FUNDAMENTA
MATHEMATICAE
142 (1993)

Hyperspaces of Peano continua of euclidean spaces

by

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Abstract. If X is a space then $L(X)$ denotes the subspace of $C(X)$ consisting of all Peano (sub)continua. We prove that for $n \geq 3$ the space $L(\mathbb{R}^n)$ is homeomorphic to B^∞, where B denotes the pseudo-boundary of the Hilbert cube Q.

Introduction. For a space X, $C(X)$ denotes the hyperspace of all nonempty subcontinua of X. It is known that for a Peano continuum X without free arcs, $C(X) \approx Q$, where Q denotes the Hilbert cube (Curtis and Schori [7]). $L(X)$ denotes the subspace of $C(X)$ consisting of all nonempty *locally connected* continua.

The spaces $L(X)$ were first studied by Kuratowski in [11]. He proved that $L(X)$ is an $F_{\sigma\delta}$-subset of $C(X)$, i.e., a countable intersection of σ-compact subsets. A little later, Mazurkiewicz [12] proved that for $n \geq 3$, $L(\mathbb{R}^n)$ belongs to the Borel class $F_{\sigma\delta} \setminus G_{\delta\sigma}$. Our main result is that for $n \geq 3$ the spaces $L(\mathbb{R}^n)$ are homeomorphic to the countable infinite product of copies of the pseudo-boundary B of Q. Our methods do not apply to the case $n = 2$. We use the theory of absorbing sets in the Hilbert cube and some ideas from Dijkstra, van Mill and Mogilski [9]. In fact, we prove that for $n \geq 3$, $L([-1,1]^n)$ is an $F_{\sigma\delta}$-absorber in $C([-1,1]^n)$. Our main result then follows easily.

We are indebted to R. Cauty for finding an inaccuracy in an earlier version of this manuscript.

1991 *Mathematics Subject Classification*: Primary 57N20.

Key words and phrases: Hilbert cube, Hilbert space, absorbing system, Z-set, $F_{\sigma\delta}$, hyperspace, Peano continuum, $\mathbb{R}^n$.

Research of the first author partially supported by the Netherlands Organization for Scientific Research (NWO). She is pleased to thank the Department of Mathematics of Wesleyan University for generous hospitality during the spring semester of 1992.

The second author is pleased to thank the Department of Mathematics of Wesleyan University for generous hospitality and support during the spring semester of 1992.

Terminology. *All spaces under discussion are separable and metrizable.* For any space X we let d denote an *admissible* metric on X, i.e., a metric that generates the topology. If $x \in X$ and $\varepsilon > 0$ then

$$B(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\} \quad \text{and} \quad \widehat{B}(x,\varepsilon) = \{y \in X : d(x,y) \le \varepsilon\}$$

denote the open and closed ball about x of radius ε, respectively.

As usual I denotes the interval $[0,1]$ and Q the Hilbert cube $\prod_{i=1}^{\infty}[-1,1]_i$ with metric $d(x,y) = \sum_{i=1}^{\infty} 2^{-(i+1)}|x_i - y_i|$. In addition, s is the *pseudo-interior* of Q, i.e., $s = \{x \in Q : (\forall i \in \mathbb{N})(|x_i| < 1)\}$. The complement B of s in Q is called the *pseudo-boundary* of Q. Any space that is homeomorphic to Q is called a *Hilbert cube.* If X is a set then the identity function on X will be denoted by 1_X.

Let A be a closed subset of a space X. We say that A is a *Z-set* provided that every map $f : Q \to X$ can be approximated arbitrarily closely by a map $g : Q \to X \setminus A$. A countable union of Z-sets is called a *σZ-set*. A *Z-embedding* is an embedding the range of which is a Z-set.

Let $\mathcal{M}$ be a class of spaces that is topological and closed hereditary.

