty set). Now again since A is strongly $\mathscr{C}$-universal we may approximate g arbitrarily closely by a Z-embedding $h: B \to E$ that restricts to g on $K \cup C$, while moreover $h^{-1}(A) \backslash (K \cup C) = B \backslash (K \cup C)$. We conclude that h is as required. $\square$

Remark 4.2. Theorem 4.1 is false when $\mathscr{M}$ only contains the class $\mathscr{F}\mathscr{D}$ of all *finite-dimensional* compact spaces. This can be seen as follows. The Hilbert cube Q contains a strongly $\mathscr{F}\mathscr{D}$-universal subset σ which is a countable union of finite-dimensional compacta (see Bessaga and Pełczyński [2]). But its complement is not strongly $\mathscr{F}\mathscr{D}$-universal. For if it were, then it could "absorb" the empty set, so that the identity from Q to Q could be approximated by an embedding from Q into σ. But this is impossible because σ is a countable union of finite-dimensional compacta, but Q is not.

Proposition 4.3. *Let E be an* ANR *and $\mathscr{M}$ a class of spaces that contains the collection of all (finite-dimensional) compact spaces and let A be a strongly $\mathscr{M}$-universal set in E. In addition, let B be (finite-dimensional and) compact, $K \subseteq B$ be closed and let $f: B \to E$ be continuous. Then for every $\varepsilon > 0$ there exist continuous functions $g, h: B \to E$ such that*

(1) $\hat{d}(f, g), \hat{d}(f, h) < \varepsilon$,

(2) $g(B\backslash K)\subseteq A$ *and* $h(B\backslash K)\subseteq E\backslash A$,
(3) $g|K=f|K=h|K$.

Proof. Write $B\backslash K$ as $\bigcup_{i=0}^{\infty} C_i$, where $C_0=\emptyset$, each C_i is (finite-dimensional and) compact, and $C_0\subseteq C_1\subseteq\cdots$. Put $g_0=f=h_0$, and assume that we constructed g_i, $h_i: B\to E$ such that
(1) $\hat{d}(g_i, g_{i-1})<2^{-i}\cdot\varepsilon$ and $\hat{d}(h_i, h_{i-1})<2^{-i}\cdot\varepsilon$,
(2) $g_i|C_{i-1}=g_{i-1}|C_{i-1}$ and $h_i|C_{i-1}=h_{i-1}|C_{i-1}$,
(3) $g_i(C_i)\in\mathscr{Z}(E)$ and is contained in A,
(4) $h_i(C_i)\in\mathscr{Z}(E)$ and is contained in $E\backslash A$,
(5) $g_i|C_i$ and $h_i|C_i$ are embeddings,
(6) $g_i|K=f|K=h_i|K$.
Since A is strongly $\mathscr{C}$-universal ($\mathscr{F}\mathscr{D}$-universal), the function $g_i|C_{i+1}$ can be approximated arbitrarily closely by a Z-embedding $\xi: C_{i+1}\to A$ such that $\xi|C_i=g_i|C_i$ (Proposition 3.3). Observe that C_{i+1} is being absorbed by A. Since E is an ANR, we can moreover assume that ξ and $g_i|C_{i+1}$ are so close that $\xi\cup(g_i|K): C_{i+1}\cup K\to E$ can be extended to a continuous function $g_{i+1}: B\to E$ such that $\hat{d}(g_{i+1}, g_i)<2^{-(i+1)}\cdot\varepsilon$.

Now h_{i+1} can be constructed precisely such as g_{i+1} by absorbing the empty set instead of C_{i+1}.

Put $g=\lim_{i\to\infty} g_i$ and $h=\lim_{i\to\infty} h_i$, respectively. Then g and h are clearly as required. □

Corollary 4.4. *Let E be an* ANR *and $\mathscr{M}$ a class of spaces that contains the class of all finite-dimensional compact spaces. Then for every strongly $\mathscr{M}$-universal set A in E, A and $E\backslash A$ are locally homotopy negligible in E.*

Proof. Fix $n\in\mathbb{N}$ for the rest of the proof. Let $f: I^n\to E$ be continuous, and let $\varepsilon>0$. Define the homotopy $H: I^n\times I\to E$ by $H(x,t)=f(x)$, $x\in I^n$, $t\in I$. Now apply Proposition 4.3. □

Corollary 4.5. *Let E be an* A(N)R *and $\mathscr{M}$ a class of spaces that contains the collection of all compact spaces. Then for every strongly $\mathscr{M}$-universal set A in E, the spaces A and $E\backslash A$ are* A(N)R's. *Moreover, A, $E\backslash A$ and E have the same homotopy type.*

Proof. Since $E\backslash A$ is locally homotopy negligible (Corollary 4.4) and E is an A(N)R, A is an A(N)R as well (Toruńczyk [22]). That A and E have the same homotopy type follows easily from the existence of (small) homotopies "from E into A". Since $E\backslash A$ is strongly $\mathscr{C}$-universal by Theorem 4.1, the statements for $E\backslash A$ follow by the same reasoning. □

Corollary 4.6. *Let E be an* ANR *and $\mathcal{M}$ a class of spaces that contains the collection of all compact spaces and let Ω be a strongly $\mathcal{M}$-universal set in E. Then every subset of Ω which is closed in E is a Z-set in E and is a Z-set in Ω.*

Proof. Let A be a subset of Ω which is closed in E. That $A \in \mathscr{Z}(E)$ is clear since A is locally homotopy negligible (Corollary 4.4). We will now show that $A \in \mathscr{Z}(\Omega)$. To this end, let $f: Q \to \Omega$ be continuous, and let $\varepsilon > 0$. Since, as we just observed, $A \in \mathscr{Z}(E)$, there is a map $g: Q \to E\setminus A$ such that $\hat{d}(f, g) < \frac{1}{2}\varepsilon$. Since $d(A, g(Q)) > 0$, by Proposition 4.3, it now follows that there exists a map $h: Q \to \Omega\setminus A$ such that $\hat{d}(h, g) < \frac{1}{2}\varepsilon$. Then h is clearly as required. □

5. F_σ-absorbers in infinite-dimensional manifolds

As stated in the introduction, we are interested in absorbing systems in infinite-dimensional manifolds. To be specific, in absorbing systems in infinite-dimensional manifolds modeled on the spaces Q, s and Σ. Observe that manifolds modeled on Q and s are topologically complete, whereas manifolds modeled on Σ are σ-compact and noncompact, and hence incomplete. So in some results to come, we will have to distinguish between two cases: the "complete case" and the "incomplete case". From now on, by an *infinite-dimensional manifold* we mean a manifold modeled on one of the spaces Q, s or Σ.

In this section we are especially interested in F_σ-absorbing sets in manifolds modeled on Q and s. Throughout this section, let M denote a Q- or s-manifold.

Definition 5.1. An element $A \in \mathscr{Z}_\sigma(M)$ is called a *capset* for M provided that A can be written as $\bigcup_{n=1}^{\infty} A_n$, where each A_n is compact, while moreover the following absorption property holds: $(\forall \varepsilon > 0)$ $(\forall$ compact $K \subseteq M)$ $(\forall n \in \mathbb{N})$ $(\exists m \in \mathbb{N}$ and an embedding $h: K \to A_m)$ such that:

(1) $\hat{d}(h, 1) < \varepsilon$,

(2) $h | A_n \cap K = 1$.

It is well known that M contains a capset (Chapman [9]). In addition, $B = Q\setminus s$ is a capset for Q. Moreover, capsets have the following properties:

(I) If A and B are capsets for M then for every open cover $\mathscr{U}$ of M there exists a $\mathscr{U}$-close to the identity homeomorphism $h: M \to M$ such that $h(A) = B$.

(II) If A is a capset for M and if Z is a Z-set in M then $A\setminus Z$ is a capset for M.

(III) If A is a capset for M and if $B \in \mathscr{Z}_\sigma(M)$ is σ-compact, then $A \cup B$ is a capset.

(IV) If $U \subseteq M$ is open and if A is a capset for M then $A \cap U$ is a capset for U.

(V) If A is a capset for M then A is a Σ-manifold.

For proofs of these facts, we refer the reader to Chapman [9].

Lemma 5.2. *Let E be an* ANR *and let $A \subseteq E$ be an F_σ which is a strongly $\mathscr{C}$-universal set in E. Then A is a σZ-set.*

Proof. Since every closed subset of A which is closed in E is a Z-set in E (Corollary 4.6), this is clear. $\square$

We emphasize that in the following result we do *not* require the set Ω to be a σZ-set.

Theorem 5.3. *Let M be a Q- or s-manifold, and let Ω be a σ-compact subset of M. Then the following statements are equivalent*:

(1) *$M \backslash \Omega$ is strongly $\mathscr{C}$-universal,*
(2) *Ω is strongly $\mathscr{C}$-universal,*
(3) *Ω is strongly F_σ-universal,*
(4) *Ω is an F_σ-absorber,*
(5) *Ω is a capset.*

Proof. We first prove that (5)$\Rightarrow$(4). Let A be an F_σ-subset of Q. In addition, let $\varepsilon > 0$ and let $f: Q \to M$ be a map that restricts to a Z-embedding on some compact subset $K \subseteq Q$. Since M has the disjoint-cells property, by [24, Theorem 7.3.5] there is a Z-embedding $g: Q \to M$ such that

(1) $g | K = f | K$,
(2) $\hat{d}(f, g) < \frac{1}{2}\varepsilon$,
(3) $g(Q \backslash K) \cap \Omega = \emptyset$.

By the above, we know that $\Omega \backslash g(K)$ and $(\Omega \cup g(A)) \backslash g(K)$ are capsets in M. By choosing a Dugundji cover for $M \backslash g(K)$ with mesh smaller than $\frac{1}{2}\varepsilon$, an application of (IV) followed by (I) above yields the existence of a homeomorphism $h \in \mathscr{H}(M)$ such that

(4) $h | g(K) = 1$,
(5) $\hat{d}(h, 1) < \frac{1}{2}\varepsilon$,
(6) $h((\Omega \cup g(A)) \backslash g(K)) = \Omega \backslash g(K)$.

Then $h \circ g: Q \to M$ is the required approximation of f. For this, it suffices to verify that

$$(h \circ g)^{-1}(\Omega) \backslash K = g^{-1}(\Omega \cup g(A) \backslash g(K)) = A \backslash K.$$

Next observe that (4)$\Rightarrow$(3)$\Rightarrow$(2) are trivialities. Moreover, (1)$\Leftrightarrow$(2) follows from Corollary 4.1.

Finally, for (2)$\Rightarrow$(5) first observe that there exists a capset for M (Chapman [9]) and that we already proved that every capset is strongly $\mathscr{C}$-universal. Since by Lemma 5.2 every strongly $\mathscr{C}$-universal σ-compact set is a σZ-set, it suffices to prove the following: for any two σ-compact σZ-sets A and B in M such that both A and B are strongly $\mathscr{C}$-universal, there exists a homeomorphism $h: M \to M$ such that $h(A) = B$. This follows from Theorem 8.2. $\square$

We finish this section with two rather technical results, that will be important later in Section 12. The proof of the first lemma is implicit in Dijkstra, van Mill and Mogilski [17, the proof of Proposition 6.1].

Lemma 5.4. *Let* $A \subseteq B \subseteq Q$ *and* $A_2 \subseteq A_1 \subseteq Q$. *Suppose* (A_2, A_1) *is an* (F_σ, F_σ)-*absorbing pair,* A *is an* F_σ-*absorber and* B *is a* σZ-*set. Then for each* $\varepsilon > 0$ *there is a homeomorphism* $h: Q \to Q$ *such that* $\hat{d}(h, 1) < \varepsilon$ *and* $h(A) = A_2$ *and* $h(B) \subseteq A_1$.

Lemma 5.5. *Let* A *be an* F_σ-*absorber and* B *a* σZ-*set with* $A \subseteq B \subseteq Q$. *Let* C *be a* σ-*compactum. Then for each* $f: Q \to Q$ *such that for some compact* $K \subseteq Q$, $f|K$ *is a* Z-*embedding and for each* $\varepsilon > 0$ *there is a* Z-*embedding* $g: Q \to Q$ *such that*

(1) $\hat{d}(f, g) < \varepsilon$,
(2) $g^{-1}(A) \backslash K = C \backslash K$,
(3) $g(Q \backslash (C \cup K)) \cap B = \emptyset$ *and*
(4) $g|K = f|K$.

