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GENERAL TOPOLOGY

A Countable Space with a Closed Subspace Without Measurable Extender

by

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Summary. We construct an example of a countable space Δ having a closed subspace A such that no extender $e : C_p(A) \to C_p(\Delta)$ is Borel. Arhangelsky asked for the existence of a *continuous* extender. Thus our example gives a strong negative answer to this question.

1. Introduction. The vector space of continuous real-valued functions on a space X is denoted by $C(X)$ and $C_p(X)$ denotes $C(X)$ endowed with the topology of pointwise convergence.

In [1, Problem 58] Arhangelsky asked whether for every countable space X and every closed subspace $A \subseteq X$ there is an extender $e : C_p(A) \to C_p(X)$ which is linear and continuous. It can be shown that the two countable spaces Δ and Π, constructed by van Douwen and Pol [4] as examples of countable spaces not having the Dugundji Extension Property, are counterexamples to this question. In the same problem, Arhangelsky also asked whether for every countable space X and every closed subspace $A \subseteq X$ there is an extender $e : C_p(A) \to C_p(X)$ which is continuous. The aim of this note is to present a variation of the space Δ which also solves this question in the negative, in a rather strong way: there is no extender measurable with respect to the σ-algebra generated by the Souslin sets (cf. Remark 2.2).

One can interpret this example from the point of view of selection theory as follows. If X is countable then $C_p(X)$ is a linear subspace of $\mathbb{R}^X$, a countable product of real lines. As a consequence, $C_p(X)$ is a separable metriz-

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able locally convex vector space. In addition, if $A \subseteq X$ is closed then the restriction operator $\rho : C_p(X) \to C_p(A)$ defined by $\rho(f) = f \restriction A$ is bounded, linear, onto (by the Tietze Extension Theorem) and open. Arhangelsky question askes for a continuous selection for the lower-semicontinuous set-valued function $f \mapsto \rho^{-1}(f)$ $(f \in C_p(A))$. By our example, such a selection need not exist (see Marciszewski [6] for a related result). Interestingly, a result of Michael [7] implies that for every *countable* subspace $C \subseteq C_p(A)$ there is a continuous selection for the set-valued function $f \mapsto \rho^{-1}(f)$ $(f \in C)$.

After this note was completed, Marciszewski found another solution of Arhangelsky problem, based on his construction from [6]. His example differs essentially from ours because his $C_p(X)$ is Borel, hence a measurable extender exists, (cf. Remark 2.2).

As usual, a cardinal is an initial ordinal, an ordinal is the set of smaller ordinals, and ω is the first infinite cardinal. A function will sometimes be identified with its graph. In addition, the restriction of a function f to a subset S of its domain is denoted by $f \restriction S$.

2. The space Δ and its variations. Let $\mathcal{D}$ be an almost disjoint family of infinite subsets of ω, i.e. $|D| = \omega$ for all $D \in \mathcal{D}$ and $|D \cap D'| < \omega$ for all distinct $D, D' \in \mathcal{D}$. In addition, let $\mathcal{F}$ be a collection of functions from ω to ω and let $\varphi : \mathcal{D} \to \mathcal{F}$ be a surjection. Finally, let q be any point not in $\omega \times (\omega + 1)$. Topologize

$$\Delta(\varphi) = \{q\} \cup (\omega \times (\omega + 1))$$

as follows: $\omega \times (\omega + 1)$ is an open subspace of $\Delta(\varphi)$, carrying the usual product topology. For finite $\mathcal{E} \subseteq \mathcal{D}$ and for $n \in \omega$ define

$$U(\mathcal{E}, n) = \{q\} \cup \left(\left(\left(\omega \setminus \bigcup \mathcal{E} \right) \times (\omega + 1) \cup \bigcup_{E \in \mathcal{E}} \varphi(E) \restriction E \right) \setminus \left(n \times (\omega + 1) \right) \right).$$

We identify here the functions $\varphi(E) \restriction E$ with their respective graphs. The $U(\mathcal{E}, n)$'s form a neighbourhood base for q. It is easy to see that $\Delta(\varphi)$ is regular and T_1 (for details see [4]).

Let $D \in \mathcal{D}$. Then the subspace $\varphi(D) \restriction D \subseteq \omega \times \omega \subseteq \omega \times (\omega + 1)$ converges to q in $\Delta(\varphi)$. This fact will be used below in the proof of Theorem 2.1.

We will always let A denote the subspace $(\omega \times \{\omega\}) \cup \{q\}$ of $\Delta(\varphi)$. Observe that the subspace topology that A inherits from $\Delta(\varphi)$ is independent of $\mathcal{F}$. Indeed, $A \setminus \{q\}$ is discrete and a basic neighbourhood of q has the form

$$\{q\} \cup \left(\left(\omega \setminus \left(\bigcup \mathcal{E} \cup n \right) \right) \times \{\omega\} \right),$$

for certain finite $\mathcal{E} \subseteq \mathcal{D}$ and $n < \omega$.

