The Homeomorphism Extension
Theorem for $\beta \omega \setminus \omega$

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ABSTRACT. CH implies that every homeomorphism of nowhere dense closed $P$-sets in $\beta \omega \setminus \omega$ can be extended to an autohomeomorphism of $\beta \omega \setminus \omega$. CH also implies that every closed $P$-set of weight at most $c$ of a compact zero-dimensional $F$-space is a retract.

1. RESULTS

All spaces considered are Hausdorff. Here $\omega^*$ denotes $\beta \omega \setminus \omega$. A subset $P$ of a space $X$ is called a $P$-set if $\cap \mathcal{Z}$ is a neighborhood of $P$ whenever $\mathcal{Z}$ is a countable family of neighborhoods of $P$ and a point $p$ is called a $P$-point if $\{p\}$ is a $P$-set.

Our main result is the following theorem.

THEOREM 1.1 (HOMEOMORPHISM EXTENSION THEOREM): The Continuum Hypothesis (CH) implies that every homeomorphism of one (nonempty) closed nowhere dense $P$-set of $\omega^*$ to another can be extended to a homeomorphism of $\omega^*$ onto itself.

A corollary is W. Rudin's Theorem that under CH for every two $P$-points $p$ and $q$ of $\omega^*$ there is a homeomorphism of $\omega^*$ onto itself that sends $p$ to $q$ [7].

A nontrivial corollary is also that under CH every nonempty closed nowhere dense $P$-set of $\omega^*$ is a retract of $\omega^*$. We do not state this as a theorem since the following result is much more general.

THEOREM 1.2 (RETraction THEOREM): CH implies that if $X$ is a zero-dimensional compact $F$-space, and if $P$ is a nonempty nowhere dense closed $P$-set in $X$ with $w(P) \leq c$, then $P$ is a retract of $X$.

2. PROOF OF THE RETraction THEOREM

If $X$ is a space, then $\mathcal{B}(X)$ denotes the Boolean algebra of clopen (= open and closed) subsets of $X$. If $A$ and $B$ are sets, then $A \Delta B$ denotes their symmetric difference, that is, $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Also, "$\subset$" denotes proper inclusion. A

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compact space is an \textit{F-space} if disjoint open $F_\sigma$-subsets have disjoint closures. Finally, a continuous function will be called a \textit{map}.

Let $Y$ be a compact zero-dimensional $F$-space, and let $P$ be a nowhere dense closed $F$-set of $Y$ with $w(P) \leq \omega_1$. We will construct a (Boolean algebra) embedding $f: \mathcal{B}(P) \to \mathcal{B}(Y)$ such that $f(A) \cap P = A$ for every $A \in \mathcal{B}(P)$. Such an embedding we call a \textit{lifting}. By Stone duality this implies that $P$ is a retract of $Y$.

Let $\mathcal{B}(P) = \{X_\alpha : \alpha < \omega_1\}$ and for $\alpha < \omega_1$ let $\mathcal{A}_\alpha$ be the subalgebra of $\mathcal{B}(P)$ generated by $\{X_\xi : \xi < \alpha\}$ (so $\mathcal{A}_0 = \{\emptyset\}$). We construct a partial lifting $f_\alpha : \mathcal{A}_\alpha \to \mathcal{B}(Y)$ for $\alpha < \omega_1$ such that $f_\alpha = f_\beta \upharpoonright \mathcal{A}_\alpha$ whenever $\alpha < \beta < \omega_1$. Then $f = \bigcup_{\alpha < \omega_1} f_\alpha$ is our lifting $\mathcal{B}(P) \to \mathcal{B}(Y)$. The definition of $f_\alpha$ from $|f_\alpha : \alpha < \lambda|$ for $\lambda < \omega_1$ a limit ordinal (including $\lambda = 0$) is trivial. This reduces the proof to proving the following

\textbf{Lemma 2.1:} Let $\mathcal{A}$ be a countable subalgebra of $\mathcal{B}(P)$. Let $A \in \mathcal{B}(P)$ and let $\mathcal{A}'$ be the subalgebra of $\mathcal{B}(P)$ generated by $\mathcal{A} \cup \{A\}$. For each partial lifting $f : \mathcal{A} \to \mathcal{B}(Y)$ there is a partial lifting $f' : \mathcal{A}' \to \mathcal{B}(Y)$ that extends $f$.