Proof. Let (A_2, A_1) be an (F_σ, F_σ)-absorbing pair in Q. That such a pair exists was proved in Dijkstra, van Mill and Mogilski [17, the proof of Proposition 6.1]. See also the proof of the first part of Proposition 10.2 in the present paper.

Let $h: Q \to Q$ be a homeomorphism such that $h(A) = A_2$, $h(B) \subseteq A_1$ and $\hat{d}(h, 1) < \varepsilon/3$. Let $g: Q \to Q$ be a Z-embedding such that

$$g^{-1}(A_1) \backslash K = g^{-1}(A_2) \backslash K = C \backslash K,$$

$$g|K = (h \circ f)|K,$$

and

$$\hat{d}(g, h \circ f) < \varepsilon/3.$$

Let $p = h^{-1} \circ g$. Then $p|K = f|K$. Furthermore

$$\begin{aligned} \hat{d}(p, f) = \hat{d}(h^{-1} \circ g, f) &\leq \hat{d}(h^{-1} \circ g, g) + \hat{d}(g, h \circ f) + \hat{d}(h \circ f, f) \\ &\leq \hat{d}(h^{-1}, 1) + \hat{d}(g, h \circ f) + \hat{d}(h, 1) \\ &< \varepsilon. \end{aligned}$$

Finally one can easily verify that $p^{-1}(A) \backslash K = C \backslash K$ and $p(Q \backslash (C \cup K)) \cap B = \emptyset$. □

6. Strongly $\mathcal{M}_\Gamma$-universal systems in certain subsets are strongly $\mathcal{M}_\Gamma$-universal in the whole space

In this section we investigate the question when strongly $\mathcal{M}_\Gamma$-universal systems in certain subsets of an ANR E are in fact strongly $\mathcal{M}_\Gamma$-universal in E.

Proposition 6.1. *Let E be an* ANR *and let $F \subseteq E$ be such that $E \backslash F$ is locally homotopy negligible. In addition, let $\mathscr{X} = (X_\gamma)_{\gamma \in \Gamma}$ be strongly $\mathscr{M}_\Gamma$-universal in F. Finally, let B be a compact space, $(A_\gamma)_{\gamma \in \Gamma}$ an $\mathscr{M}_\Gamma$-system in B, and $f: B \to E$ a continuous function that restricts to a Z-embedding on some compact subset K of B. Then for every $\varepsilon > 0$ there exists a Z-embedding $g: B \to E$ such that*

(1) $\hat{d}(f, g) < \varepsilon$,
(2) $g(B \backslash K) \subseteq F$,
(3) $(\forall \gamma \in \Gamma)[g^{-1}(X_\gamma) \backslash K = A_\gamma \backslash K]$,
(4) $g | K = f | K$.

Proof. Let d be an admissible metric on E that is bounded by 1. Write $B \backslash K$ as countable union of compact sets T_i such that $\emptyset = T_0 \subseteq T_1 \subseteq \cdots$. Put $g_0 = f$, and assume that we constructed $g_i: B \to E$ such that

(1) $\hat{d}(g_i, g_{i-1}) < 2^{-i} \cdot \varepsilon$,
(2) $g_i(T_i) \in \mathscr{Z}(E)$, $g_i | T_i$ is an embedding, and $g_i(T_i) \subseteq F \backslash f(K)$,
(3) $g_i | K = f | K$,
(4) $g_i | T_{i-1} = g_{i-1} | T_{i-1}$,
(5) $(\forall \gamma \in \Gamma)[g_i^{-1}(X_\gamma) \cap T_i = A_\gamma \cap T_i]$.

Since $f(K) \in \mathscr{Z}(E)$ and $g_i(T_i) \cap f(K) = \emptyset$, we may approximate $g_i | T_{i+1}$ arbitrarily closely by a continuous function $\zeta: T_{i+1} \to E$ such that ζ restricts to g_i on T_i while moreover $\zeta(T_{i+1}) \cap f(K) = \emptyset$ [24, Lemma 7.2.1 and Corollary 7.2.6].

There is a "small" homotopy $H: E \times I \to E$ such that $H_0 = 1$ and $H(E \times (0, 1]) \subseteq F$. For every n, define $\zeta_n: T_{i+1} \to F$ by

$$\zeta_n(x) = H\left(\zeta(x), \frac{d(x, T_i)}{n}\right).$$

Then each ζ_n restricts to g_i on T_i, and $\lim_{n \to \infty} \zeta_n = \zeta$. There consequently exists $n \in \mathbb{N}$ such that $\zeta_n(T_{i+1}) \cap f(K) = \emptyset$. Since $\mathscr{X} = (X_\gamma)_{\gamma \in \Gamma}$ is strongly $\mathscr{M}_\Gamma$-universal in F, we can approximate ζ_n arbitrarily closely by a Z-embedding $\eta: T_{i+1} \to F$ such that $\eta | T_i = g_i$ and $\eta(T_{i+1}) \cap f(K) = \emptyset$ while moreover, for every $\gamma \in \Gamma$,

$$\eta^{-1}(X_\gamma) \backslash T_i = (A_\gamma \cap T_{i+1}) \backslash T_i$$

(Proposition 3.3). By (5) this clearly implies that for every $\gamma \in \Gamma$,

$$\eta^{-1}(X_\gamma) = A_\gamma \cap T_{i+1}.$$

Since $\eta(T_{i+1}) \in \mathscr{Z}(F)$ is compact and $E \backslash F$ is locally homotopy negligible, it follows that $\eta(T_{i+1})$ is a Z-set in E. Since E is an ANR, we can choose η so close to $g_i | T_{i+1}$ that the function $(f | K) \cup \eta: K \cup T_{i+1} \to E$ can be extended to a continuous function $g_{i+1}: B \to E$ such that $\hat{d}(g_{i+1}, g_i) < 2^{-(i+1)} \cdot \varepsilon$. Then g_{i+1} is clearly as required.

Now put $g = \lim_{i \to \infty} g_i$. Then by (1), g is continuous and $\hat{d}(f, g) < \varepsilon$, and by (3), $g | K = f | K$. Moreover, by (2), $g(B \backslash K) \cup g(K)$ is a countable union of Z-sets of E, i.e., $g(B) \in \mathscr{Z}(E)$ because $g(B)$ is compact. Observe that g is one-to-one and therefore an embedding and that moreover $g(B \backslash K) \subseteq F$. Now fix an arbitrary $\gamma \in \Gamma$.

Precisely such as in the proof of Lemma 3.2 it can now be shown that $g^{-1}(X_\gamma)\backslash K = A_\gamma\backslash K$. $\square$

Corollary 6.2. *Let E be an* ANR *and let $F \subseteq E$ be such that $E\backslash F$ is locally homotopy negligible. If $\mathscr{X} = (X_\gamma)_{\gamma\in\Gamma}$ is strongly $\mathscr{M}_\Gamma$-universal in F, then $\mathscr{X}$ is strongly $\mathscr{M}_\Gamma$-universal in E.*

Remark 6.3. Let F be a subspace of a space E. If $\mathscr{X}$ is a strongly $\mathscr{M}_\Gamma$-universal system in E that is contained in F then it need not follow that $\mathscr{X}$ is strongly $\mathscr{M}_\Gamma$-universal in F. Even in case both E and F are AR's and $E\backslash F$ is locally homotopy negligible in E there is a simple counterexample. For consider s and Q. Then $B = Q\backslash s$ is a capset in Q and hence is an F_σ-absorber in Q (Theorem 5.3). Thus s is strongly G_δ-universal in Q. But s is of course not strongly G_δ-universal in s.

7. Local behaviour of universal systems

We will now present some simple results on the local behaviour of strongly $\mathscr{M}_\Gamma$-universal systems.

Lemma 7.1. *Let E be an* ANR, *U a nonempty open subset of E and $(X_\gamma)_{\gamma\in\Gamma}$ a strongly $\mathscr{M}_\Gamma$-universal system in E. Then the system $(X_\gamma \cap U)_{\gamma\in\Gamma}$ is strongly $\mathscr{M}_\Gamma$-universal in U.*

Proof. We may assume that $U \neq E$. Let $f: Q \to U$ be a continuous function that restricts to a Z-embedding on some closed set K. In addition, let $\varepsilon > 0$ and let $(A_\gamma)_{\gamma\in\Gamma}$ be an $\mathscr{M}_\Gamma$-system in Q. Since E is an ANR, $f(K) \in \mathscr{Z}(U)$ and U is open in E, it follows that $f(K) \in \mathscr{Z}(E)$ [24, Theorem 7.2.5]. By compactness of $f(Q)$, $\delta = d(f(Q), E\backslash U) > 0$. We may approximate f by a Z-embedding $g: Q \to E$ such that $\hat{d}(f, g) < \min\{\varepsilon, \delta\}$ while moreover for every $\gamma \in \Gamma$, $g^{-1}(X_\gamma)\backslash K = A_\gamma\backslash K$. Then g is clearly as required because $g(Q) \subseteq U$. $\square$

Proposition 7.2. *Let E be an* ANR *and let $\mathscr{U}$ be an open cover of E. If an $\mathscr{M}_\Gamma$-system $\mathscr{X} = (X_\gamma)_{\gamma\in\Gamma}$ has the property that $(X_\gamma \cap U)_{\gamma\in\Gamma}$ is strongly $\mathscr{M}_\Gamma$-universal in U for every $U \in \mathscr{U}$, then $\mathscr{X}$ is strongly $\mathscr{M}_\Gamma$-universal in E.*

Proof. Let $(A_\gamma)_{\gamma\in\Gamma}$ be an $\mathscr{M}_\Gamma$-system in Q, and $f: Q \to E$ a map that restricts to a Z-embedding on some compact subset K of Q. In addition, let $\varepsilon > 0$. By the fact that E is Lindelöf we may assume that $\mathscr{U}$ is countable. Enumerate $\mathscr{U}$ as $\{U_0, U_1, \ldots\}$, where $U_0 = \emptyset$. There exists for every i an open set $V_i \subseteq \bar{V}_i \subseteq U_i$ such that $\mathscr{V} = \{V_0, V_1, \ldots\}$ covers E [24, Proposition 4.3.3].

By compactness of $f(Q)$ there exists $n \geq 0$ such that $f(Q) \subseteq \bigcup_{i=0}^n V_i$. For every $0 \leq i \leq n$ define $C_i = f^{-1}(\bar{V}_i)$, and $W_i = f(C_i)$, respectively. Then $W_i \subseteq \bar{V}_i \subseteq U_i$, so by compactness of W_i it follows that

$$\xi = \min\{d(W_i, E\backslash U_i): 0 \leq i \leq n\} > 0.$$

Put $f_0=f$ and assume that for certain $i<n$ we constructed $f_i:Q\to E$ such that

(1) $f_i|K\cup C_0\cup\cdots\cup C_{i-1}=f_{i-1}|K\cup C_0\cup\cdots\cup C_{i-1}$,

(2) $f_i|K\cup C_0\cup\cdots\cup C_i$ is a Z-embedding,

(3) $(\forall\gamma\in\Gamma)[f_i^{-1}(X_\gamma\cap U_i)\backslash K=(C_i\cap A_\gamma)\backslash K]$,

(4) $\hat{d}(f_{i-1},f_i)<\min\{\xi,\varepsilon\}/n$.

