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Hyperspaces of infinite-dimensional compacta

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Abstract. If X is an infinite product of non-degenerate Peano continua then the set $\dim_\infty(X) = \{A \in 2^X : \dim A = \infty\}$ is an $F_{\sigma\delta}$-absorber in 2^X. As a consequence, there is a homeomorphism $f: 2^X \to Q^\infty$ such that $f[\dim_\infty(X)] = B^\infty$, where B denotes the pseudo-boundary of the Hilbert cube Q. There is a locally infinite-dimensional Peano continuum X such that for every n, $\dim_\infty(X^n)$ is not homeomorphic to B^∞.

Keywords: Hilbert cube, absorbing system, $F_{\sigma\delta}$, hyperspace, Peano continuum, cohomological dimension.

AMS Subject Classification: 57N20.

1. Introduction

If X is a compact metric space then 2^X denotes the hyperspace of all nonempty closed subsets of X topologized by the Hausdorff metric. For $k \in \{0, 1, \ldots, \infty\}$ let $\dim_{\geqslant k}(X)$ denote the subspace consisting of all $\geqslant k$-dimensional elements of 2^X. We define $\dim_k(X)$ and $\dim_{\leqslant k}(X)$ in the same way.

Let Q denote the Hilbert cube. In Dijkstra et al. [9] it is proved that there exists a homeomorphism $f: 2^Q \to Q^\infty$ such that for every $k \geqslant 0$,

$$f[\dim_{\geqslant k}(Q)] = \underbrace{B \times \cdots \times B}_{k \text{ times}} \times Q \times Q \times \cdots, \tag{1}$$

where B is the pseudo-boundary of Q. As a consequence,

$$f[\dim_\infty(Q)] = B^\infty. \tag{2}$$

The proof of (1) is based in an essential way on the "convex" structure of Q as well as on the technique of absorbing systems.

Since for every non-degenerate Peano continuum X we have $2^X \approx Q$ (Curtis and Schori [7]; see also [13, Chapter 8]), it is natural to ask for which Peano continua X there is a homeomorphism of pairs $(2^X, \dim_\infty(X)) \approx (Q^\infty, B^\infty)$. Since $\dim_\infty(X) \neq \phi$ iff $\dim X = \infty$, this question is of interest only when X is infinite-dimensional.

In this paper we will prove that if X is an infinite product of non-degenerate Peano continua then there is a homeomorphism of pairs

$$(2^X, \dim_\infty(X)) \approx (Q^\infty, B^\infty).$$

This generalizes (2) above. We do not know whether (1) can also be generalized. We come back to this in §4, where we will explain why our proof does not work for obtaining a generalization of (1).

The proof of our result is different from the proof of (1) in Dijkstra et al. [9] because a "convex" structure is not available on an aribtrary infinite product of Peano continua.

We also present an example of an everywhere infinite-dimensional Peano continuum X such that for every $n \in \mathbb{N}$, $\dim_\infty(X^n) \not\approx B^\infty$. So our result is in a sense "best possible".

2. Terminology

All spaces under discussion are separable and metrizable. For any space X we let d denote an *admissible* metric on X, i.e., a metric that generates the topology. All our metrics are bounded by 1.

As usual I denotes the interval $[0,1]$ and Q the Hilbert cube $\Pi_{i=1}^\infty [-1,1]_i$ with metric $d(x,y) = \Sigma_{i=1}^\infty 2^{-(i+1)}|x_i - y_i|$. In addition s is *the pseudo-interior* of Q, i.e., $\{x \in Q : (\forall i \in \mathbb{N})(|x_i| < 1)\}$. The complement B of s in Q is called the *pseudo-boundary* of Q. Any space that is homeomorphic to Q is called a *Hilbert cube*. If X is a set then the identity function on X will be denoted by 1_X.

Let A be a closed subset of a space X. We say that A is a Z-set provided that every map $f: Q \to X$ can be approximated arbitrarily closely by a map $g: Q \to X \setminus A$. The collection of all Z-sets in X will be denoted by $\mathscr{Z}(X)$. A countable union of Z-sets is called a *σZ-set*. A *Z-embedding* is an embedding the range of which is a Z-set.