2.1. Definition. Let X be a Hilbert cube. A subset $A \subseteq X$ is called *strongly $\mathcal{M}$-universal* in X if for every $M \in \mathcal{M}$ with $M \subseteq Q$, every embedding $f : Q \to X$ that restricts to a Z-embedding on some compact subset K of Q, can be approximated arbitrarily closely by a Z-embedding $g : Q \to X$ such that $g|K = f|K$ while moreover $g^{-1}[A] \setminus K = M \setminus K$.

2.2. Definition. Let X be a Hilbert cube. A subset $A \subseteq X$ is called an *$\mathcal{M}$-absorber* in X if:

(1) $A \in \mathcal{M}$;
(2) there is a σZ-set $S \subseteq X$ with $A \subseteq S$;
(3) A is strongly $\mathcal{M}$-universal in X.

2.3. Theorem ([9]). *Let X be a Hilbert cube and let A and B be $\mathcal{M}$-absorbers for X. Then there is a homeomorphism $h : X \to X$ with $h[A] = B$. Moreover, h can be chosen arbitrarily close to the identity.* ∎

Absorbers for the class F_σ of all σ-compact spaces were first constructed by Anderson and Bessaga and Pełczyński. A basic example of such an absorber in Q is B. For details, see [2] and [14, Chapter 6]. The space B^∞ in Q^∞ is an absorber for the Borel class $F_{\sigma\delta}$. This was shown in Bestvina and Mogilski [3]; see also [9].

2.4. Corollary. *Let X be a Hilbert cube and let A be an absorber in X for the Borel class $F_{\sigma\delta}$. Then there is a homeomorphism of pairs $(Q^\infty, B^\infty) \approx (X, A)$. In particular, A is homeomorphic to B^∞.* ∎

In Dijkstra, van Mill and Mogilski [9] it was shown that the subspace

$$c_0 = \{x \in s : \lim_{n\to\infty} x_n = 0\}$$

is an $F_{\sigma\delta}$-absorber in Q.

The following result is of crucial importance in the proof of our main result.

2.5. Theorem. *The subspace*

$$\widehat{c}_0 = \{x \in Q : \lim_{n\to\infty} x_n = 0\}$$

is an $F_{\sigma\delta}$-absorber in Q.

Proof. This follows by a trivial modification of the proof of the above quoted result in Dijkstra, van Mill and Mogilski [9].

The space B^∞ has been studied intensively in infinite-dimensional topology during the last years. For more information, see e.g. [3, 4, 10, 9, 8, 1].

If (X, d) is a space then 2^X denotes the hyperspace of all nonempty compact subsets of X, topologized by the Hausdorff metric d_{H}. The subspace of 2^X consisting of all nonempty subcontinua of X is denoted by $C(X)$. The subspace of $C(X)$ consisting of all *locally connected* elements of $C(X)$ is denoted by $L(X)$. Finally, $\mathcal{F}(X) = \{F \in 2^X : F \text{ is finite}\}$.

We will need the following basic result.

2.6. Theorem (Curtis [6]). *Let X be a nondegenerate Peano continuum. Then there is a homotopy $H : 2^X \times I \to 2^X$ such that*

(1) $H_0 = 1_{2^X}$;
(2) *for all $t \in (0, 1]$, $H_t[2^X] \subseteq \mathcal{F}(X)$.* ■

For background information on hyperspaces see Nadler [13].

3. The space $L(X)$. Let X be a continuum. As was mentioned in the introduction, Kuratowski [11] showed that $L(X)$ is an $F_{\sigma\delta}$-subset of $C(X)$. For the sake of completeness and also for later use we will reprove this theorem.

To this end, for a continuum X and $n \in \mathbb{N}$ define

$$\mathcal{A}(X)^m_n = \{C \in C(X) : C \text{ can be covered by } \le m \text{ subcontinua of diameter } \le 1/n\}.$$

A routine verification shows that each $\mathcal{A}(X)^m_n$ is compact, and that

$$L(X) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{A}(X)^m_n .$$

So $L(X)$ is an $F_{\sigma\delta}$-subset of $C(X)$.