Note that by construction, $f_i(C_{i+1})\subseteq U_{i+1}$ and that f_i restricts to a Z-embedding on $C_{i+1}\cap(K\cup C_0\cup\cdots\cup C_i)$. Since $(X_\gamma\cap U_{i+1})_{\gamma\in\Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in U_{i+1} and $(A_\gamma\cap C_{i+1})_{\gamma\in\Gamma}$ is an $\mathcal{M}_\Gamma$-system in C_{i+1}, it follows that we may approximate $f_i|C_{i+1}$ arbitrarily closely by a Z-embedding $g:C_{i+1}\to U_{i+1}$ such that for every $\gamma\in\Gamma$, $g^{-1}(X_\gamma\cap U_{i+1})\backslash K=(C_{i+1}\cap A_\gamma)\backslash K$ and $g|C_{i+1}\cap(K\cup C_0\cup\cdots\cup C_i)=f_i$. Since E is an ANR it follows that $g(C_{i+1})$ is a Z-set in E [24, Theorem 7.2.5]. Again since E is an ANR, we can choose g so close to $f_i|C_{i+1}$ that $g\cup(f_i|K\cup C_0\cup\cdots\cup C_i)$ extends to a map f_{i+1} such that (1)-(4) hold.

Now define $h=f_n$. Then h is a Z-embedding, $h|K=f|K$, and for every $\gamma\in\Gamma$, $h^{-1}(X_\gamma)\backslash K=A_\gamma\backslash K$. Since clearly $\hat{d}(h,f)<\varepsilon$, we conclude that h is as required. $\square$

8. Basic properties of absorbing systems: I

In this section we derive some basic properties of absorbing systems in infinite-dimensional manifolds. Observe that every infinite-dimensional manifold satisfies the Z-approximation property: For details, see Bessaga and Pełczyński [2], Chapman [10] and van Mill [24]. This property of infinite-dimensional manifolds will be used without explicit reference in the remaining part of this paper.

To begin with, we prove a result on pairs of absorbing systems that will be used later on in this section, but that is also interesting in its own right.

Theorem 8.1. *Let E be an* ANR *and let Ω be a strongly $\mathscr{C}$-universal subset of E. Let $\mathscr{X}=(X_\gamma)_{\gamma\in\Gamma}$ be an order preserving indexed collection of subsets of Ω. Then for every system $\mathcal{M}_\Gamma$ the following statements are equivalent*:

(1) *$\mathscr{X}$ is strongly $\mathcal{M}_\Gamma$-universal in Ω.*

(2) *$(\mathscr{X},\Omega)$ is strongly $(\mathcal{M}_\Gamma,\mathscr{C})$-universal in E.*

Proof. For (1)$\Rightarrow$(2), first observe that $(\mathscr{X},\Omega)$ is order preserving (with respect to the appropriate ordering, see Section 3). Next, observe that $E\backslash\Omega$ is locally homotopy negligible (Corollary 4.4). Finally, let A be a compact subset of Q, and $(A_\gamma)_{\gamma\in\Gamma}$ an $\mathcal{M}_\Gamma$-system of subsets of A. Let $f:Q\to E$ be a continuous function that restricts to a Z-embedding on some compact subset K of Q, and let $\varepsilon>0$. Since $\mathscr{X}$ is strongly $\mathcal{M}_\Gamma$-universal in Ω, we can approximate by Proposition 6.1 and Corollary 4.4 the function $f|A\cup K$ arbitrarily closely by a Z-embedding $g:A\cup K\to E$ such that $g(A\backslash K)\subseteq\Omega$, $g|K=f|K$, while moreover, for every $\gamma\in\Gamma$, $g^{-1}(X_\gamma)\backslash K=A\gamma\backslash K$. Since E is an ANR we may assume that $f|A\cup K$ and g are so close that g can be extended to a continuous function $\bar{g}:Q\to E$ such that $\hat{d}(f,\bar{g})<\frac{1}{2}\varepsilon$. Since the empty set is compact and is contained in Q, and Ω is strongly $\mathscr{C}$-universal, we can approximate $\bar{g}$ by a Z-embedding $h: Q \to E$ such that $\hat{d}(h, \bar{g}) < \frac{1}{2}\varepsilon$, $h | A \cup K = \bar{g} | A \cup K$ and $h^{-1}(\Omega) \setminus (A \cup K) = \emptyset \setminus (A \cup K) = \emptyset$. Since $h(A \setminus K) = \bar{g}(A \setminus K) = g(A \setminus K) \subseteq \Omega$, this implies that $h^{-1}(\Omega) \setminus K = A \setminus K$. We conclude that h is as required.

For (2)$\Rightarrow$(1), let $(A_\gamma)_{\gamma \in \Gamma}$ be an $\mathscr{M}_\Gamma$-system in Q, $f: Q \to \Omega$ be continuous such that f restricts to a Z-embedding on some compact subset K of Q, and let $\varepsilon > 0$. Since $f(K)$ is compact it follows from Corollary 4.6 that $f(K) \in \mathscr{Z}(E)$. There consequently exists an approximation $h: Q \to E$ such that

(1) $\hat{d}(f, g) < \varepsilon$,

(2) h is a Z-embedding,

(3) $(\forall \gamma \in \Gamma)[h^{-1}(X_\gamma) \setminus K = A_\gamma \setminus K]$,

(4) $h^{-1}(\Omega) \setminus K = Q \setminus K$,

(5) $h | K = f | K$.

Since $h(Q)$ is a compact subset of Ω, it follows from Corollary 4.6 that $h(Q) \in \mathscr{Z}(\Omega)$. So we are done. $\square$

We now come to the main result in this section, that will be proved by a standard back and forth technique, cf. Bessaga and Pełczyński [2]. See also Dijkstra, van Mill and Mogilski [17] where an easier case was proved for absorbing systems in Q.

Theorem 8.2. (a) *Let E be an s- or Q-manifold. In addition, for $j = 1, 2$, let $\mathscr{X}_j = (X_{j,\gamma})_{\gamma \in \Gamma}$ be an $\mathscr{M}_\Gamma$-absorbing system in E and let K be a Z-set in E that misses $\bigcup_{\gamma \in \Gamma} X_{1,\gamma} \cup X_{2,\gamma}$. Then for every open cover $\mathscr{U}$ of E there exists a $\mathscr{U}$-close to the identity homeomorphism $h: E \to E$ that restricts to the identity on K while moreover for every $\gamma \in \Gamma$, $h(X_{1,\gamma}) = X_{2,\gamma}$.*

(b) *If moreover $\mathscr{X}_j$ is contained in a σ-compact set $\Omega_j \subseteq E$ satisfying*

(i) *Ω_j is a strongly $\mathscr{C}$-universal set in E,*

(ii) *$\mathscr{X}_j$ is strongly $\mathscr{M}_\Gamma$-universal in Ω_j,*

then h can be chosen such that moreover $h(\Omega_1 \setminus K) = \Omega_2 \setminus K$.

Proof. We first prove the theorem in the special case that $K = \emptyset$. Our plan is to prove both statements in the theorem simultaneously. Pick for $j = 1, 2$ compact Z-sets $(A_i^j)_{i=0}^\infty$ in E such that $\bigcup_{\gamma \in \Gamma} X_{j,\gamma} \subseteq \bigcup_{i=0}^\infty A_i^j$, while moreover $\emptyset = A_0^j \subseteq A_1^j \subseteq \cdots$. In the case that we have to deal with the Ω_j, we pick the A_i^j in such a way that $\Omega_j = \bigcup_{i=0}^\infty A_i^j$ (here we use Lemma 5.2). In the remaining part of the proof, we will mark statements concerning the Ω_j with an asterisk in the left margin. The reader should ignore these statements in case he is only interested in the first part of the theorem.

Let d be an admissible complete metric on E such that the family of all open d-balls of radius 1 refines $\mathscr{U}$ (Bessaga and Pełczyński [2, Theorem II.4.1]). We will inductively define sequences of homeomorphisms f_i and g_i of E satisfying the following conditions:

(1) $L\prod_{i=0}^\infty f_i \in \mathscr{H}(E)$,

(2) $\hat{d}(f_i, 1) < 2^{-(i+2)}$,
(3) $g_i = f_i \circ f_{i-1} \circ \cdots \circ f_0$,
(4) $(\forall \gamma \in \Gamma)[A_i^1 \cap g_i^{-1}(X_{2,\gamma}) = A_i^1 \cap X_{1,\gamma}]$,
(5) $(\forall \gamma \in \Gamma)[A_i^2 \cap g_i(X_{1,\gamma}) = A_i^2 \cap X_{2,\gamma}]$,
(*) (6) $g_i(A_i^1) \subseteq \Omega_2$,
(*) (7) $g_i^{-1}(A_i^2) \subseteq \Omega_1$,
(8) $f_i \mid g_{i-1}(A_{i-1}^1) \cup A_{i-1}^2 = 1$.

Let $f_0 = 1$ and assume that f_i has been constructed. Put $T = g_i(A_i^1) \cup A_i^2$ and $\tilde{T} = g_i(A_{i+1}^1) \cup A_i^2$, respectively. Then both T and $\tilde{T}$ are compact Z-sets in E. We have for every $\gamma \in \Gamma$, $g_i(X_{1,\gamma}) \cap \tilde{T} \in \mathcal{M}_\gamma$ and

$$\begin{aligned} g_i(X_{1,\gamma}) \cap T &= g_i(X_{1,\gamma}) \cap (g_i(A_i^1) \cup A_i^2) \\ &= g_i(X_{1,\gamma} \cap A_i^1) \cup (g_i(X_{1,\gamma}) \cap A_i^2) \\ &= g_i(g_i^{-1}(X_{2,\gamma}) \cap A_i^1) \cup (X_{2,\gamma} \cap A_i^2) \\ &= X_{2,\gamma} \cap (g_i(A_i^1) \cup A_i^2) \\ &= X_{2,\gamma} \cap T. \end{aligned}$$

Let $\delta > 0$ be such that (1) and (2) force f_{i+1} to be δ-close to the identity. If we do not have to worry about the Ω_j then we are about to use the fact that $(X_{2,\gamma})_{\gamma \in \Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in E. In the other case, $(X_{2,\gamma})_{\gamma \in \Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in Ω_2, thus $(X_{2,\gamma}, \Omega_2)_{\gamma \in \Gamma}$ is strongly $(\mathcal{M}_\Gamma, \mathscr{C})$-universal in E (Theorem 8.1). So in either case we can approximate $1 : \tilde{T} \to E$ by a Z-embedding $\alpha : \tilde{T} \to E$ such that (remember our convention about the asterisks in the left margin)

- α and 1 are as close as we please,
- $(\forall \gamma \in \Gamma)[\alpha^{-1}(X_{2,\gamma}) \setminus T = (g_i(X_{1,\gamma}) \cap \tilde{T}) \setminus T = (g_i(X_{1,\gamma}) \cap g_i(A_{i+1}^1)) \setminus T]$,
- (*) $\alpha^{-1}(\Omega_2) \setminus T = g_i(A_{i+1}^1) \setminus T$, thus $\alpha \circ g_i(A_{i+1}^1) \subseteq \Omega_2$,
- $\alpha \mid T = 1$.

The Z-set Unknotting Theorem for E now implies that α can be extended to a homeomorphism $\bar{\alpha} : E \to E$ which is $\frac{1}{2}\delta$-close to 1_E. Observe the following facts:

- $\bar{\alpha} \mid T = 1_T$,
- (*) $\bar{\alpha}(g_i(A_{i+1}^1)) \subseteq \Omega_2$,
- $(\forall \gamma \in \Gamma)[\bar{\alpha}^{-1}(X_{2,\gamma}) \cap g_i(A_{i+1}^1) = g_i(A_{i+1}^1 \cap X_{1,\gamma})]$.