Let $\mathscr{M}$ be a class of spaces that is topological and closed hereditary.

2.1. DEFINITION. Let X be a Hilbert cube. A subset $A \subseteq X$ is called *strongly $\mathscr{M}$-universal* in X if for every $M \in \mathscr{M}$ with $M \subseteq Q$ and every compact set $K \subseteq Q$, every Z-embedding $f: Q \to X$ can be approximated arbitrarily closely by a Z-embedding $g: Q \to X$ such that $g|K = f|K$ while moreover $g^{-1}[A] \setminus K = M \setminus K$.

2.2. DEFINITION. Let X be a Hilbert cube. A subset $A \subseteq X$ is called an *$\mathscr{M}$-absorber* in X if:

(1) $A \in \mathscr{M}$;

(2) there is a σZ-set $S \subseteq X$ with $A \subseteq S$;
(3) *A is strongly $\mathscr{M}$*-universal in X.

2.3. THEOREM ([9]). *Let X be a Hilbert cube and let A and B be $\mathscr{M}$-absorbers for X. Then there is a homeomorphism $h: X \to X$ with $h[A] = B$. Moreover, h can be chosen arbitrarily close to the identity.*

There are three absorbers that are important for us in the present paper. The first one is an absorber for the class consisting of all finite-dimensional compacta. Such absorbers were first constructed by Anderson and Bessaga and Pełczyński [2] and were called *fd-capsets* by Anderson. A basic example of an fd-capset in Q is

$$\{x \in Q : (\exists N \in \mathbb{N})(\forall n \geqslant N)(x_n = 0)\}.$$

For details, see [2]. The second one is an absorber for the class of all compacta. Again, such absorbers were first constructed by Anderson and Bessaga and Pełczyński: they were called *capsets* by Anderson. A basic example of a capset in Q is B. For details, see [2] and [13, Chapter 6]. The third one is an absorber for the Borel class $F_{\sigma\delta}$. Such absorbers were first constructed in Bestvina and Mogilski [3]; see also Dijkstra et al. [9]. A basic example of an $F_{\sigma\delta}$-absorber is B^∞ in the Hilbert cube Q^∞. The space B^∞ has been studied intensitively in infinite-dimensional topology during the last years. For more information, see e.g. [3, 4, 10, 9, 8, 1].

2.4. DEFINITION. A subset A of a space X is *locally homotopy negligible in X* if for every map $f: M \to X$, where M is any ANR, and for every open cover $\mathscr{U}$ of X there exists a homotopy $H: M \times I \to X$ such that $\{H(\{x\} \times I)\}_{x \in M}$ refines $\mathscr{U}$, $H_0 = f$ and $H[M \times (0, 1]] \subseteq X \backslash A$.

By a result of Toruńczyk [14], if X is an ANR, for a set $A \subseteq X$ to be locally homotopy negligible it suffices to consider maps $f: M \to X$, where $M \in \{\{\mathrm{pt}\}, I, I^2, \ldots, I^n, \ldots\}$.

If A is an $\mathscr{M}$-absorber in X and if $\mathscr{M}$ contains the class of all finite-dimensional compacta then both A and $X \backslash A$ are locally homotopy negligible in X (Baars et al. [1, Corollary 4.4]). So it follows in particular that if E is one of the absorbers for Q mentioned above, then there is a homotopy $H: Q \times I \to Q$ such that $H_0 = 1_Q$ and $H[Q \times (0, 1]] \subseteq E$. This can be seen also by noting that on the one hand for the three basic examples of absorbers such homotopies exist and that on the other hand every absorber is "equivalent" to its own model (Theorem 2.3).

Let X be a non-degenerate Peano continuum. The subspace of 2^X consisting of all subsets of X of cardinality at most n is denoted by $\mathscr{F}_n(X)$ and $\mathscr{F}(X)$ denotes $\bigcup_{n=1}^\infty \mathscr{F}_n(X)$.