\textbf{Proof:} We claim that there is $B \in \mathcal{B}(Y)$ with $B \cap P = A$ such that

1. $\forall X \in \mathcal{A} \ [X \subseteq A \Rightarrow f(X) \subseteq B]$,
2. $\forall X \in \mathcal{A} \ [X \subseteq P \setminus A \Rightarrow f(X) \subseteq Y \setminus B]$.

Given such a $B$, one defines $f'$ by

$$f'((X \cap A) \cup (Z \setminus A)) = (f(X) \cap B) \cup (f(Z) \setminus B) \quad (X, Z \in \mathcal{A}).$$

It follows from Sikorski's Theorem (see, e.g., Koppelberg [3, corollary 5.8]) that $f'$ is a well-defined embedding of $\mathcal{A}'$ into $\mathcal{B}(Y)$.

We now construct $B$. Define

$$\mathcal{D} = \{f(X) : X \in \mathcal{A}, X \subseteq A\} \quad \text{and} \quad \mathcal{S} = \{f(X) : X \in \mathcal{A}, X \subseteq P \setminus A\}.$$ 

Pick any $B' \in \mathcal{B}(Y)$ with $B' \cap P = A$. Then $\mathcal{D}' = \{E \cap B' : E \in \mathcal{S}\}$ and $\mathcal{S}' = \{D \setminus B' : D \in \mathcal{D}\}$ are countable families of clopen subsets of $Y$ that miss $P$. Since $P$ is a $P$-set there consequently exists a clopen neighborhood $U$ of $P$ such that

3. $U \cup \mathcal{D}' \cup \mathcal{S}' \subseteq Y \setminus U$.

Since $U \cup \mathcal{D}$ and $U \cup \mathcal{S}$ are trivially disjoint and $Y$ is a zero-dimensional compact $F$-space, there exists a clopen subset $S$ of $Y$ such that $S \subseteq Y \setminus U$ and

4. $U \cup \mathcal{D} \setminus U \subseteq S$ \quad and \quad $U \cup \mathcal{S} \setminus U \subseteq Y \setminus S$.

Therefore we can define $B$ satisfying (1) and (2) by

$$B = (B' \cap U) \cup S.$$ 

As $B' \cap P = A$ and $S \cap P = \emptyset$ we have $B \cap P = A$. \hfill $\Box$

\textbf{Remark 2.2:} CH is essential in Theorem 1.2 because under $\text{MA} + \omega = \omega_2$ there exists a compact zero-dimensional $F$-space $Y$ (of weight $\omega_1$) with a $P$-set copy of $\beta_\omega$ that is not a retract, van Douwen and van Mill [2].
3. PROOF OF THE HOMEOMORPHISM EXTENSION THEOREM

In the proof of the Homeomorphism Extension Theorem we will use two known results on constructing homeomorphisms.

The first one is Parovičenko’s Theorem in [5] that under CH all compact zero-dimensional $F$-spaces of weight $c$ in which every nonempty $G_δ$ have infinite interior are homeomorphic (to $ω^*$) (see also [1, sec. 6], [4, corollary 1.2.4], and [9, p. 81]). This result can be proved as follows. Take two arbitrary compact zero-dimensional $F$-spaces of weight $c$ in which nonempty $G_δ$ have infinite interior, say $X$ and $Y$. It can be shown that if $\mathcal{B}$ is a countable subalgebra of $\mathcal{B}(X)$ and if $A \in \mathcal{B}(X)$ and if $f: \mathcal{B} \to \mathcal{B}(Y)$ is an embedding (of Boolean algebras), then this embedding can be extended to an embedding from the subalgebra of $\mathcal{B}(X)$ that is generated by $\mathcal{B} \cup \{A\}$ into $\mathcal{B}(Y)$. Once this has been proved it is possible under CH to construct an isomorphism $\eta : \mathcal{B}(X) \to \mathcal{B}(Y)$ by a standard back-and-forth technique. As a consequence, $X \approx Y$ by Stone duality. So the proof of Parovičenko’s Theorem shows that an arbitrary isomorphism between countable subalgebras of $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ can be extended to a full isomorphism from $\mathcal{B}(X)$ to $\mathcal{B}(Y)$. It is this form of Parovičenko’s Theorem that we will use in our proof.