This is so because for every γ,

$$\begin{aligned} \alpha^{-1}(X_{2,\gamma}) &= (\alpha^{-1}(X_{2,\gamma}) \cap T) \cup (\alpha^{-1}(X_{2,\gamma}) \setminus T) \\ &= (X_{2,\gamma} \cap T) \cup (g_i(X_{1,\gamma}) \cap g_i(A_{i+1}^1) \setminus T) \\ &= (g_i(X_{1,\gamma}) \cap T) \cup (g_i(X_{1,\gamma} \cap A_{i+1}^1) \setminus T), \end{aligned}$$

so that

$$\begin{aligned} \bar{\alpha}^{-1}(X_{2,\gamma}) \cap g_i(A_{i+1}^1) &= \alpha^{-1}(X_{2,\gamma}) \cap g_i(A_{i+1}^1) \\ &= (g_i(X_{1,\gamma} \cap A_{i+1}^1) \cap T) \cup (g_i(X_{1,\gamma} \cap A_{i+1}) \setminus T) \\ &= g_i(X_{1,\gamma} \cap A_{i+1}^1). \end{aligned}$$

Now put $P=\bar{\alpha}\circ g_i(A^1_{i+1})\cup A^2_i$ and $\tilde{P}=\bar{\alpha}\circ g_i(A^1_{i+1})\cup A^2_{i+1}$. Then both P and $\tilde{P}$ are compact Z-sets in E. Since every $\mathcal{M}_\gamma$ is closed hereditary, $(X_{2,\gamma}\cap\tilde{P})_{\gamma\in\Gamma}$ is an $\mathcal{M}_\Gamma$-system in $\tilde{P}$. If we do not have to worry about the Ω_j then we are about to use the fact that $((\bar{\alpha}\circ g_i)(X_{1,\gamma}))_{\gamma\in\Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in E. In the other case, $((\bar{\alpha}\circ g_i)(X_{1,\gamma}))_{\gamma\in\Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in $\bar{\alpha}\circ g_i(\Omega_2)$, thus $(\bar{\alpha}\circ g_i(X_{1,\gamma}),\ \bar{\alpha}\circ g_i(\Omega_2))_{\gamma\in\Gamma}$ is strongly $(\mathcal{M}_\Gamma, \mathscr{C})$-universal in E (Theorem 8.1). So in either case we can approximate $1:\tilde{P}\to E$ by a Z-embedding $\beta:\tilde{P}\to E$ such that

- β and 1 are as close as we please,
- $\beta^{-1}(\bar{\alpha}\circ g_i(X_{1,\gamma}))\backslash P=(X_{2,\gamma}\cap\tilde{P})\backslash P=(X_{2,\gamma}\cap A^2_{i+1})\backslash P$,
- (∗) $\beta^{-1}(\bar{\alpha}\circ g_i(\Omega_1))\backslash P=A^2_{i+1}\backslash P$, thus $(\beta^{-1}\circ\bar{\alpha}\circ g_i)^{-1}(A^2_{i+1})\subseteq\Omega_1$,
- $\beta\,|\,P=1$.

We now put $f_{i+1}=\bar{\beta}^{-1}\circ\bar{\alpha}$. It is easily seen that f_{i+1} satisfies our inductive requirements.

Let $g=L\prod_{i=0}^{\infty}f_i$. Then g has the following properties:

- $g\in\mathscr{H}(E)$,
- $\hat{d}(g,1_E)<1$, so g is $\mathscr{U}$-close to 1_E,
- (∗) $g(\Omega_1)=\Omega_2$,
- $(\forall\gamma\in\Gamma)[g(X_{1,\gamma})=X_{2,\gamma}]$.

We conclude that g is as required.

We now turn to the case that $K\neq\emptyset$. Then we let $\mathscr{V}$ be a Dugundji cover of $E\backslash K$. Observe that any homeomorphism of $E\backslash K$ that is $\mathscr{V}$-close to the identity can be extended to a homeomorphism of E that restricts to the identity on K. By Lemma 7.1 we conclude that for $j=1,2$, $\mathscr{X}_j$ is an $\mathcal{M}_\Gamma$-universal system in the Q- or s-manifold $E\backslash K$. In addition, in case we have to worry about the Ω_j, observe that by Lemma 7.1, $\mathscr{X}_j$ is strongly $\mathcal{M}_\Gamma$-universal in $\Omega_j\backslash K$ and $\Omega_j\backslash K$ is strongly $\mathscr{C}$-universal in $E\backslash K$. □

Theorem 8.2. (c) *Let Ω be a Σ-manifold. In addition, for $j=1,2$, let $\mathscr{X}_j=(X_{j,\gamma})_{\gamma\in\Gamma}$ be an $\mathcal{M}_\Gamma$-absorbing system in Ω and let K be a Z-set in Ω that missed $\bigcup_{\gamma\in\Gamma}X_{1,\gamma}\cup X_{2,\gamma}$. Then for every open cover $\mathscr{U}$ of Ω there exists a $\mathscr{U}$-close to the identity homeomorphism $h:\Omega\to\Omega$ that restricts to the identity on K while moreover for every $\gamma\in\Gamma$, $h(X_{1,\gamma})=X_{2,\gamma}$.*

Proof. As in the proof of Theorem 8.2(a), (b) it suffices to prove the corollary for the special case $K=\emptyset$. By Chapman [9, Theorem 10.2] there exists an s-manifold E and an embedding $i:\Omega\to E$ such that $i(\Omega)$ is a capset for E. So without loss of generality we may assume that Ω is a subset of E. Let $\mathscr{U}$ be an open cover of Ω. There is a cover $\mathscr{V}$ of Ω consisting of open subsets of E such that $\mathscr{U}=\mathscr{V}\cap\Omega$. Put $F=\bigcup\mathscr{V}$. Then F is an s-manifold and Ω is a capset for F, hence by Theorem 5.3, Ω is strongly $\mathscr{C}$-universal in F. The desired result therefore follows by an application of Theorem 8.2(b). □

Corollary 8.3. *Let E be an s- or a Σ-manifold. Similarly for F. Let $\mathcal{M}$ be a class of spaces containing the class of all compact spaces. Finally, let A and B be $\mathcal{M}$-absorbers in E and F, respectively. Then A and B are homeomorphic iff they have the same homotopy type.*

Proof. Assume that A and B have the same homotopy type. By using the same technique as in the proof of Theorem 8.2(c), we may assume that A and B are $\mathcal{M}$-absorbers in the s-manifolds $\hat{E}$ and $\hat{F}$, respectively. By Corollary 4.5, $\hat{E}$ and $\hat{F}$ have the same homotopy type. Consequently, $\hat{E}$ and $\hat{F}$ are homeomorphic (Bessaga and Pełczyński [2 Theorem IX.7.3]). The desired result now follows from Theorem 8.2(a). □

We do not know whether this result can also be proved for $\mathcal{M}$-absorbers in Q-manifolds. For a partial result, see Proposition 10.7.

9. Basic properties of absorbing systems: II

There are several results that guarantee that subsets or supersets of capsets are again capsets. For details, see Chapman [9] and Section 5. In this section we shall show that our absorbing systems share some of the nice properties that capsets have. It is possible for example under certain natural conditions to add or to remove a Z-set from the absorbing system without destroying its absorption properties. The results in this section are not endresults, but they are useful tools in working with absorbing systems. For capsets some known properties will follow from our general theory about $\mathcal{M}$-absorbing systems, by taking for $\mathcal{M}$ the class of all σ-compact spaces.

Proposition 9.1. *Suppose that every $\mathcal{M}_\gamma \in \mathcal{M}_\Gamma$ is open hereditary. In addition, let E be an infinite-dimensional manifold, which contains an $\mathcal{M}_\Gamma$-absorbing system in E. Then for every open subset U of E and every $\mathcal{M}_\Gamma$-absorbing system $(Y_\gamma)_{\gamma\in\Gamma}$ in U there is an $\mathcal{M}_\Gamma$-absorbing system $(Z_\gamma)_{\gamma\in\Gamma}$ in E such that for every $\gamma\in\Gamma$, $Z_\gamma \cap U = Y_\gamma$.*

Proof. Suppose that $\mathscr{X} = (X_\gamma)_{\gamma\in\Gamma}$ is an $\mathcal{M}_\Gamma$-absorbing system in E. By Lemma 7.1 the system $(X_\gamma \cap U)_{\gamma\in\Gamma}$ is $\mathcal{M}_\Gamma$-universal in U. Since $\mathscr{X}$ is contained in a σ-compact σZ-set of E, say Z, the system $(X_\gamma \cap U)_{\gamma\in\Gamma}$ is contained in the σ-compact σZ-set $Z \cap U$ of U. By Theorem 8.2 there consequently exists a homeomorphism $h: U \to U$ such that for every $\gamma \in \Gamma$, $h(X_\gamma \cap U) = Y_\gamma$, while moreover h can be extended to a homeomorphism $\bar{h}$ of E that restricts to the identity on $E \setminus U$ (pick a Dugundji cover $\mathscr{V}$ of U and let h be $\mathscr{V}$-close to 1_U). Then the system $(\bar{h}(X_\gamma))_{\gamma\in\Gamma}$ is clearly as desired. □

By the same method it is possible to derive the following proposition.

Proposition 9.2. *Suppose that every $\mathcal{M}_\gamma \in \mathcal{M}_\Gamma$ is open hereditary and let E be an infinite-dimensional manifold. Then for all $\mathcal{M}_\Gamma$-absorbing systems $(X_\gamma)_{\gamma\in\Gamma}$ and $(Y_\gamma)_{\gamma\in\Gamma}$*

and for every open subset U of E,

$$((X_\gamma \cap U) \cup (Y_\gamma \setminus U))_{\gamma \in \Gamma}$$

is an $\mathcal{M}_\Gamma$-absorbing system in E.

We will now prove that under certain conditions one can remove a Z-set from an absorbing system.

Theorem 9.3. *Let E be an* ANR, *let $A \in \mathscr{Z}(E)$, and let $(X_\gamma)_{\gamma \in \Gamma}$ be strongly $\mathcal{M}_\Gamma$-universal in E. Then $(X_\gamma \setminus A)_{\gamma \in \Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in E as well.*

Proof. By Lemma 7.1, $(X_\gamma \setminus A)_{\gamma \in \Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in $E \setminus A$. Since A is locally homotopy negligible in E, by Corollary 6.2 it therefore follows that $(X_\gamma \setminus A)_{\gamma \in \Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in E. □

Corollary 9.4. *Assume that each $\mathcal{M}_\gamma \in \mathcal{M}$ is open hereditary. Let E be an* ANR, *let $A \in \mathscr{Z}(E)$, and let $\mathscr{X} = (X_\gamma)_{\gamma \in \Gamma}$ be an $\mathcal{M}_\Gamma$-absorbing system in E. Then $(X_\gamma \setminus A)_{\gamma \in \Gamma}$ is an $\mathcal{M}_\Gamma$-absorbing system in E as well.*

Our next goal is to show that we can always add a subset of a Z-set to a strongly $\mathcal{M}_\Gamma$-universal system.

Theorem 9.5. *Let E be an* ANR, *let $A \in \mathscr{Z}(E)$, and let $(X_\gamma)_{\gamma \in \Gamma}$ be strongly $\mathcal{M}_\Gamma$-universal in E. Then for every subset $B \subseteq A$, $(X_\gamma \cup B)_{\gamma \in \Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in E as well.*

Proof. Let $(A_\gamma)_{\gamma \in \Gamma}$ be an $\mathcal{M}_\Gamma$-system in Q. In addition, let $f: Q \to E$ be continuous such that f restricts to a Z-embedding on some closed set $K \subseteq Q$, and let $\varepsilon > 0$. By Lemma 7.1, $(X_\gamma \setminus A)_{\gamma \in \Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in $E \setminus A$. Since A is locally homotopy negligible, there consequently is by Proposition 6.1 a Z-embedding $g: Q \to E$ such that

(1) $\hat{d}(f, g) < \varepsilon$,
(2) $g(Q \setminus K) \subseteq E \setminus A$,
(3) $(\forall \gamma \in \Gamma)[g^{-1}(X_\gamma \setminus A) \setminus K = A_\gamma \setminus K]$,
(4) $g \mid K = f \mid K$.

We claim that g is as required. To this end, pick an arbitrary $\gamma \in \Gamma$. Then

$$\begin{aligned} g^{-1}(X_\gamma \cup B) \setminus K &= (g^{-1}(X_\gamma) \setminus K) \cup (g^{-1}(B) \setminus K) \\ &= (g^{-1}(X_\gamma \setminus A) \setminus K) \cup (g^{-1}(X_\gamma \cap A) \setminus K) \\ &= g^{-1}(X_\gamma \setminus A) \setminus K \\ &= A_\gamma \setminus K. \end{aligned}$$

We are done. □

Remark 9.6. Let $\mathcal{M}$ be a class of spaces. We define its dual class $\hat{\mathcal{M}}$ in the following way:

$$X \in \hat{M} \quad \Leftrightarrow \quad \text{for some compact space } B \text{ containing } X\text{: } B\backslash X \in \mathcal{M}.$$

In addition, we call $\mathcal{M}$ *dually absolute* if it has the following property:

$$(\forall \text{ compact } B)(\forall X \subseteq B)[X \in \mathcal{M} \Leftrightarrow B\backslash X \in \hat{\mathcal{M}}].$$

Observe that every Borel class of spaces is dually absolute.