We will make use of the following basic result.

2.5. THEOREM (Curtis [5]). *Let X be a non-degenerate Peano continuum. Then $\mathscr{F}(X)$ contains an fd-capset for 2^X.* □

The following corollary to this result is probably well-known.

2.6. COROLLARY. *Let X be a non-degenerate Peano continuum. If $\mathscr{F}(X) \subseteq Y \subseteq 2^X$ then Y is an AR.*

Proof. This follows immediately from Theorem 2.5 and Toruńczyk [14]. □

Let X be a compact space. It is easy to see that for every k the set $\dim_{\geqslant k}(X)$ is an F_σ-subset of 2^X (Dijkstra et al. [9, the proof of Theorem 4.6]). Consequently, $\dim_\infty(X)$ is an $F_{\sigma\delta}$-subset of 2^X.

We will also need the following result.

2.7. THEOREM (Curtis and Michael [6]). *Let X be a non-degenerate Peano continuum. Then $\dim_{\geqslant 1}(X)$ is a capset for 2^X.*

If A is a non-empty closed subset of X then the inclusion $A \hookrightarrow X$ induces a natural embedding $2^A \to 2^X$. We will always identify 2^A and the range $\{B \in 2^X : B \subseteq A\}$ of this embedding.

Let X be a compact space with admissible metric d. Then the formula

$$d_H(A, B) = \min\{\varepsilon \geqslant 0 : (\forall x \in A)(d(x, B) \leqslant \varepsilon) \text{ and } (\forall x \in B)(d(x, A) \leqslant \varepsilon)\}$$

defines an admissible metric on 2^X. It is called the *Hausdorff metric* on 2^X.

3. The space of infinite-dimensional subcompacta

In this section we will present a proof of our main result that for an infinite product of non-degenerate Peano continua X, the space $\dim_\infty(X)$ is an $F_{\sigma\delta}$-absorber for 2^X.

3.1. LEMMA. *Let X be a non-degenerate Peano continuum. Then for every $c \in X$ there are a continuous injective map $\rho: I \to [0, 2]$, and a map $\phi: X \times 2^I \to 2^X$ such that:*

(1) $\rho(0) = 0$,
(2) $\rho(t) \geqslant t$ *for all* t,
(3) $\phi \mid \{c\} \times \mathscr{F}_1(I) \to \mathscr{F}_1(X)$ *is an embedding the range of which is a proper subset of* $\mathscr{F}_1(X)$,
(4) *If* $A \in 2^I$ *then* $\phi(c, A) = \bigcup\{\phi(c, \{a\}) : a \in A\}$, *i.e.,* $\phi \mid \{c\} \times 2^I$ *is the hyperspace embedding induced by the embedding* $\phi \mid \{c\} \times \mathscr{F}_1(I) \to \mathscr{F}_1(X)$,
(5) $\phi(x, \{0\}) = \{x\}$, *for all* $x \in X$,

(6) $\dim \phi(x, A) \leqslant 1$ *for all* $A \in 2^I$ *and* $x \in X$,
(7) $d_H(\{x\}, \phi(x, A)) \leqslant \rho(\max(A))$ *for all* $A \in 2^I$ *and* $x \in X$.

Proof. Pick an arbitrary $c \in X$. There is an embedding $\varphi: I \to X$ such that $\varphi(0) = c$ ([13, Theorem 5.3.13]). We may assume without loss of generality that $\varphi[I] \neq X$. Let $\mathscr{A}$ be the subspace

$$(X \times \{\{0\}\}) \cup (\{c\} \times 2^I)$$

of $X \times 2^I$ and define $\psi: \mathscr{A} \to \mathscr{F}(X) \cup 2^{\varphi[I]}$ as follows:

$$\begin{cases} \psi(x, \{0\}) = \{x\} & (x \in X), \\ \psi(c, A) = \varphi[A] & (A \in 2^I). \end{cases}$$

Note that $\mathscr{A}$ is closed in $X \times 2^I$. Since $\mathscr{F}(X) \cup 2^{\varphi[I]}$ is an AR (Corollary 2.6) we can extend ψ over $X \times 2^I$ to a map $\phi: X \times 2^I \to \mathscr{F}(X) \cup 2^{\varphi[I]}$. It is clear that ϕ satisfies (3) through (6) above.