For technical reasons that will be clear later, we need an "$F_{\sigma\delta}$-approximation" of $L(X)$ consisting of nowhere dense compacta (in fact of Z-sets). Since the $\mathcal{A}(X)_n^m$ have nonempty interior in $C(X)$, we have to redefine them in order to have a chance to meet this criterion. For a continuum X and $n \in \mathbb{N}$ define

$$\mathcal{B}(X)_n^m = \{C \in C(X) : C \text{ can be covered by } \le m \text{ subcontinua of diameter } \le (1/n)\operatorname{diam}(C)\}.$$

A routine verification shows again that each $\mathcal{B}(X)_n^m$ is compact, and that

$$L(X) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{B}(X)_n^m .$$

In §4 we will show that for $n \ge 3$ the compacta $\mathcal{B}([-1,1]^n)_3^m$ are Z-sets in $C([-1,1]^n)$.

3.1. Theorem. (1) *$L(\mathbb{R})$ is σ-compact.*

(2) *For $n \ge 2$, $L(\mathbb{R}^n)$ and $L([-1,1]^n)$ belong to the Borel class $F_{\sigma\delta} \setminus G_{\delta\sigma}$.*

Proof. The proof of (1) is left as an exercise to the reader.

For (2), for every $x \in Q$ define $S(x) \subseteq [-1,1]^2$ by

$$S(x) = (\{0\} \times [-1,1]) \cup ([0,1] \times \{0\}) \cup \bigcup_{n=1}^{\infty} \{1/n\} \times \begin{cases} [0, x_n] & (x_n \ge 0), \\ [x_n, 0] & (x_n \le 0). \end{cases}$$

It is clear that the function $S : Q \to C([-1,1]^2) \subseteq C(\mathbb{R}^2)$ defined by $x \mapsto S(x)$ is an embedding. Now if $x \in Q$ does not belong to $\widehat{c}_0$ then there exists an infinite subset $E \subseteq \mathbb{N}$ and an $\varepsilon > 0$ such that $|x_i| \ge \varepsilon$ for all $i \in E$. Then $S(x)$ is not locally connected at some point of $\{0\} \times \{-\varepsilon, \varepsilon\}$. Also, if x does belong to $\widehat{c}_0$ then $S(x)$ is locally connected. Consequently,

$$S[Q] \cap L([-1,1]^2) = S[Q] \cap L(\mathbb{R}^2) = S[\widehat{c}_0].$$

So $L(\mathbb{R}^2)$ and $L([-1,1]^2)$ contain a closed copy of $\widehat{c}_0$, which does not belong to the Borel class $G_{\delta\sigma}$ because it is an $F_{\sigma\delta}$-absorber (Theorem 2.5). This proves that $L(\mathbb{R}^2)$ and $L([-1,1]^2)$ are not absolute $G_{\delta\sigma}$'s. Since $L(\mathbb{R}^n)$ contains a closed copy of $L(\mathbb{R}^2)$ for every $n \ge 2$, this also proves that $L(\mathbb{R}^n)$ is not an absolute $G_{\delta\sigma}$. Similarly for $L([-1,1]^n)$. ■

As remarked in the introduction, that $L(\mathbb{R}^n)$ for $n \ge 3$ belongs to the Borel class $F_{\sigma\delta} \setminus G_{\delta\sigma}$ was proved by Mazurkiewicz [12]. For $n = 2$ this was first proved by Cauty [5].