For dually absolute classes, there is an easier proof of an important special case of Theorem 9.5. Indeed, assume that $\mathcal{M}$ is dually absolute and let $\hat{\Gamma}$ be Γ with the reverse ordering. Then if $(X_\gamma)_{\gamma\in\Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal in a space E, $(E\backslash X_\gamma)_{\gamma\in\hat{\Gamma}}$ is strongly $\hat{\mathcal{M}}_{\hat{\Gamma}}$-universal. So if $A \subseteq E$ is a Z-set, then by Theorem 9.3,

$$((E\backslash X_\gamma)\backslash A)_{\gamma\in\hat{\Gamma}} \text{ is strongly } \hat{\mathcal{M}}_{\hat{\Gamma}}\text{-universal.}$$

We conclude that $(X_\gamma \cup A)_{\gamma\in\Gamma}$ is strongly $\mathcal{M}_\Gamma$-universal.

There is no natural analogue to Corollary 9.4 because instead of the natural condition "each $\mathcal{M}_\gamma$ is open hereditary", one has to assume something like the "union of an element in $\mathcal{M}_\gamma$ and a compact set is again in $\mathcal{M}_\gamma$", which is unnatural. For several natural classes however, this unnatural condition is trivially satisfied, as the next corollary shows.

Corollary 9.7. *Let A be an $\mathcal{M}$-absorber in an* ANR *E and suppose that $\mathcal{M}$ is a Borel class of spaces. Then for every Z-set $Z \subseteq E$ and subset $B \subseteq Z$ such that $B \in \mathcal{M}$, $B \cup A$ is an $\mathcal{M}$-absorber in E.*

Proof. In view of Theorem 9.5, it remains to be shown that $B \cup Z \in \mathcal{M}$. But this is a triviality. □

Remark 9.8. Corollary 9.7 can also be formulated for $\mathcal{M}_\Gamma$-absorbing systems $\mathcal{X}$ such that every $\mathcal{M}_\gamma$ is a Borel class of spaces. We leave this to the reader.

Remark 9.9. As we saw in Section 5, statement (III), the union of a capset and a σ-compact σZ-set is again a capset. In view of Corollary 9.7 it is naturally to ask whether the same result holds for the other absorbing systems as well. The answer to this question is in the negative. If A is an $\mathcal{M}$-absorber in an infinite-dimensional manifold E, then A is contained in a σ-compact σZ-set. Consequently, if it were possible to add a σ-compact σZ-set to A, then A would be σ-compact itself. Simple counterexamples show that this need not be the case.

It is also natural to ask whether for an $\mathcal{M}$-absorber A in an infinite-dimensional manifold E, the union $A \cup B$ is again an $\mathcal{M}$-absorber for an arbitrary $B \subseteq E$ such that $B \in \mathcal{M}$. The first obvious condition for B is that it is contained in a σ-compact σZ-set. For some classes the answer to this question is in the affirmative. Let for example $E = Q$ and $\mathcal{M} = \{$all countable spaces$\}$; it is easy to see that each countable dense set in E is an $\mathcal{M}$-absorber. For most interesting classes however the answer is in the negative for the same reason as above. For example, let $\mathcal{M}$ be a Borel class other than the class F_σ and the class G_δ. Then $F_\sigma \subseteq \mathcal{M}$, from which it follows as above that A must be σ-compact, which is easily seen to be a contradiction.

Question 9.10. Is the union of two $\mathcal{M}$-absorbers in an infinite-dimensional manifold again an $\mathcal{M}$-absorber, where $\mathcal{M}$ is a Borel class of spaces?

10. New absorbing systems from old

We shall now consider the special case that the system $\mathcal{X}$ is a decreasing sequence $X_1 \supseteq X_2 \supseteq \cdots$ (so $\Gamma = \mathbb{N}$ with the inverted ordering) of absorbers, and we assume that all the classes $\mathcal{M}_i$ are equal to a fixed class $\mathcal{M}$. In this situation we shall use the term *$\mathcal{M}$-absorbing sequence*. Some results and proofs in this section are similar to the corresponding ones in Dijkstra, van Mill and Mogilski [17].

Let $\mathcal{M}$ be a class of spaces. We say that $\mathcal{M}_\delta$ is *absolute* provided that for every element $A \in \mathcal{M}_\delta$ that is contained in Q there exists a sequence $\{A_n\}_n$ of elements of $\mathcal{M}$ in Q such that $A = \bigcap_{n=1}^{\infty} A_n$. Easy examples show that $\mathcal{M}_\delta$ in general need not be absolute. Observe that each Borel class is absolute.

Lemma 10.1. *Let E be a space and let $(X_i)_i$ be an $\mathcal{M}$-absorbing sequence, where $\mathcal{M}$ is closed under finite intersections and $\mathcal{M}_\delta$ is absolute. Then $X_\infty = \bigcap_{i=1}^{\infty} X_i$ is an $\mathcal{M}_\delta$-absorber in E.*

Proof. Let $A \in \mathcal{M}_\delta$ and $f: Q \to E$ a map that restricts to a Z-embedding on some compact subset K of Q. There exist $A_n \in \mathcal{M}$ for every n such that $A = \bigcap_{n=1}^{\infty} A_n$. Since $\mathcal{M}$ is closed under finite intersections, we may assume without loss of generality that $A_1 \supseteq A_2 \supseteq \cdots$. We can approximate f arbitrarily closely by a Z-embedding $g: Q \to E$ that restricts to f on K, while moverover, for every n, $f^{-1}(X_n) \setminus K = A_n \setminus K$. We conclude that $f^{-1}(A) \setminus K = X_\infty \setminus K$. Since X_∞ is clearly contained in a σ-compact σZ-set (because X_1 is), we are done. $\square$

For spaces E_n and subspaces $X_n \subseteq E_n$, $n \in \mathbb{N}$, we define a decreasing sequence of subsets of $E = \prod_{n=1}^{\infty} E_n$:

$$S_n = \prod_{i=1}^{n} X_i \times \prod_{i=n+1}^{\infty} E_i \quad (n \in \mathbb{N}).$$

Proposition 10.2. *For each $n \in \mathbb{N}$ let E_n be a space such that $E = \prod_{n=1}^{\infty} E_n$ is an* ANR *with the Z-approximation property. If for every n, $X_n \subseteq E_n$ is strongly $\mathcal{M}$-universal, then the sequence $(S_n)_n$ is strongly $\mathcal{M}$-universal in E.*

If moreover $E_n \approx Q$ for every n, $\mathcal{M} = F_\sigma$ and every X_n is an F_σ-absorber which is contained in a σZ-set $C_n \subseteq E_n$, then every subset $X \subseteq E$ such that

$$\prod_{n=1}^{\infty} X_n \subseteq X \subseteq \prod_{n=1}^{\infty} C_n$$

is strongly $F_{\sigma\delta}$-universal.

Proof. For every n let d_n be an admissible metric on E_n that is bounded by 1. Then $\rho(x, y) = \max_n d_n(x_n, y_n)$ is an admissible metric on E. Denote the projection $E \to E_n$ by π_n ($n \in \mathbb{N}$).

We will first prove that $(S_n)_n$ is strongly $\mathcal{M}$-universal.

Consider a map $f: Q \to E$ that restricts to a Z-embedding on some compact subset $K \subseteq Q$, and let $\mathcal{U}$ be an open cover of E. In addition, let $A_1 \supseteq A_2 \supseteq \cdots$ be a sequence of subsets of Q consisting of elements of $\mathcal{M}$. We may assume without loss of generality that f is a Z-embedding. There is an $\varepsilon > 0$ such that every subset of E that intersects $f(Q)$ and has diameter less than ε, is contained in an element of $\mathcal{U}$ [24, Lemma 1.1.1]. Write $Q \backslash K$ as $\bigcup_{i=0}^{\infty} F_i$, where $F_0 = \emptyset$, each F_i is compact, and $F_0 \subseteq F_1^\circ \subseteq F_1 \subseteq F_2^\circ \subseteq F_2 \subseteq \cdots$. For every $i \geq 0$ put

$$\varepsilon_i = \min\{2^{-i} \cdot \varepsilon, \tfrac{1}{2}\rho(f(K), f(F_i))\}$$

and observe that

$$\varepsilon_0 \geq \varepsilon_1 \geq \cdots \geq \varepsilon_i \geq \cdots > 0, \quad \lim_{i \to \infty} \varepsilon_i = 0.$$

Now consider the nth component function $f_n : Q \to E_n$. Put $\alpha_{n,0} = f_n$ and assume that we constructed $\alpha_{n,i} : Q \to E_n$ such that

(1) $\hat{d}(\alpha_{n,i}, \alpha_{n,i-1}) < \varepsilon_{i+1}$,
(2) $\alpha_{n,i} | F_{i-1} = \alpha_{n,i-1} | F_{i-1}$,
(3) $\alpha_{n,i} | Q \backslash F_{i+1}^\circ = f_n | Q \backslash F_{i+1}^\circ$,
(4) $\alpha_{n,i} | F_i$ is a Z-embedding,
(5) $\alpha_{n,i}^{-1}(X_n) \cap F_i = A_n \cap F_i$.

For the construction of $\alpha_{n,i+1}$ satisfying our inductive hypotheses, use the same technique as in the proofs of several earlier results in this paper.

Now put $g_n = \lim_{i \to \infty} \alpha_{n,i}$. Then g_n has the following properties:

(i) $\hat{d}(g_n, f_n) < \varepsilon$,
(ii) if $x \in F_{i+1} \backslash F_i$ then $d(g_n(x), f_n(x)) < d(f(K), f(F_{i+1}))$,
(iii) $g_n | K = f_n$,
(iv) $g_n | F_i$ is a Z-embedding for every i,
(v) $g_n^{-1}(X_n) \backslash K = A_n \backslash K$.

Define $g = (g_n)_n : Q \to E$. It is easily seen that g is one-to-one, and therefore an embedding. The compact set $g(Q)$ is contained in the σZ-set $f(K) \cup \bigcup_{i=1}^{\infty} (\pi_i^{-1}(g_i(F_i)))$ and is therefore a Z-set. Moreover, the maps f and g are clearly $\mathcal{U}$-close, because $\hat{\rho}(f, g) < \varepsilon$, and have the property that $f | K = g | K$.

If x is an element of $A_n \setminus K$, then $x \in \bigcap_{j=1}^n A_j \setminus K$ so $g_j(x) \in X$ for $j = 1, \ldots, n$. We conclude that in that case, $g(x) \in S_n$. On the other hand, if $g(x) \in S_n$ then $x \in A_n$.

We next assume that $E_n \approx Q$ for every n, that $\mathcal{M} = F_\sigma$ and finally that every X_n is an F_σ-absorber which is contained in a σZ-set $C_n \subseteq E_n$. Pick an arbitrary subset $X \subseteq E$ such that

$$\prod_{n=1}^{\infty} X_n \subseteq X \subseteq \prod_{n=1}^{\infty} C_n.$$

We will show how the above proof can be modified so that we can conclude that X is strongly $F_{\sigma\delta}$-universal. By Lemma 5.5, we may assume that the function $\alpha_{n,i} : Q \to E_n$ additionally satisfies the following property:

(6) $\alpha_{n,i}(F_i \setminus A_n) \cap C_n = \emptyset$.

This has the effect that for the function g_n we have

(vi) $g_n(Q \setminus (A_n \cup K)) \cap C_n = \emptyset$.