Define $\sigma: I \to I$ by

$$\sigma(t) = \max\{d_H(\{x\}, \phi(x, A)) : (x, A) \in X \times 2^{[0,t]}\}.$$

Note that σ is increasing and that $\sigma(t) = 0$ iff $t = 0$. Define ρ by $\rho(t) = \sigma(t) + t$ $(t \in I)$.

Then ϕ and ρ are as desired. □

3.2. LEMMA. *Let X be a non-degenerate Peano continuum. Then for every $c \in X$ there is a homotopy $r: X \times I \to \mathscr{F}(X)$ such that for every $x \in X$,*

(1) $r(x, 0) = \{x\}$,
(2) $r(x, 1) = \{c\}$.

Proof. Pick an arbitrary $c \in X$. By Corollary 2.6 it follows that $\mathscr{F}(X)$ is an AR. So $\mathscr{F}(X)$ can be contracted to the point $\{c\}$, say by the homotopy H. It is clear that the function $r(x, t) = H(\{x\}, t)$ $(t \in I)$ is as desired. □

On a product space $X = \Pi_{n=1}^{\infty} X_n$ we will always use the admissible metric d on X given by

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x_n, y_n) \qquad (x, y \in X),$$

where d_n is an arbitrary admissible metric on X_n. Note that if two points agree on their first n coordinates, then their distance is $\leqslant 2^{-n}$. We also let for every m, π_m be the projection from X onto X_m.

The following result is one of the main tools in our proof.

3.3. LEMMA. *Let $X = \Pi_{n=1}^{\infty} X_n$, where each X_n is a non-degenerate Peano continuum. Then for every $c = (c_1, c_2, \ldots) \in X$ there is homotopy $\Gamma: X \times I \to \mathscr{F}(X)$ such that for all $x \in X$ and $t \in I$:*

(1) $\Gamma(x, 0) = \{x\}$,
(2) *If* $t \in [2^{-n}, 2^{-(n-1)}]$ *then for all* $m \geqslant n + 1$, $\pi_m[\Gamma(x, t)] = \{c_m\}$,
(3) *If* $t \in [2^{-n}, 2^{-(n-1)}]$ *then for all* $m \leqslant n - 1$, $\pi_m[\Gamma(x, t)] = \{x_m\}$,
(4) $d_H(\{x\}, \Gamma(x, t)) \leqslant 2t$.

Proof. Pick an arbitrary element $c = (c_1, c_2, \ldots) \in X$. Define for $n \in \mathbb{N}$ a map $\gamma_n: [2^{-n}, 2^{-(n-1)}] \to I$ such that $\gamma_n(2^{-n}) = 0$ and $\gamma_n(2^{-(n-1)}) = 1$.

For every n let $r_n: X_n \times I \to \mathscr{F}(X_n)$ be as in Lemma 3.2 for the point $c_n \in X_n$. Now define a homotopy $\Gamma: X \times I \to \mathscr{F}(X)$ as follows:

$$\begin{cases} \Gamma(x, 0) = \{x\}, \\ \Gamma(x, t) = \{(x_1, \ldots, x_{n-1}, y_i, c_{n+1}, c_{n+2}, \ldots): y_i \in r_n(x_n, \gamma_n(t))\} \\ \hfill (x \in X,\ t \in [2^{-n}, 2^{-(n-1)}]). \end{cases}$$

Claim 1. Γ is well-defined and continuous on $X \times (0, 1]$. Note that the points in which Γ is not trivially well-defined are of the form $(x, 2^{-n})$, $(x \in X,\ n \in \mathbb{N})$. So pick arbitrary $x \in X$ and $n \in \mathbb{N}$. Then on the one hand,

$$\Gamma(x, 2^{-n}) = \{(x_1, \ldots, x_{n-1}, y_i, c_{n+1}, c_{n+2}, \ldots): y_i \in r_n(x_n, \gamma_n(2^{-n}))\}.$$