4. $L([-1,1]^n)$ is contained in a σZ-set. The aim of this section is to prove that for $n \ge 3$, $L([-1,1]^n)$ is contained in a σZ-set in $C([-1,1]^n)$. The strategy of the proof is roughly speaking the following. First we push $C([-1,1]^n)$ by a small movement into $C(\Gamma)$ for a certain finite connected graph $\Gamma \subseteq [-1,1]^n$. Then we carefully "blow up" each subcontinuum of Γ to a close subcontinuum of $[-1,1]^n$ that has more or less the following shape:

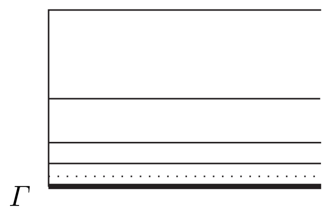

Fig. 1

Then we consider the collection

$$\mathcal{B} = \{C \in C([-1,1]^n) : C \text{ can be covered by finitely many subcontinua of diameter } \le \tfrac{1}{3}\operatorname{diam}(C)\}$$

and note that in §3 it was shown that $L([-1,1]^n) \subseteq \mathcal{B}$ and that $\mathcal{B}$ is σ-compact. We then prove that $\mathcal{B}$ is a σZ-set by observing that continua C of the type as shown in Figure 1 cannot be covered by finitely many subcontinua of diameter $\le \frac{1}{3}\operatorname{diam}(C)$.

As usual, let S^1 denote $\{x \in \mathbb{R}^2 : \|x\| = 1\}$. We will use the well-known and easily established fact that there is a homeomorphism $\varphi : D = \{x \in \mathbb{R}^2 : \|x\| \le 1\} \to C(S^1)$ such that for every $x \in S^1$, $\varphi(x) = \{x\}$. So if we identify S^1 and the subspace $\{\{x\} : x \in S^1\}$ of $C(S^1)$ then φ is the identity on S^1. If A is a square then we let ∂A denote its boundary.

Consider the square $[0,1] \times [0,1]$ and its subspace

$$G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^n} \partial([(m-1)\cdot 2^{-n}, m \cdot 2^{-n}] \times [2^{-n}, 2^{-(n-1)}]) \cup (I \times \{0\}).$$

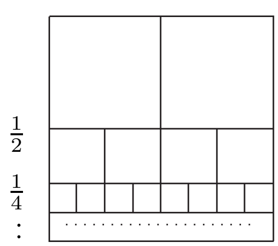

Fig. 2

We claim that there is a continuous function $\widetilde{e} : [0,1]^2 \to C(G)$ such that for every $x \in G$, $\widetilde{e}(x) = \{x\}$. This follows easily from the observation above that $C(S^1) \approx D$ and from the fact that the diameters of the sets of type $\partial([(m-1)\cdot 2^{-n}, m\cdot 2^{-n}] \times [2^{-n}, 2^{-(n-1)}])$ tend to 0 as $n \to \infty$ and $1 \le m \le 2^n$.

Observe that if $t \in I$ and $2^{-n} \le t \le 2^{-(n-1)}$ then $\bigcup \widetilde{e}[I \times \{t\}] \subseteq I \times [2^{-n}, 2^{-(n-1)}]$. This observation will be used in the proof of Proposition 4.1.

Let Γ be any finite connected graph (with a fixed triangulation) and let $x_1, \ldots, x_N$ be the vertices of Γ. In addition, let ϱ be an arbitrary admissible convex metric on Γ (not necessarily the path-length metric) and let $0 < \delta < 1$. We will associate with Γ, with the set $\{x_1, \ldots, x_N\}$ and with the number δ a certain space $D(\Gamma)$ that will be important in the proof of Theorem 4.6; (Γ, ϱ), $x_1, \ldots, x_N$ and δ will be specified there.

For every $n \le N$ let t_n denote the point $(\cos\frac{\pi}{n}, \sin\frac{\pi}{n}) \in \mathbb{R}^2$. For every $n \le N$ let L_n be the straight line segment connecting t_n and $(0,0)$. For notational simplicity we will denote the point $(0,0)$ in $\mathbb{R}^2$ by $\mathbf{0}$ from now on. By abuse of notation, for $n \le N$ and $\alpha \in [0,1]$ we will write $[\mathbf{0}, \alpha t_n]$ for $\{z \in L_n : \|z\| \le \alpha\}$. The union $\bigcup_{i=1}^N L_i$ is denoted by S. Observe that it is a compact 1-dimensional subspace of $\mathbb{R}^2$.