The second one is a result due to M. E. Rudin [6] (see also Walker [9, p. 173]). Let $\mathcal{U} = \{U_n : n < ω\}$ and $\mathcal{V} = \{V_n : n < ω\}$ be pairwise disjoint families of clopen subsets of $ω^*$. In addition, for every $n < ω$ let $h_n : ω^* \to ω^*$ be a homeomorphism such that $h_n[U_n] = V_n$. Then, as was shown by M. E. Rudin, under CH there exists a homeomorphism $h : ω^* \to ω^*$ such that for every $n, h \upharpoonright U_n = h_n$. We use this result in the following form. Suppose that $E$ and $F$ are open $F_σ$ subsets of $ω^*$ and that $f : E \to F$ is a homeomorphism. We claim that $f$ can be extended to a homeomorphism $\tilde{f} : ω^* \to ω^*$. There clearly exists a pairwise disjoint clopen family $\{E_n : n < ω\}$ in $ω^*$ whose union is $E$ (use that $E$ is Lindelöf). Then $\{f[E_n] : n < ω\}$ is a pairwise disjoint clopen family in $ω^*$ whose union is $F$. Since all nonempty clopen subsets of $ω^*$ are homeomorphic it is possible for every $n$ to extend the homeomorphism $f \upharpoonright E_n \to f[E_n]$ to a homeomorphism $g_n$ of $ω^*$ (let $g_n$ be the union of $f_n$ and an arbitrary homeomorphism from $ω^* \setminus E_n \to ω^* \setminus f[E_n]$). So M. E. Rudin’s Theorem implies that under CH the desired extension $\tilde{f}$ can be found.

We now turn to the proof of the Homeomorphism Extension Theorem. To this end, let $A$ and $B$ be nowhere dense closed $P$-sets in $ω^*$ and let $h : A \to B$ be a homeomorphism. As to be expected, we will extend $h$ in an inductive process of length $ω_1$.

By CH we can list $\{C \in \mathcal{B}(ω^*) : C \cap A = \emptyset\}, \{C \in \mathcal{B}(ω^*) : C \cap B = \emptyset\}$, and $\mathcal{B}(A)$ as $\{A_α : α < ω_1\}, \{B_α : α < ω_1\}$, and $\{X_α : α < ω_1\}$, respectively. Without loss of generality, $A_0 = \emptyset = B_0$. The subalgebra of $\mathcal{B}(A)$ generated by $\{X_β : β < α\}$ will be denoted by $\mathcal{A}_α$. By Theorem 1.2 there exist retractions $r : ω^* \to A$ and $ρ : ω^* \to B$. If $X \subseteq A$, then we let $X_α$ and $X_β$ denote $r^{-1}[X]$ and $ρ^{-1}[h[X]]$, respectively.

By transfinite induction on $α < ω_1$ we now construct clopen subsets $E_α$ and $F_α$ of $ω^*$ and a homeomorphism $f_α : E_α \to F_α$ such that

1. $A_α \subseteq E_α$ and $B_α \subseteq F_α$.
2. $E_α \subseteq ω^* \setminus A$, $F_α \subseteq ω^* \setminus B$, and if $β < α$, then $E_β \subseteq E_α$ and $F_β \subseteq F_α$.
3. If $β < α$, then $f_α \upharpoonright E_β = f_β$.
(4) if $\beta < \alpha$ and $X \in \mathcal{A}_\beta$, then

$$f_\alpha[X_r \cap (E_\alpha \backslash E_\beta)] = X_r \cap (F_\alpha \backslash F_\beta).$$

Assume that we defined $E_\beta, F_\beta$, and $f_\beta$ for every $\beta < \alpha, \alpha < \omega_1$ (possibly $\alpha = 0$).

There are two cases to consider: $\alpha$ is a successor and $\alpha$ is a limit. We first show that the limit case can be reduced to a case similar to the successor case.