Note that $r_n(x_n, \gamma_n(2^{-n})) = r_n(x_n, 0) = \{x_n\}$, thus

$$\Gamma(x, 2^{-n}) = \{(x_1, \ldots, x_{n-1}, x_n, c_{n+1}, c_{n+2}, \ldots)\}.$$

On the other hand,

$$\Gamma(x, 2^{-n}) = \{(x_1, \ldots, x_n, y_i, c_{n+2}, c_{n+3}, \ldots): y_i \in r_{n+1}(x_{n+1}, \gamma_{n+1}(2^{-n}))\}.$$

Note that $r_{n+1}(x_{n+1}, \gamma_{n+1}(2^{-n})) = r_{n+1}(x_{n+1}, 1) = \{c_{n+1}\}$, thus

$$\Gamma(x, 2^{-n}) = \{(x_1, \ldots, x_{n-1}, x_n, c_{n+1}, c_{n+2}, \ldots)\}.$$

We conclude that Γ is well-defined and continuous on $X \times (0, 1]$.

Claim 2. For every $x \in X$ and $t \in I$, $d_H(\{x\}, \Gamma(x, t)) \leqslant 2t$.

If $t = 0$ this is clear. If $t > 0$, find $n \in \mathbb{N}$ such that $t \in [2^{-n}, 2^{-(n-1)}]$. Then $d_H(\{x\}, \Gamma(x, t)) \leqslant \Sigma_{i=n}^{\infty} 2^{-i} = 2^{-(n-1)} \leqslant 2t$.

Thus Γ is also continuous on $X \times \{0\}$. Since we already know from Claim 1 that Γ is continuous on $X \times (0, 1]$, we conclude that Γ is continuous. $\square$

We now come to the main result in this section.

3.4. THEOREM. *For every n let X_n be a non-degenerate Peano continuum and let $X = \Pi_{n=1}^{\infty} X_n$. Then $\dim_\infty(X)$ is strongly $F_{\sigma\delta}$-universal in 2^X.*

Proof. Let $A \subseteq Q$ be an $F_{\sigma\delta}$, let $K \subseteq Q$ be compact, let $0 < \varepsilon < 1$ and let $f: Q \to 2^X$ be a Z-embedding. For every n, let $A_n \subseteq Q$ be σ-compact such that $\bigcap_{n=1}^{\infty} A_n = A$. We may assume without loss of generality that if $n \leqslant m$ then $A_n \supseteq A_m$.

For every $n \in \mathbb{N}$ there are a $c_n \in X_n$, a homotopy $\phi_n: X_n \times 2^I \to 2^{X_n}$ and an injective continuous map $\rho_n: I \to [0, 2]$ satisfying the conditions (1) through (7) of Lemma 3.1.

Define $\delta: Q \to I$ and for every $n \in \mathbb{N}$, $\delta_n: Q \to I$ by

$$\delta(x) = \frac{\varepsilon}{6} d(f(x), f[K]) \qquad \text{and} \qquad \delta_n(x) = \rho_n^{-1}(\delta(x)).$$

Observe that δ_n is well-defined because of the properties of the map ρ_n. By (2) in Lemma 3.1 we have for every n and $x \in Q$,

$$\delta_n(x) = \rho_n^{-1}(\delta(x)) \leqslant \delta(x).$$

By Theorem 2.5 there is a homotopy $H: 2^X \times I \to 2^X$ such that:

(1) H_0 is the identity,
(2) $d(H_t, 1) \leqslant 2t$,
(3) $H[2^X \times (0, 1]] \subseteq \mathscr{F}(2^X)$.

Define $g: Q \to 2^X$ by

$$g(x) = H(f(x), \delta(x)).$$

Observe that

(4) for every $x \in Q$, $d_H(f(x), g(x)) \leqslant 2\delta(x)$ (so in particular, $g \mid K = f \mid K$),
(5) $g[Q \backslash K] \subseteq \mathscr{F}(X)$.