We claim that the function g has the property that $g^{-1}(X) \setminus K = A_\infty \setminus K$. Let $x \in g^{-1}(X) \setminus K$. Then $g(x) \in X \subseteq \prod_{i=1}^{\infty} C_i$. So for every $n \in \mathbb{N}$, $g_n(x) \in C_n$, hence $g_n(x) \notin g_n(Q \setminus (A_n \cup K))$. This gives $x \notin Q \setminus (A_n \cup K)$. Since $x \notin K$ we consequently obtain $x \in A_n$ and therefore, since n was arbitrary, $x \in A_\infty$. Conversely, if $x \in A_\infty \setminus K$ then for each n, $x \in A_n \setminus K$. Consequently, $g_n(x) \in B_n$ which implies that $g(x) \in \prod_{i=1}^{\infty} B_i \subseteq X$, i.e., $x \in g^{-1}(X)$. □

Corollary 10.3. *Let E be a topologically complete nondegenerate* AR. *If $X \subseteq E$ is strongly $\mathcal{M}$-universal, then the sequence $(S_n)_n$ is strongly $\mathcal{M}$-universal in E^∞.*

Proof. By observing that E^∞ trivally satisfies the disjoint-cells property, it follows that E has the Z-approximation property [24, Theorem 7.3.5]. So now we can apply Proposition 10.2. □

The proof of the following proposition is analogous to that of Propositon 10.2 and is therefore left as an exercise to the reader.

Propositon 10.4. *Let $N_0, N \in \mathbb{N} \cup \{\infty\}$ be such that $N_0 \leq N$, and let $M \in \mathbb{N}$. Moreover, for $n \leq M$, let E_n be an* ANR, *and for $n > M$, let E_n be an* AR, *such that $E = \prod_{n=1}^{N} E_n$ has the Z-approximation property. Finally, let for each $n \leq N_0$, $(X_\gamma^n)_{\gamma \in \Gamma}$ be a strongly $\mathcal{M}_\Gamma$-universal system in E_n. Then the system*

$$\left(\prod_{i=1}^{N_0} X_\gamma^i \times \prod_{i=N_0+1}^{N} E_i \right)_{\gamma \in \Gamma}$$

is strongly $\mathcal{M}_\Gamma$-universal in E.

The following result generalizes Dijkstra, van Mill and Mogilski [17, Lemma 6.4].

Corollary 10.5. *Assume that Y is an* ANR *with the Z-approximation property. If A is strongly $\mathcal{M}$-universal in Y and X is locally homotopy negligible in an* ANR *M, then $A \times (M \setminus X)$ is strongly $\mathcal{M}$-universal in $Y \times M$.*

Proof. Since X is locally homotopy negligible, $M \backslash X$ is an ANR (Toruńczyk [22]). Consequently, Lemma 2.1 implies that $Y \times (M \backslash X)$ has the Z-approximation property. So by Proposition 10.4, $A \times (M \backslash X)$ is strongly $\mathcal{M}$-universal in $Y \times (M \backslash X)$. But $Y \times X$ is clearly locally homotopy negligible in $Y \times M$. So by Corollary 6.2 it follows that $A \times (M \backslash X)$ is strongly $\mathcal{M}$-universal in $Y \times M$. □

We are now in a position to present an example of two "different" strongly $F_{\sigma\delta}$-universal sets in s.

Example 10.6. There exist subsets A and B in s such that

(1) A and B are strongly $F_{\sigma\delta}$-universal,
(2) A and B are homeomorphic,
(3) A is an $F_{\sigma\delta}$-absorber,
(4) B is not an $F_{\sigma\delta}$-absorber.

Proof. Let $\Omega \subseteq Q$ be the capset in van Mill [24, Proposition 6.5.4]. Then by Theorem 5.3(4), Ω is an F_σ-absorber in Q. From Corollary 10.3 and Lemma 10.1 we conclude that $E = \Omega^\infty$ is an $F_{\sigma\delta}$-absorber in $Q^\infty \approx Q$.

By Corollary 10.3 it follows that $E \times \Omega$ is a strongly $\mathcal{F}_{\sigma\delta}$-universal in $Q^\infty \times s \approx s$, [24, Corollary 6.5.10]. Now observe that $E \times \Omega$ is contained in a σ-compact subset of $Q^\infty \times s \approx s$ and that all compact subsets of s are Z-sets. So we conclude that there exists an $F_{\sigma\delta}$-absorber A in s, and that A is homeomorphic to Ω^∞.

The set Ω is also a capset in s, and it is therefore also an F_σ-absorber in s. Now observe that $B = A^\infty$ is strongly $F_{\sigma\delta}$-universal in $s^\infty \approx s$ (Corollary 10.3). Clearly B and A are homeomorphic, but B is not contained in a σ-compact subset of s^∞. For take arbitrary compact sets $A_n \subseteq s^\infty$. Since A is not compact, we can pick for each n a point $x_n \in A \backslash \pi_n(A_n)$ (here π_n is the projection onto the nth factor of s^∞). Then

$$(x_1, x_2, \ldots) \in B \backslash \bigcup_{n=1}^{\infty} A_n.$$

So B is not an $F_{\sigma\delta}$-absorber. □

A similar example was constructed independently by Cauty [7]. He constructed two G_δ-absorbers (in the sense of Mogilski and Bestvina [3]) A and B in l^2 such that the pairs (l^2, A) and (l^2, B) are not homeomorphic.

So despite the fact that $A \approx B$, there does not exist a homeomorphism $f: s \to s$ with $f(A) = B$. This example explains why in the definition of absorber we require the set under consideration to be contained in a *σ-compact σZ*-set.

We now partly generalize Corollary 8.3 for the case of Q-manifolds.

Proposition 10.7. *Let E and F be Q-manifolds. In addition, let $\mathcal{M}$ be a class of spaces containing the class of all compact spaces such that moreover for every $A \in \mathcal{M}$, $A \times \Sigma \in \mathcal{M}$. Finally, let A and B be $\mathcal{M}$-absorbers in E and F, respectively. Then A and B are homeomorphic iff they have the same homotopy type.*

Proof. Let $\Omega \subseteq Q$ be a capset [24, Proposition 6.5.4]. Since $\Omega \approx \Sigma$ (Section 5, Statements (IV) and (V)), it follows that for every $A \in \mathcal{M}$, $A \times \Omega \in \mathcal{M}$. First observe that by Lemma 2.1, $E \times \Omega$ has the Z-approximation property.

Now by Proposition 10.4, $A \times \Omega$ is an $\mathcal{M}$-absorber in $E \times \Omega$ and by Corollary 10.3, that $A \times \Omega$ is an $\mathcal{M}$-absorber in $E \times Q$. But E and $E \times Q$ are homeomorphic (Chapman [10, Theorem 15.1]). So we conclude by Theorem 8.2 that A and $A \times \Omega$ are homeomorphic. Similarly it follows that B and $B \times \Omega$ are homeomorphic. Since $A \times \Omega$ is an $\mathcal{M}$-absorber in the Σ-manifold $E \times \Omega$ (this is so because $E \times \Omega$ is a capset in the Q-manifold $E \times Q$, Chapman [9, the proof of Lemma 5.6]; alternatively, use Mogilski's characterization of Σ-manifolds from [21]), and similarly $B \times \Omega$ is an $\mathcal{M}$-absorber in the Σ-manifold $F \times \Omega$ we can apply Corollary 8.3 to conclude that A and B are homeomorphic if and only if they have the same homotopy type. $\square$

Piecing everything together for Borel classes of spaces other than the class G_δ yields the following satisfactory result. (Observe that there does not exist a G_δ-absorber in an infinite-dimensional manifold (use the Baire Category Theorem), so that the following corollary is "best possible".)

Corollary 10.8. *Let E and F be infinite-dimensional manifolds. In addition, let $\mathcal{M}$ be a Borel class of spaces other than the class G_δ. Finally, let A and B be $\mathcal{M}$-absorbers in E and F, respectively. Then A and B are homeomorphic iff they have the same homotopy type.*

Proof. First observe that $\mathcal{F}_\sigma \subseteq \mathcal{M}$, which easily implies that for every $M \in \mathcal{M}$ and σ-compact space T, the product $M \times T$ also belongs to M.

We will prove that A is homeomorphic to an $\mathcal{M}$-absorber is an s-manifold. For this, we only need to consider the case that E is a Q- or a Σ-manifold. If E is a Q-manifold, then by the proof of Proposition 10.7 it follows that A is homeomorphic to an $\mathcal{M}$-absorber in a Σ-manifold. So in fact we only need to consider the case that E is a Σ-manifold. By Chapman [9], there exists an s-manifold $\hat{E}$ such that E is a capset in $\hat{E}$. By Theorem 5.3 and Corollary 4.4 it follows that $\hat{E} \setminus E$ is locally homotopy negligible in $\hat{E}$. This implies, by Corollary 6.2, that A is strongly $\mathcal{M}$-universal in $\hat{E}$. Since compact sets in an s-manifold are Z-sets, it even follows that A is an $\mathcal{M}$-absorber in $\hat{E}$.

By a similar argumentation it also follows that B is homeomorphic to an $\mathcal{M}$-absorber in an s-manifold $\hat{F}$. Now we are in a position to apply Corollary 8.3 to conclude that A and B are homeomorphic iff they have the same homotopy type. $\square$

Proposition 10.9. *Let X be an* AR, *and let A be a strongly $\mathcal{M}$-universal subset of X. Assume that $\mathcal{M}$ contains all compact spaces. Then for every $p \in A$ the weak product $W(A, (p, p, \ldots))$ is strongly $\mathcal{M}$-universal in X^∞.*

Proof. We first observe that A is an AR because X is (Corollary 4.5). As a consequence, A is path-connected. Let $G: X \times I \to X$ be a contraction. In addition, let $F: X \times I \to X$ be a homotopy such that $F_0 = 1_X$ and $F(X \times (0, 1]) \subseteq A$ (Corollary 4.4). For every $t \in I$, put $H_t = F_t \circ G_t$. Then H is a homotopy having the following properties:

- $H_0 = 1_X$,
- $H(X \times (0, 1]) \subseteq A$,
- H_1 is constant.

Since A is path-connected, we may even assume that

- H_1 is the constant function with value p.

We can now follow the proof of the corresponding result in Bestvina and Mogilski [3, Proposition 2.5] verbatim. □

The results in this section can be used to prove along the lines of Bestvina and Mogilski [3, Proposition 6.4] that for every Borel class of spaces other than the class G_δ there exist absorbers in every infinite-dimensional manifold.

11. More definitions

The concepts in this section are of importance if one wishes to prove in a concrete situation that a certain subset of an ANR is an absorber of some sort. See Section 12 for details.

As before, let $\mathscr{X} = (X_\gamma)_{\gamma \in \Gamma}$ be an order preserving indexed collection of subsets of a space E.

Definition 11.1. The system $\mathscr{X}$ is called *$\mathscr{M}_\Gamma$-universal* in E if for every $\mathscr{M}_\Gamma$-system $(A_\gamma)_{\gamma \in \Gamma}$ in Q, there is an embedding $f: Q \to E$ such that for every $\gamma \in \Gamma, f^{-1}(X_\gamma) = A_\gamma$.

Observe that if $\mathscr{X}$ is strongly $\mathscr{M}_\Gamma$-universal in E, then it is $\mathscr{M}_\Gamma$-universal in E.

Definition 11.2. The $\mathscr{M}_\Gamma$-system $\mathscr{X}$ is called *reflexively $\mathscr{M}_\Gamma$-universal* in E if for every $\varepsilon > 0$ and every map $f: E \to E$ such that

(1) $\overline{f(E)}$ is compact,

(2) f restricts to a Z-embedding on some compact set K,

there exists an embedding $g: E \to E$ such that

(3) $g|K = F|K$ and $\hat{d}(F, g) < \varepsilon$,

(4) $\overline{g(E)}$ is a Z-set,

(5) $(\forall \gamma \in \Gamma)[g^{-1}(X_\gamma) \setminus K = X_\gamma \setminus K]$.