So assume that $\alpha$ is a limit and put $E = \bigcup_{\beta < \alpha} E_\beta, F = \bigcup_{\beta < \alpha} F_\beta$, and $f = \bigcup_{\beta < \alpha} f_\beta$. Observe that (2) implies that both $E$ and $F$ are noncompact open $F_\sigma$-subsets of $\omega^*$ and that $f : E \to F$ is a homeomorphism. By the preceding we can extend $f$ to a homeomorphism $\tilde{f} : \omega^* \to \omega^*$. Pick an arbitrary element $X \in \mathcal{A}_\alpha = \bigcup_{\beta < \alpha} \mathcal{A}_\beta$, say $X \in \mathcal{A}_\beta$. By (4),

$$X_r \cap (E \backslash E_\beta) = \tilde{f}^{-1}[X_r] \cap (E \backslash E_\beta).$$

Consequently,

$$(X, \Delta \tilde{f}^{-1}[X_r]) \backslash E_\beta$$

is a clopen subset of $\omega^*$ that misses $E$. Since $\bigcup_{\beta < \alpha} \mathcal{A}_\beta$ is countable, and $E$ is an open $F_\sigma$ in $\omega^*$, and $\omega^*$ is an $F$-space, there is a clopen subset $S$ of $\omega^*$ such that

(5) $E \subseteq S$,

(6) $\forall \beta < \alpha \forall X \in \mathcal{A}_\beta [(X, \Delta \tilde{f}^{-1}[X_r]) \cap (S \backslash E_\beta) = \emptyset]$.

Since $A$ and $B$ are $P$-sets, and $E$ is an $F_\sigma$-subset of $\omega^*$ that misses $A \cup \tilde{f}^{-1}[B]$, we can additionally assume that

(7) $S \cap (A \cup \tilde{f}^{-1}[B]) = \emptyset$.

So by putting $T = \tilde{f}|S$ we conclude that for the clopen sets $S \subseteq \omega^* \backslash A$ and $T \subseteq \omega^* \backslash B$ and for the homeomorphism $g = \tilde{f} \upharpoonright S : S \to T$ we have for every $\beta < \alpha$ and for every $X \in \mathcal{A}_\beta$ that

(8) $g[X_r \cap (S \backslash E_\beta)] = X_r \cap (T \backslash F_\beta)$.

So our task is now to enlarge $S$ and $T$ and to extend the homeomorphism $g$ so that the new $S$ contains $A_\alpha$, the new $T$ contains $B_\alpha$, and the new $g$ still satisfies (8) (with $S$ replaced by the new $S$, etc.). In the case that $\alpha$ is a successor, we also have to achieve this so that without loss of generality we can now restrict ourselves to the case that $\alpha$ is a successor.

Assume that $\alpha = \beta + 1$ (possibly, $\beta = -1$). We will construct nonempty clopen subsets $M$ and $N$ of $\omega^*$ such that $M \subseteq \omega^* \setminus (A \cup E_\beta), N \subseteq \omega^* \setminus (B \cup F_\beta)$ such that $E_\beta \cup M \supseteq A_\alpha, F_\beta \cup N \supseteq B_\alpha$, and a homeomorphism $\xi : M \to N$ such that for every $X \in \mathcal{A}_\beta$:

$$\xi[X \cap M] = X \cap N.$$

Then we put $E_\alpha = E_\beta \cup M, F_\alpha = F_\beta \cup N$, and $f_\alpha = f_\beta \cup \xi$. Then $E_\alpha, F_\alpha$, and $f_\alpha$ are clearly as required.

So in fact $E_\beta, F_\beta$, and $f_\beta$ play no role of importance in the remaining part of the construction and for convenience we therefore assume that $E_\beta = \emptyset = F_\beta$. 