By Theorem 2.7 there is for every n an embedding $j_n: Q \to 2^I$ such that $j_n^{-1}[\dim_1(I)] = A_n$. For every $n \in \mathbb{N}$ and $x \in X$ let

$$S_n(x) = \left\{\frac{\delta_n(x)}{4}\right\} \cup \left\{\frac{\delta_n(x)}{2}(1 + y): y \in j_n(x)\right\}.$$

Observe that $S_n(x) \subseteq I$ is compact and non-empty and depends continuously on x. Let $\Gamma: X \times I \to \mathscr{F}(X)$ be as in Lemma 3.3 for the point $(c_1, c_2, \ldots) \in X$. Now define a function $h: Q \to 2^X$ by the formula:

$$h(x) = \bigcup \left\{ \prod_{n=1}^{\infty} \phi_n(y_n, S_n(x)) : (y_1, y_2, \dots) \in \{\Gamma(z, \delta(x)) : z \in g(x)\} \right\}.$$

Note that this map is well-defined because for every $x \in Q$ and $n \in \mathbb{N}$ we have $S_n(x) \subseteq I$. Also, h is clearly continuous. We claim that h is the approximation of f that we are looking for.

Claim 1. For every $x \in Q$, $d_H(f(x), h(x)) \leqslant 5\delta(x) < \varepsilon$. Take an arbitrary $x \in Q$ and put $G = \bigcup\{\Gamma(y, \delta(x)) : y \in g(x)\}$. By (4), $d_H(f(x), g(x)) \leqslant 2\delta(x)$. In addition, by Lemma 3.3(4) it follows that $d_H(g(x), G) \leqslant 2\delta(x)$. We conclude that $d_H(f(x), G) \leqslant 4\delta(x)$. Now take an arbitrary element $(y_1, y_2, \dots) \in G$. Observe that by (7) and by the fact that ρ_n is increasing,

$$d_H(\{y_n\}, \phi_n(y_n, S_n(x)) \leqslant \rho_n(\max(S_n(x)) \leqslant \rho_n(\delta_n(x)) = \delta(x).$$

This clearly implies that $d_H(G, h(x)) \leqslant \delta(x)$. We conclude that $d_H(f(x), h(x)) \leqslant 5\delta(x) < \varepsilon$, which is as required.

Claim 2. $h \mid K = f \mid K$

Note that for all $k \in K$ and $n \in \mathbb{N}$, $\delta(k) = 0 = \delta_n(k)$. This implies that

$$\begin{aligned} h(k) &= \bigcup \left\{ \prod_{n=1}^{\infty} \phi_n(y_n, \{0\}) : y \in \{\Gamma(z, 0) : z \in g(k)\} \right\} \\ &= \bigcup \left\{ \prod_{n=1}^{\infty} \{y_n\} : y \in \{\Gamma(z, 0) : z \in f(k)\} \right\} \\ &= \bigcup \left\{ \prod_{n=1}^{\infty} \{y_n\} : y \in f(k) \right\} \\ &= f(k). \end{aligned}$$

Claim 3. h is an embedding.

First observe that $h[K] \cap h[Q \backslash K] = 0$. This follows easily from the fact that f is an embedding and from Claims 1 and 2 because for every $x \in Q$ we have $d_H(f(x), h(x)) \leqslant 5\delta(x) \leqslant \frac{5}{6} d_H(f(x), f[K]) = \frac{5}{6} d_H(f(x), h[K])$. Since by Claim 2, $h \mid K = f \mid K$ and $f \mid K$ is an embedding, it follows that $h \mid K$ is an embedding. The only thing left to prove is therefore that if $x, y \in Q \backslash K$ and $h(x) = h(y)$, then $x = y$. To this end, pick such x and y and put $r = \delta(x)$ and $t = \delta(y)$. Observe that by the above, r, $s > 0$. There consequently exists m such that $2^{-(m-1)} < \min\{r, t\}$. Then $\pi_m[\cup\{\Gamma(z, r) : z \in g(x)\}] = c_m = \pi_m[\cup\{\Gamma(z, t) : z \in g(y)\}]$. Also, $\pi_m[h(x)] = \phi_m(c_m, S_m(x))$ and $\pi_m[h(y)] = \phi_m(c_m, S_m(y))$. Since $\phi_m \mid \{c_m\} \times 2^I$ is an embedding, it follows that $S_m(x) = S_m(y)$. Since $\min(S_m(x)) = \delta_m(x)/4$ and $\min(S_m(y)) = \delta_m(y)/4$ we obtain that $\delta_m(x) = \delta_m(y)$. Observe that $\delta_m(x) = \delta_m(y) > 0$ so that