The space Δ mentioned above is the space $\Delta(\varphi)$, where $\mathcal{D}$ is any almost disjoint family of infinite subsets of ω of cardinality $\mathfrak{c}$, $\mathcal{F} = \omega^\omega$ and $\varphi : \mathcal{D} \to$

$\mathcal{F}$ is any surjection.

We will now describe a variation of Δ that will solve Arhangelsky problem. We first fix some notation. Let Σ denote $\bigcup_{n<\omega} 2^n$, the set of finite sequences of 0's and 1's. If $f \in 2^n$ then n is called the *length* of f. For each $f \in 2^\omega$ we define

$$I(f) = \{f \restriction n : n < \omega\},$$

the set of initial sequences of f; $I(f)$ can be seen as the set of finite approximations to f. It is clear that

$$\text{if } f, g \in 2^\omega \text{ are distinct, then } I(f) \cap I(g) \text{ is finite.}$$

It will be convenient to identify ω and Σ. Thus we think of

$$\mathcal{D} = \{I(f) : f \in 2^\omega\}$$

as an almost disjoint collection of infinite subsets of ω.

We are interested in $\Delta = \Delta(\varphi)$, where $\mathcal{D}$ is as above, $\mathcal{F} = \omega^\omega$; the function $\varphi : \mathcal{D} \to \mathcal{F}$ will be determined below. Since the subspace topology that A inherits from $\Delta(\varphi)$ is independent of $\mathcal{F}$, we already know A. For every $D \in \mathcal{D}$ we let D_ω denote the subset $D \times \{\omega\}$ of A. Observe that D_ω is clopen in A; we let $D^*_\omega \in C_p(A)$ denote its characteristic function. Our choice of $\mathcal{D}$ easily implies that the set $\mathcal{D}^* = \{D^*_\omega : D \in \mathcal{D}\} \subseteq C_p(A)$ is homeomorphic to the Cantor set 2^ω. We now let

$$\varphi : \mathcal{D} \to \omega^\omega$$

be a function such that for each $f \in \omega^\omega$ the set $\{D^*_\omega : \varphi(D) = f\}$ intersects every Cantor set in $\mathcal{D}^*$. It is not hard to construct such a function. Simply observe that the cardinality of $\mathcal{D}^*$ is equal to $\mathfrak{c}$, that the family of all Cantor subsets of $\mathcal{D}^*$ has size $\mathfrak{c}$ and that every Cantor set has size $\mathfrak{c}$. Thus by the Disjoint Refinement Lemma, [3, Lemma 7.5], there is a disjoint family $\mathcal{E}$ of subsets of $\mathcal{D}^*$ such that every Cantor subset of $\mathcal{D}^*$ contains an element of $\mathcal{E}$; moreover, each element $E \in \mathcal{E}$ has size $\mathfrak{c}$. For every $E \in \mathcal{E}$ let $\varphi_E : E \to \omega^\omega$ be a surjection. Now define $\varphi : \bigcup \mathcal{E} \to \omega^\omega$ by the rule $\varphi(x) = \varphi_E(x)$ iff $x \in E$ and extend φ over $\mathcal{D}^*$ in an arbitrary way. Then this function is clearly as desired.

THEOREM 2.1. *If $e : \mathcal{D}^* \to C_p(\Delta)$ is an extender and $S \subseteq \mathcal{D}^*$ is a dense G_δ-subset then $e \restriction S$ is not continuous.*

Proof. Striving for a contradiction, assume that there exist a subcollection $\mathcal{G}$ of $\mathcal{D}$ and an extender $e : \mathcal{D}^* \to C_p(\Delta)$ such that

(i) $\mathcal{G}^* = \{G^*_\omega : G \in \mathcal{G}\}$ is a dense G_δ-subset of $\mathcal{D}^*$;

(ii) $e \restriction \mathcal{G}^*$ is continuous.

For every $\sigma \in \Sigma$, the set

$$\mathcal{D}^*(\sigma) = \{D^*_\omega \in \mathcal{D}^* : \sigma \subseteq D\}$$

is a basic clopen subset of $\mathcal{D}^*$ and consequently intersects $\mathcal{G}^*$. For convenience, put

$$\mathcal{G}^*(\sigma) = \mathcal{D}^*(\sigma) \cap \mathcal{G}^* \quad (\sigma \in \Sigma).$$

We will now construct a function $f : \Sigma \to \omega$ having some special properties. To this end, take an arbitrary $\sigma \in \Sigma$. The value $f(\sigma)$ is determined as follows. Take an arbitrary $G^*_\omega \in \mathcal{G}^*(\sigma)$. Since $e(G^*_\omega)(\langle\sigma,\omega\rangle) = 1$, there exists $n \in \omega$ such that $e(G^*_\omega)(\langle\sigma,n\rangle) > \frac{1}{2}$. Put $f(\sigma) = n$.