We claim that there is a nonempty clopen subset $M$ of $\omega^* \setminus A$ such that the function $\mathcal{A}_\beta \to \mathcal{B}(M)$ defined by

$$X \mapsto X \cap M \quad (X \in \mathcal{A}_\beta)$$

is an embedding of Boolean algebras. Indeed, for every nonempty $X \in \mathcal{A}_\beta$ pick a point in $X \setminus A$ (here we use that $A$ is nowhere dense). The union of these points is a countable subset of $\omega^* \setminus A$. Since $A$ is a $P$-set, there consequently exists a nonempty clopen set $M$ in $\omega^* \setminus A$ such that $\forall X \in \mathcal{A}_\beta \setminus \{\emptyset\}[X \cap M \neq \emptyset]$. Then $M$ is clearly as required. Observe that without loss of generality we may assume that $A, C \subseteq M$. There similarly exists a nonempty clopen subset $N$ of $\omega^* \setminus B$ such that the function $\mathcal{A}_\beta \to \mathcal{B}(N)$ defined by

$$X \mapsto X \cap N \quad (X \in \mathcal{A}_\beta)$$

is an embedding (here we use that $h$ is a homeomorphism). We also assume without loss of generality that $B, C \subseteq N$. The function

$$\eta : X \cap M \to X \cap N \quad (X \in \mathcal{A}_\beta)$$

is an isomorphism of countable subalgebras of $\mathcal{B}(M)$ and $\mathcal{B}(N)$. By the remarks at the beginning of this section it follows that this isomorphism can be extended to an isomorphism

$$\hat{\eta} : \mathcal{B}(M) \to \mathcal{B}(N).$$

By Stone duality $\hat{\eta}$ corresponds to a homeomorphism $\xi : M \to N$, which clearly has the property that

$$\forall X \in \mathcal{A}_\beta[\xi[X \cap M] = X \cap N]$$

(see that $\hat{\eta}$ extends $\eta$). This completes the transfinite construction.

The function $f = \bigcup_{\alpha < \omega_1} f_\alpha : \omega^* \setminus A \to \omega^* \setminus B$ is a homeomorphism, and the function $\bar{\eta} = h \cup f$ is one-to-one. We claim that $\bar{\eta}$ is open at the points of $A$. By compactness of $\omega^*$ this then implies that $\bar{\eta}$ is a homeomorphism. To this end, pick an arbitrary clopen neighborhood $C$ of an arbitrary point $x \in A$. Pick $\beta < \omega_1$ such that $C \cap A \in \mathcal{A}_\beta$. Observe that the family $\mathcal{C} = \{(C \cap A)_\alpha \setminus E_\alpha : \alpha < \omega_1\}$ is decreasing and that $\bigcap \mathcal{C} = C \cap A$. Since $C$ is a neighborhood of $C \cap A$, by compactness of $\omega^*$ there consequently exists $\alpha < \omega_1$ such that $\beta < \alpha$ and $P = (C \cap A)_\alpha \setminus E_\alpha$ is contained in $C$. Observe that $P$ is a clopen neighborhood of $x$. By (4) and the definition of $\bar{\eta}$ it follows that

$$\bar{\eta}[P] = (C \cap A)_\beta \setminus F_\alpha$$

and consequently $\bar{\eta}[P]$, and hence $\bar{\eta}[C]$, is a neighborhood of $\bar{\eta}(x)$. 

**Remark 3.1:** CH is essential in Theorem 1.1. It is consistent with $\text{MA} + \neg \text{CH}$ that every autohomeomorphism of $\omega^*$ is trivial, Shelah and Steprāns [8]. In this model there are $c$ autohomeomorphisms but $2^c P_e$-points.
4. REMARKS BY JAN VAN MILL

Theorems 1.1 and 1.2 were proved in the Spring of 1980 (when Eric van Douwen visited me in Baton Rouge) by a complexification of W. Rudin's method from [7]. These results were not published because we derived Theorem 1.2 for the special case $X = \omega^*$ from Theorem 1.1 and the proof of Theorem 1.1 was technically very complicated: we felt that it was not in its final form yet. The simple proof of Theorem 1.2 presented here I found in Eric van Douwen's unpublished manuscripts and the simple proof of Theorem 1.1 was found recently by me. It is interesting to note that Theorem 1.2 was used here in the proof of Theorem 1.1. In our original proof we did everything in opposite order.

Remarks like these usually are not made in a mathematical paper. The reader is interested in the theorems only and not in their history. I felt, however, that I should offer the reader an explanation of why this joint paper was written four years after the death of the first-named author.

REFERENCES