$$\frac{\delta_m(x)}{4} < \min\left\{\frac{\delta_m(x)}{2}(1+z): z \in j_m(x)\right\},$$

and

$$\frac{\delta_m(y)}{4} < \min\left\{\frac{\delta_m(y)}{2}(1+z): z \in j_m(y)\right\}.$$

Therefore, $j_m(x) = j_m(y)$ because $S_m(x) = S_m(y)$ and $\delta_m(x) = \delta_m(y)$. So $x = y$ because j_m is an embedding.

Claim 4. $h[Q] \in \mathscr{Z}(2^X)$.

For every $n \in \mathbb{N}$ let $J_n = \cup\{\phi_n(c_n, \{t\}): t \in I\}$. Observe that for every n, $J_n \neq X_n$. For $N \in \mathbb{N}$ define $\mathscr{B}_N = \{B \in 2^X: (\forall n \geqslant N)(\pi_n[B] \subseteq J_n)\}$. Then $\mathscr{B}_N$ is clearly closed and we claim that it is a Z-set in 2^X. Indeed, for a very large index $i > N$ let $f: X \to X$ be the function that is the product of the identity functions on all factors of X, except for the i-th factor, and a constant function $X_i \to X_i$ the range of which misses J_i. Then the induced hyperspace function $2^f: 2^X \to 2^X$ ([13, §5.3]) is close to the identity and its range misses $\mathscr{B}_N$.

Now note that for $x \in Q\backslash K$ there is an $N \in \mathbb{N}$ such that for all $n \geqslant N$, $\pi_n[h(x)] \subseteq J_n$. So since $h[K]$ is a Z-set in 2^X, we conclude that $h[Q]$ is contained in a σZ-set, and is therefore a Z-set itself.

Claim 5. $h^{-1}[\dim_\infty(X)]\backslash K = \bigcap_{k=1}^\infty A_k\backslash K$.

Note that for $x \in Q\backslash K$ there is an $N_x \in \mathbb{N}$ such that for every $n \geqslant N_x$, $\pi_n[h(x)] = \phi_n(c_n, S_n(x))$. If $x \notin \bigcap_{k=1}^\infty A_k \cup K$, there is a $K \geqslant N_x$ such that for all $k \geqslant K$, $\dim j_k(x) = 0$. So by (4) and (6) it follows that $h(x)$ is a finite union of products of K at most one-dimensional spaces and infinitely many zero-dimensional spaces. This means that $h(x)$ is at most K-dimensional, and hence not infinite-dimensional. If $x \in \bigcap_{k=1}^\infty A_k\backslash K$ then for all i we have that $j_i(x)$ is one-dimensional, thus there are $y_1, \ldots, y_{N_x-1}$ such that $h(x)$ contains the infinite dimensional set $(y_1, \ldots, y_{N_x-1}) \times \prod_{i=N_x}^\infty \phi_i(c_i, S_i(x))$.