For $\sigma \in \Sigma$ put

$$(1) \qquad \mathcal{U}(\sigma) = \left\{u \in C_p(\Delta) : u(\langle\sigma,\omega\rangle) > \frac{1}{2} \text{ and } u(\langle\sigma, f(\sigma)\rangle) > \frac{1}{2}\right\}.$$

Then $\mathcal{U}(\sigma)$ is clearly open and is nonempty since by construction it contains $e(G^*_\omega)$. In particular, $e[\mathcal{G}^*(\sigma)] \cap \mathcal{U}(\sigma) \neq \emptyset$, or, equivalently, $\mathcal{G}^*(\sigma) \cap e^{-1}[\mathcal{U}(\sigma)] \neq \emptyset$.

For every $k \in \omega$ define

$$(2) \qquad \mathcal{U}_k = \bigcup\{\mathcal{U}(\sigma) : \text{ length } \sigma \geqslant k\}.$$

By the above and the continuity of e on $\mathcal{G}^*$, for every k we have that $e^{-1}[\mathcal{U}_k] \cap \mathcal{G}^*$ is dense and open in $\mathcal{G}^*$. We conclude that

$$\bigcap_{k\in\omega} e^{-1}[\mathcal{U}_k] \cap \mathcal{G}^*$$

is a dense G_δ-subset of $\mathcal{G}^*$, and hence of $\mathcal{D}^*$, and it, therefore, contains a Cantor set. Thus by construction, there exists

$$D^*_\omega \in \bigcap_{k\in\omega} e^{-1}[\mathcal{U}_k] \cap \mathcal{G}^* \quad \text{with } \varphi(D) = f.$$

It follows that $e(D^*_\omega) \in \mathcal{U}_k$ for all k, and hence, by (1) and (2),

$$e(D^*_\omega)(\langle\sigma,\omega\rangle) > \frac{1}{2} \quad \text{and} \quad e(D^*_\omega)(\langle\sigma, f(\sigma)\rangle) > \frac{1}{2}$$

for infinitely many σ. However,

$$e(D^*_\omega)(\langle\sigma,\omega\rangle) = D^*_\omega(\langle\sigma,\omega\rangle) > \frac{1}{2}$$

implies that $\sigma \in D$, so in fact,

$$(3) \qquad e(D^*_\omega)(\langle\sigma, f(\sigma)\rangle) > \frac{1}{2}$$

for infinitely many $\sigma \in D$. However $e(D^*_\omega)(q) = 0$ and $f = \varphi(D)$ and so $f \restriction D$ converges to q. By continuity of $e(D^*_\omega)$ we must consequently have that

$$(4) \qquad e(D^*_\omega)(\langle\sigma, f(\sigma)\rangle) < \frac{1}{2}$$

for all but finitely many $\sigma \in D$. Conditions (3) and (4) provide a contradiction, which establishes the proof.

Remark 2.2. Suppose that $A \subseteq X$ is a closed subset of a countable space X such that the function space $C_p(X)$ is Borel in $\mathbb{R}^X$. Then by the Yankov-von Neumann Theorem the restriction map $f \mapsto f \restriction A$ from $C_p(X)$ to $C_p(A)$ has a selection $e : C_p(A) \to C_p(X)$ (an extender) measurable with respect to the σ-algebra generated by the analytic sets in $C_p(A)$, cf. Rogers *et al.* [8, p. 212] or Arveson [2, Theorem 3.4.3]. Therefore, for each $\mathcal{E} \subseteq C_p(A)$ which is a G_δ-set in $\mathbb{R}^A$, the restriction $e : \mathcal{E} \to C_p(X)$ is measurable with respect to the σ-algebra of sets in $\mathcal{E}$ that are open modulo a first category set (in $\mathcal{E}$). As a consequence, e is continuous on a dense G_δ-subset of $\mathcal{E}$.

Let us now return to the space Δ and its closed subset A. Since $\mathcal{D}^*$ is a Cantor set, by the above remarks and Theorem 2.1 it follows that $C_p(\Delta)$ is not Borel. Also, there does not even exist a measurable extender from $C_p(A)$ to $C_p(\Delta)$. Interestingly, $C_p(A)$ *is* Borel since the space A is similar to one of the examples considered in Lutzer, van Mill and Pol [5].

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