Proposition 11.3. *Let E be an* ANR. *The following statements are equivalent*:

(1) *$\mathscr{X}$ is strongly $\mathscr{M}_\Gamma$-universal in E*,

(2) *$\mathscr{X}$ is $\mathscr{M}_\Gamma$-universal and reflexively $\mathscr{M}_\Gamma$-universal in E.*

Proof. We first prove that (2)$\Rightarrow$(1). To this end, let $(A_\gamma)_{\gamma\in\Gamma}$ be an $\mathcal{M}_\Gamma$-system in Q, and let $f: Q\to E$ be a continuous function that restricts to a Z-embedding on some compact subset K of Q. Let $\varepsilon>0$. Since $\mathscr{X}$ is $\mathcal{M}_\Gamma$-universal, there exists a Z-embedding $j: Q\to E$ with $j^{-1}(X_\gamma)=A_\gamma$ for all γ. Since Q is an AR, we can extend $j^{-1}: j(Q)\to Q$ to a map $\tau: E\to Q$. Since $\mathscr{X}$ is reflexively $\mathcal{M}_\Gamma$-universal, there is an embedding $g: E\to E$ such that

(1) $\overline{g(E)}$ is a Z-set,

(2) $\hat{d}(g, f\circ\tau)<\varepsilon$,

(3) $(\forall\gamma\in\Gamma)[g^{-1}(X_\gamma)\backslash j(K)=X_\gamma\backslash j(K)]$,

(4) $g\,|\,j(K)=f\circ\tau\,|\,j(K)$.

Now put $h=g\circ j: Q\to E$. Observe that

$$\varepsilon>\hat{d}(g, f\circ\tau)\geqslant\hat{d}(g\circ j, f\circ\tau\circ j)=\hat{d}(h, f).$$

It is clear that h is a Z-embedding since $h(Q)\subseteq g(E)$, and that $h\,|\,K=f\,|\,K$. Finally, for every $\gamma\in\Gamma$ we have

$$\begin{aligned} h^{-1}(X_\gamma)\backslash K &= j^{-1}(g^{-1}(X_\gamma))\backslash j^{-1}(j(K)) \\ &= j^{-1}(g^{-1}(X_\gamma)\backslash j(K)) \\ &= j^{-1}(X_\gamma\backslash j(K)) \\ &= A_\gamma\backslash K. \end{aligned}$$

We next prove (1)$\Rightarrow$(2). We only need to prove that $\mathscr{X}$ is reflexively $\mathcal{M}_\Gamma$-universal. To this end, let $\varepsilon>0$ and let $f: E\to E$ be a map such that

(1) $\overline{f(E)}$ is compact,

(2) f restricts to a Z-embedding on some compact set K.

Since $\overline{f(E)}$ is compact, there is a compactification bE of E such that f can be extended to a continuous function $\bar{f}: bE\to\overline{f(E)}$. (We may assume that E is a subspace of Q. Then the closure of the graph of f in the product $Q\times\overline{f(E)}$ is the desired compactification bE of E. The restriction to bE of the projection onto the first factor of $Q\times\overline{f(E)}$ extends f.) Now by Proposition 3.3 we can approximate $\bar{f}$ arbitrarily closely by a Z-embedding, the restriction to E of which is easily seen to be as required. $\square$

Definition 11.4. Let M be a subspace of a space E. Then M is called *reflexively universal in* E if for every $\varepsilon>0$ and every map $f: E\to E$ such that

(1) $\overline{f(E)}$ is compact,

(2) f restricts to a Z-embedding on some compact set K,

there exists an embedding $g: E\to E$ such that

(3) $g\,|\,K=f\,|\,K$ and $\hat{d}(f, g)<\varepsilon$,

(4) $\overline{g(E)}$ is a Z-set,

(5) $g^{-1}(M)\backslash K=M\backslash K$.

Observe that if $M\in\mathcal{M}$, and if M is reflexively universal in E, then it is reflexively $\mathcal{M}$-universal in E.

We finish this section with the following rather technical result, that will be important in the next section.

Proposition 11.5. *Let Y be a dense linear subspace of s. Then $s\times Y$ is reflexively universal in $Q\times Q$.*

Proof. Let $f=(f_1,f_2):Q\times Q\to Q\times Q$ be a map that restricts to a Z-embedding on some compact subset K of $Q\times Q$. Without loss of generality we may assume that f is a Z-embedding [24, Theorem 6.4.8]. Let $\varepsilon>0$. As in the proof of Proposition 10.2 we can find a map $g_1:Q\times Q\to Q$ such that

(1) $f_1|K=g_1|K$,
(2) $\hat{d}(f_1,g_1)<\varepsilon/2$,
(3) for $x\notin K$, $d(f_1(x),g_1(x))<\frac{1}{2}d(f(x),f(K))$,
(4) $g_1|(Q\times Q)\backslash K$ is one-to-one,
(5) $g_1((Q\times Q)\backslash K)\in\mathscr{Z}_\sigma(Q)$, and
(6) $g_1^{-1}(s)\backslash K=(s\times Q)\backslash K$.

(We use that s is strongly G_δ-universal in Q.)

Since Y is a dense linear subspace of s, by Curtis [11], $Q\backslash Y$ is locally homotopy negligible in Q. We can therefore find a homotopy $H:Q\times I\to Q$ such that

(7) $H_0=1$,
(8) $H_t(Q)\subseteq Y\cap[-1+t,1-t]^\infty$ if $t>0$, and
(9) $\hat{d}(H_t,1)\leq 2t$ for all $t\in I$.

Define $g_2:Q\times Q\to Q$ by

$$g_2(x,y)=H(f_2(x,y),\tfrac{1}{6}\varepsilon\cdot d(f(x,y),f(K)))+\tfrac{1}{6}\varepsilon\cdot d(f(x,y),f(K))\cdot y.$$

Then g_2 satisfies

(10) $g_2|K=f_2|K$,
(11) $\hat{d}(f_2,g_2)<\varepsilon/2$,
(12) for $x\notin K$, $d(f_2(x),g_2(x))<\frac{1}{2}d(f(x),f(K))$,
(13) for $(x,y)\notin K$ we have $g_2(x,y)\in Y$ if and only if $y\in Y$.

Put $g=(g_1,g_2):Q\times Q\to Q\times Q$. Then $\hat{d}(f,g)<\varepsilon$ and $f|K=g|K$. By (1), (3), (4), (10) and (12) we have that g is one-to-one and hence an embedding. By (1), (5) and (10) we obtain that g is a Z-embedding. Finally by (6) and (13) it follows that $g^{-1}(s\times Y)\backslash K=(s\times Y)\backslash K$. □

12. Applications

Each subset A of $\mathbb{N}$ can be identified with a point in the Cantor set C, namely with the point χ_A defined by $(\chi_A)_i=1$ if and only if $i\in A$ (so A is in fact identified with its characteristic function). In this way we associate to each $\mathscr{F}\subseteq\mathscr{P}(\mathbb{N})$ the subset $F=\{\chi_A\colon A\in\mathscr{F}\}$ of C. In the sequel a subset of $\mathscr{P}(\mathbb{N})$ will be denoted by a script letter and the corresponding italic capital character denotes the corresponding subset of C. We are interested in free filters $\mathscr{F}$ on $\mathbb{N}$. Let $\mathscr{N}_0$ be the free filter on $\mathbb{N}$ consisting of all cofinite subsets of $\mathbb{N}$. It is easily seen that a filter $\mathscr{F}$ on $\mathbb{N}$ is free if and only if $\mathscr{N}_0 \subseteq \mathscr{F}$.

Lemma 12.1 (Dobrowolski, Marciszewski and Mogilski [19]). *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$. Then the following conditions are equivalent:*

(i) *F is an element of the σ-algebra generated by the open subsets and first category subsets of C,*

(ii) *F is a first category subset of C.*

Lemma 12.2 (Dobrowolski, Marciszewski and Mogilski [19]). *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ such that F is a first category subset of C. Then there is a decomposition $\bigcup_{i=1}^{\infty} \mathbb{N}_i$ of $\mathbb{N}$ into infinite pairwise disjoint sets $\mathbb{N}_i$ such that if $\mathscr{F}_i = \{A \cap \mathbb{N}_i : A \in \mathscr{F}\}$, then*

(1) *for every $n \in \mathbb{N}$ and for every $A_i \in \mathscr{F}_i$ $(i \leq n)$, $\bigcup_{i=1}^{n} A_i \cup \bigcup_{i>n} \mathbb{N}_i \in \mathscr{F}$,*

(2) *F_i embeds as a closed subset of F, and*

(3) *$\mathscr{F}_i$ is a free filter on $\mathbb{N}_i$.*

For each filter on $\mathbb{N}$ we can topologize the set $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ as follows: each element of $\mathbb{N}$ is isolated and the collection $\{U \cup \{\infty\} : U \in \mathscr{F}\}$ is a neighborhood base at ∞. This topological space will be denoted by $\mathbb{N}_F$. It is an easy exercise to verify that $\mathbb{N}_F$ is Tychonov if and only if $\mathscr{F}$ is a free filter on $\mathbb{N}$.

Lemma 12.3. *Let A be a closed and nowhere dense subset of C. Then for every $n \in \mathbb{N}$, there is $a \in \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}}$ such that $(\{0,1\}^{\{1,2,\dots,n\}} \times \{a\}) \cap A = \emptyset$.*

Proof. Suppose to the contrary that for each $a \in \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}}$, $(\{0,1\}^{\{1,2,\dots,n\}} \times \{a\}) \cap A \neq \emptyset$. For $s \in \{0,1\}^{\{1,2,\dots,n\}}$, let $A_s = \{x \in \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}} : (s,x) \in A\}$. Then by assumption $\bigcup_{s\in\{0,1\}^{\{1,2,\dots,n\}}} A_s = \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}}$. We claim that each A_s is closed in $\{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}}$. Let $x \in \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}} \setminus A_s$. Then $(s,x) \notin A$. Since A is closed in C we can find a neighborhood V of x such that $(\{s\} \times V) \cap A = \emptyset$. This gives $V \cap A_s = \emptyset$. Since $\{0,1\}^{\{1,2,\dots,n\}}$ is finite, there is $s \in \{0,1\}^{\{1,2,\dots,n\}}$ such that A_s has nonempty interior. Let U be a nonempty open subset of A_s. Then $\{s\} \times U$ is open in C and $\{s\} \times U \subseteq A$. This contradicts the fact that A is nowhere dense. □

Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ and let F be the corresponding subset of C. Define

$$c_F = \{x \in \mathbb{R}^{\mathbb{N}} : (\forall \varepsilon > 0)(\exists A \in \mathscr{F})(\forall i \in A)[|x_i| < \varepsilon]\}.$$

Since $\mathscr{F}$ is a filter it is easily seen that

$$c_F = \{x \in \mathbb{R}^{\mathbb{N}} : (\forall \varepsilon > 0)[\{i : |x_i| < \varepsilon\} \in \mathscr{F}]\}.$$

If $\mathscr{F} = \mathscr{N}_0$, then c_F will be denoted by c_0. Note that for any free filter $\mathscr{F}$, $c_0 \subseteq c_F$. In Calbrix [4, 5] it is proved that c_F is an absolute Borel set if and only if F is an absolute Borel set. Let Q be represented by $\hat{\mathbb{R}}^{\mathbb{N}}$. We use the following arithmetic:

$1/0=\infty$ and $\infty+a=\infty$, if a is finite. We define the continuous function $\Psi:\hat{\mathbb{R}}\to\hat{\mathbb{R}}^{\hat{\mathbb{N}}}$ by

$$\Psi(r)_n=\operatorname{sign}(r)\min\{|r|,n\}.$$

For $f\in\hat{\mathbb{R}}^{\mathbb{N}}$ let $\hat{f}$ be the extension of f over $\hat{\mathbb{N}}$ that assigns 0 to ∞. Define a map $\Phi:\hat{\mathbb{R}}^{\mathbb{N}}\times\hat{\mathbb{R}}\to\hat{\mathbb{R}}^{\hat{\mathbb{N}}}$ by $\Phi(f,r)=\hat{f}+\Psi(r)$. As in Dijkstra, van Mill and Mogilski [17] one can show that Φ is a homeomorphism. Moreover, a straightforward calculation yields the following.

Lemma 12.4. $\Phi(c_F)\times\mathbb{R}=C_p(\mathbb{N}_F)$.