This completes the proof of the Theorem. □

3.5. COROLLARY. *For every n let X_n be a non-degenerate Peano continuum and let $X = \prod_{n=1}^\infty X_n$. Then* $\dim_\infty(X)$ *is an $F_{\sigma\delta}$-absorber in 2^X.*

Proof. In the light of Theorem 3.4, it suffices to prove that $\dim_\infty(X)$ is an $F_{\sigma\delta}$ and is contained in a σZ-set. But this follows immediately from the remark following Corollary 2.6 and Theorem 2.7. □

3.6. COROLLARY. *For every n let X_n be a non-degenerate Peano continuum and let $X = \Pi_{n=1}^{\infty} X_n$. Then there is a homeomorphism $f: 2^X \to Q^\infty$ such that $f[\dim_\infty(X)] = B^\infty$.*

Proof. Since as was observed in §2 that B^∞ is an $F_{\sigma\delta}$-absorber in Q^∞, this follows from Corollary 3.5 and Theorem 2.3. □

4. The Example

In this section we will present an example of an everywhere infinite-dimensional Peano continuum X such that for every n, $\dim_\infty(X^n) \not\approx B^\infty$.

By Dranišnikov [11], there exists an infinite-dimensional compactum with cohomological dimension equal to 3. There even exists an infinite-dimensional compactum with cohomological dimension equal to 2 (Dydak and Walsh [12]).

Let D be any infinite-dimensional compactum with finite cohomological dimension, say n. We will use D as a building block for the construction of our example.

By Edwards and Walsh [15], there is a cell-like map $f: Y \to D$, for some compact space Y with $\dim Y \leqslant n$. There also is a null sequence $\mathscr{Y} = (Y_i)_{i=1}^{\infty}$ of copies of Y in I^{2n+1} such that

(1) if $i \neq j$ then $Y_i \cap Y_i = \phi$;
(2) $\bigcup_{i=1}^{\infty} Y_i$ is dense in I^{2n+1}.

([13, Theorem 4.4.4]). We now replace every Y_i in I^{2n+1} by a copy D_i of D by using maps of the form $f \circ \xi_i$, where $\xi_i: Y_i \to Y$ is any homeomorphism. The resulting quotient space X is compact and metrizable because $\mathscr{Y}$ is null.

Observe that X is an everywhere infinite-dimensional Peano continuum: X is a continuous image of I^{2n+1} and every non-empty open subset of X contains a homeomorph of the infinite-dimensional space D. Also, the cohomological dimension of X is finite because X is a cell-like image of I^{2n+1}. If $m \in \mathbb{N}$ then the cohomological dimension of X^m is also finite, for example because X^m is a cell-like image of $I^{m(2n+1)}$. So in order to prove that X is as required, it suffices to verify the following:

4.1. PROPOSITION. *Let X be a compactum with finite cohomological dimension. Then $\dim_\infty(X)$ is σ-compact, and hence not homeomorphic to B^∞.*

Proof. This is easy. Let n be the cohomological dimension of X. Since cohomological dimension and covering dimension agree on finite dimensional compact spaces, it follows that if $A \subseteq X$ is closed then either $\dim A \leqslant n$ or $\dim A = \infty$. Consequently, $\dim_\infty(X) = \dim_{\geqslant n+1}(X)$, which is known to be σ-compact; see the remarks following Corollary 2.6. That B^∞ is not σ-compact is left as an exercise to the reader. □

It is also possible to construct an example of an everywhere infinite-dimensional Peano continuum Y such that for no n, $\dim_\infty(Y^n)$ is homeomorphic to B^∞, while yet for every n, $\dim_\infty(Y^n)$ is a "true" $F_{\sigma\delta}$. Simply take the space X constructed here and let Y be the space obtained from X and Q by glueing them together in one point.

4.2. REMARK. In the introduction we promised to explain here why our method does not work for obtaining a generalization of (1) in §1. For a generalization of (1), the approximation h in the proof of Theorem 3.4 should satisfy the following:

$$(\forall k \in \mathbb{N})(h^{-1}[\dim_{\geqslant k}(X)]\setminus K = A_k \setminus K).$$

It is impossible however for an arbitrary $x \in Q\setminus K$ to calculate the dimension of $h(x)$. All we know about $h(x)$ is that it is a finite union of products of at most one-dimensional spaces. It seems that a much more delicate argument than ours is needed to control the dimension of $h(x)$.

Added in proof: It was recently shown by Gladdines that for a space X such as in Theorem 3.4 the sequence $(\dim_{\geqslant K}(X))_K$ is absorbing.

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