Lemma 12.5. *Let $\mathcal{F}$ be a free filter on $\mathbb{N}$ such that F is a first category subset of C. Then c_F is contained in a σZ-set of Q.*

Proof. Let $\hat{T}=\hat{\mathbb{R}}\setminus((-3,-1)\cup(1,3))$. Define $g:\hat{T}\to\hat{\mathbb{R}}$ by

$$g(x)=\begin{cases}x, & x\in[-1,1],\\ x-2, & x\in[3,\infty],\\ x+2, & x\in[-\infty,-3].\end{cases}$$

Let $g^\infty:\hat{T}^{\mathbb{N}}\to\hat{\mathbb{R}}^{\mathbb{N}}$ be defined by $g^\infty((x_i)_i)=(g(x_i))_i$. Then g^∞ is obviously continuous. Find nowhere dense closed subsets A_n $(n\in\mathbb{N})$ of C such that $F\subseteq\bigcup_{n=1}^\infty A_n$. For $n\in\mathbb{N}$ let $B_n=\{x\in\hat{T}^{\mathbb{N}}:\{i:|x_i|<2\}\in\mathcal{A}_n\}$.

Claim 1. *B_n is closed in $\hat{T}^{\mathbb{N}}$.*

Let $x\in\hat{T}^{\mathbb{N}}\setminus B_n$. Then $K=\{i:|x_i|<2\}\notin\mathcal{A}_n$ and consequently, $\chi_K\notin A_n$. Hence there are $m\in\mathbb{N}$ and $U_i\subseteq\{0,1\}$ such that if

$$U=U_1\times\cdots\times U_m\times\{0,1\}\times\{0,1\}\times\cdots,$$

then $\chi_K\in U$ and $U\cap A_n=\emptyset$. For $i\leqslant m$ such that $|x_i|\leqslant 1$ let $V_i=[-1,1]$ and for $i\leqslant m$ such that $|x_i|\geqslant 3$ let $V_i=[3,\infty)$. Put $V=V_1\times\cdots\times V_m\times\hat{T}^{\mathbb{N}}$. Then V is open in $\hat{T}^{\mathbb{N}}$, $x\in V$ and $V\cap B_n=\emptyset$.

Claim 2. *$g^\infty(B_n)\in\mathcal{Z}(Q)$.*

Let $\varepsilon>0$. Find $i\in\mathbb{N}$ such that $\sum_{j=i}^\infty 2^{-j}<\varepsilon/2$. By Lemma 12.3 there is $a=(a_{i+1},a_{i+2},\ldots)\in\{0,1\}^{\mathbb{N}\setminus\{1,2,\ldots,i\}}$ such that $(\{0,1\}^{\{1,2,\ldots,i\}}\times\{a\})\cap A_n=\emptyset$. Define $f:Q\to Q$ by

$$f(x)_j=\begin{cases}x_j, & j\leqslant i,\\ 0, & a_j=1,\ j>i,\\ 10, & a_j=0,\ j>i.\end{cases}$$

Then $\hat{d}(f,1)<\varepsilon$. We claim that $f(Q)\cap g^\infty(B_n)=\emptyset$. Suppose $f(y)=g^\infty(x)$ for $x\in B_n$. We then have $C=\{i:|x_i|<2\}\in\mathcal{A}_n$. So $\chi_C\notin\{0,1\}^{\{1,2,\ldots,i\}}\times\{a\}$. This gives us a $j>i$ such that $a_j=0$ and $|x_j|\leqslant 1$ or $a_j=1$ and $|x_j|\geqslant 3$. If $a_j=0$ and $|x_j|\leqslant 1$, then $|g(x_j)|\leqslant 1$, hence $|f(y)_j|\leqslant 1$. This gives $|f(y)_j|=0$, so $a_j=1$ which is a contradiction. Similarly we arrive at a contradiction when $a_j=1$ and $|x_j|\geqslant 3$.

Claim 3. $c_F\subseteq\bigcup_{n=1}^{\infty} g^{\infty}(B_n)$.

Let $x\in c_F\subset\mathbb{R}^{\mathbb{N}}$. Define the point $y=(y_i)_{i\in\mathbb{N}}$ by

$$y_i=\begin{cases} x_i, & x_i\in[-1,1],\\ x_i+2, & x_i\in(1,\infty),\\ x_i-2, & x_i\in(-\infty,-1).\end{cases}$$

Then $y\in\hat{T}^{\mathbb{N}}$ and $g^{\infty}(y)=x$. Since $x\in c_F$, $\{i\colon |x_i|<1\}\in\mathcal{F}$. Because $\{i\colon |x_i|<1\}\subseteq\{i\colon |y_i|<2\}$ and $\mathcal{F}$ is a filter, $\{i\colon |y_i|<2\}\in\mathcal{F}\subseteq\bigcup_{n=1}^{\infty}\mathcal{A}_n$. We conclude that $y\in\bigcup_{n=1}^{\infty}B_n$. $\square$

Lemma 12.6. *Let X be a countable nondiscrete Tychonov space such that $C_p(X)$ is analytic. Then $C_p(X)$ is contained in a σZ-set of Q.*

Proof. Let $x\in X$ be a nonisolated point. Identify $X\backslash\{x\}$ with $\mathbb{N}$. Define $\mathcal{F}=\{U\subseteq\mathbb{N}\colon U\cup\{x\}$ is a neighborhood of $x\}$. Then $C_p(X)\subseteq C_p(\mathbb{N}_F)$. By the proof of Corollary 3.6 from Dobrowolski, Marciszewski and Mogilski [19], $\mathcal{F}$ is a free first category filter on $\mathbb{N}$. By Lemma 12.5, c_F is contained in a σZ-set, hence by Lemma 12.4, the same holds for $C_p(\mathbb{N}_F)$. $\square$

It is easily seen that $c_0=\{x\in\mathbb{R}^{\mathbb{N}}\colon \lim_{i\to\infty}x_i=0\}$. Define for each $n\in\mathbb{N}$,

$$\Sigma_n=\{x\in\mathbb{R}^{\mathbb{N}}\colon |x_i|\leqslant 2^{-n} \text{ for all but finitely many } i\}.$$

Observe that $(\Sigma_n)_{n\in\mathbb{N}}$ is a decreasing sequence of σZ-sets in Q the intersection of which is c_0.

Lemma 12.7. *Let $\mathcal{F}$ be a free filter on $\mathbb{N}$ such that c_F is analytic. Then c_F is $\mathcal{F}_{\sigma\delta}$-universal in Q.*

Proof. As in the proof of Corollary 3.6 of Dobrowolski, Marciszewski and Mogilski [19], F is a first category subset of C (here we need that c_F is analytic). Find a decomposition $\bigcup_{i=1}^{\infty}\mathbb{N}_i$ of $\mathbb{N}$ into infinite pairwise disjoint sets $\mathbb{N}_i$ as prescribed in Lemma 12.2, it follows that c_{F_i} embeds as a closed subset of c_F, hence c_{F_i} is analytic as well. Again we conclude that F_i is of the first category. Let B_i be a nowhere dense σ-compactum in $\{0,1\}^{\mathbb{N}_i}$ such that $F_i\subseteq B_i$. It is easily seen that $c_0\subseteq c_F\subseteq\prod_{i=1}^{\infty}c_{F_i}$. For each $i\in\mathbb{N}$, let $Q_i=[-2^{-i+1},2^{-i+1}]^{\mathbb{N}_i}$. Then $\prod_{i=1}^{\infty}Q_i\subseteq Q$. As in the proof of Lemma 12.5 it follows that $c_{F_i}\cap Q_i$ is contained in a σZ-set $D_i\subseteq Q_i$.

Let $A=\bigcap_{n=1}^{\infty}A_n$ where each A_n is a σ-compact subset of Q and $A_{n+1}\subseteq A_n$ for every $n\in\mathbb{N}$. Define for every i,

$$C_i=\{x\in Q_i\colon (\forall j)[|x_j|\leqslant 2^{k-j} \text{ for some } k]\}.$$

Note that for each $x \in C_i$ we have $\lim_{j\to\infty} x_j = 0$. By Corollary 4.9 of Curtis [11], C_i is an $\mathcal{F}_\sigma$-absorber in Q_i. By Lemma 5.5, there is an embedding $f_i : Q \to Q_i$ such that $f_i^{-1}(C_i) = A_i$ and $f_i(Q \setminus A_i) \cap D_i = \emptyset$. Define $f = (f_i)_i : Q \to \prod_{i=1}^{\infty} Q_i \subseteq Q$. Then f is an embedding. We claim that $f^{-1}(c_F) = A$. First, let $x \in A$ and $n \in \mathbb{N}$. If $i > n$, then $f_i(x) \in Q_i$, so all components of $f_i(x)$ are in $[-2^{-n}, 2^{-n}]$. If $i \leqslant n$, then since $x \in A_i$ we have $f_i(x) \in C_i$. In this case only finitely many components of $f_i(x)$ are outside $[-2^{-n}, 2^{-n}]$. We conclude that only finitely many components of $f(x)$ are outside $[-2^{-n}, 2^{-n}]$. This implies that $f(x) \in \Sigma_n$. Since n was arbitrary we get $f(x) \in c_0 \subseteq c_F$. Second, let $f(x) \in c_F$ and fix $n \in \mathbb{N}$. Then $f_n(x) \in c_{F_n} \cap Q_n \subseteq D_n$, hence $x \in A_n$. We conclude that $x \in A$. $\square$

Lemma 12.8. *Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Then c_F is reflexively universal in Q.*

Proof. If $\mathcal{F} = \mathcal{N}_0$, it follows from Lemma 6.2 in Dijkstra, van Mill and Mogilski [17]. So suppose there is $U \in \mathcal{F}$ with infinite complement. Let $\mathcal{G} = \{U \cap V : V \in \mathcal{F}\}$. Then $\mathcal{G}$ is a free filter on U and $c_F = s \times c_G$. By Lemma 11.5 we obtain the desired result. $\square$

Lemma 12.9 (Dobrowolski, Marciszewski and Mogilski [19]). *Let X be a countable nondiscrete Tychonov space. Then there exists a clopen subset Y of X with exactly one accumulation point, or there exists a decomposition of X into infinitely many pairwise disjoint nondiscrete clopen subsets of X.*

We now come to the main result in this section.

Theorem 12.10. *Let X be a countable nondiscrete Tychonov space such that $C_p(X)$ is an $F_{\sigma\delta}$-subset of Q. Then $C_p(X)$ is an $\mathcal{F}_{\sigma\delta}$-absorber in Q.*

Proof. By Lemma 12.6, $C_p(X)$ is contained in a σZ-set of Q. By Lemma 12.9, there are two cases to consider. First suppose that there exists a clopen subset Y of X with exactly one nonisolated point. Then there is a free filter $\mathcal{F}$ on $\mathbb{N}$ such that $\mathbb{N}_F$ is homeomorphic to Y. It easily follows that c_F is an $F_{\sigma\delta}$, hence by Lemmas 12.7 and 12.8, c_F is strongly $F_{\sigma\delta}$-universal. By Lemmas 10.5 and 12.4 we obtain that $C_p(\mathbb{N}_F)$ is strongly $F_{\sigma\delta}$-universal. Since Y is clopen in X we have by Lemma 10.5 and the fact that the complement in Q of a dense linear subspace of s is locally homotopy negligible in Q (Curtis [11]) that $C_p(X)$ is strongly $F_{\sigma\delta}$-universal.

Second, suppose that $X = \bigcup_{n=1}^{\infty} X_n$ a disjoint union of nondiscrete clopen subsets of X. Since for each $n \in \mathbb{N}$, $C_p(X_n)$ is a dense linear subspace of $\mathbb{R}^{\mathbb{N}}$, $C_p(X_n)$ contains a copy of l_f^2. This is so because the linear span of every countable dense subset of $\mathbb{R}^\infty$ is homeomorphic to l_f^2 (Curtis, Dobrowolski and Mogilski [12]). Since we have infinitely many $C_p(X_n)$, we may assume that $C_p(X_n)$ contains a copy of σ_ω, or consequently that $C_p(X_n)$ contains an F_σ-absorber B_n. By Lemma 12.6, $C_p(X_n)$ is contained in a σZ-set C_n. We now have

$$\prod_{n=1}^{\infty} B_n \subseteq C_p(X) \subseteq \prod_{n=1}^{\infty} C_n.$$

By Proposition 10.2 we obtain that $C_p(X)$ is strongly $F_{\sigma\delta}$-universal. □

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