Fix one of the L_i for a moment and let E be an edge of Γ. Then $E \times L_i$ is a square. We think of E as the subspace $E \times \{\mathbf{0}\}$ of $E \times L_i$. We remove from $E \times L_i$ the "same" open squares that we removed from $[0,1]^2$ in order to get G. In that way we obtain a subspace $G_i(E)$ of $E \times L_i$ and we note that there is a continuous function $e_E^i : E \times L_i \to C(G_i(E))$ which is the identity on $G_i(E)$ (singletons and points are again identified here).

The union $D(\Gamma)$ of all the sets $G_i(E)$, where $i \le N$ and $E \subseteq \Gamma$ is an edge, is a 1-dimensional compact subspace of $\Gamma \times S$ which contains Γ. We will not distinguish between a function $g : \Gamma \to \Gamma$ and the function $\widehat{g} : \Gamma \times \{\mathbf{0}\} \to \Gamma \times \{\mathbf{0}\}$ defined by $\widehat{g}(x, \mathbf{0}) = (g(x), \mathbf{0})$. Let $e : \Gamma \times S \to C(D(\Gamma))$ be the union of all the functions e_E^i, where E is an edge of Γ and $i \le N$, and observe that clearly e is a continuous function which is the identity on $D(\Gamma)$.

4.1. Proposition. *Let $i \le N$ and $\alpha \in (0,1]$. For every n let*

$$C_n = \bigcup e[\Gamma \times \{2^{-2n}\alpha t_i\}] \quad \textit{and} \quad C_0 = \bigcup_{j \ne i} \Gamma \times L_j\,.$$

Then the collection $\{C_n : n \ge 0\}$ is pairwise disjoint. Moreover, if C is a subcontinuum of $\bigcup_{n=0}^\infty C_n$ then there exists $n \ge 0$ such that $C \subseteq C_n$.

Proof. Fix $n \in \mathbb{N}$. Find $p \in \mathbb{N}$ such that $2^{-p} \le \alpha \le 2^{-(p-1)}$. Then for every n, by the way e was constructed (see the above remark),

$$(1) \qquad C_n = \bigcup e[\Gamma \times \{2^{-2n}\alpha t_i\}] \subseteq \Gamma \times [2^{-(p+2n)}t_i, 2^{-(p-1+2n)}t_i]\,.$$

So (1) implies that the collection $\{C_n : n \ge 0\}$ is pairwise disjoint as well as the fact that if a continuum is contained in the union of the C_n, $n \geq 0$, then it must be contained in one of them. ■

For every $i \leq N$ define the continuous function $s_i : C(\Gamma) \to [0, \delta]$ as follows:

$$s_i(A) = \max\{\delta - \varrho(x_i, A), 0\}\,.$$

In addition, let $g : C(\Gamma) \to C(\Gamma)$ be the continuous function sending each $A \in C(\Gamma)$ onto $\widehat{B}_\delta(A)$, the *closed* ϱ-ball about A of radius δ. Moreover, define the functions $\widehat{S}^m : C(\Gamma) \to C(\Gamma \times S)$ $(m \in \mathbb{N})$ by

$$\widehat{S}^m(A) = (g(A) \times \{\mathbf{0}\}) \cup \bigcup_{x_i \in g(A)} \{x_i\} \times [\mathbf{0}, 2^{-(m-1)} s_i(A) t_i]\,.$$

4.2. Lemma. *The functions $\widehat{S}^m$ $(m \in \mathbb{N})$ are continuous. Moreover*, $\lim_{m\to\infty} \widehat{S}^m = g$.

Proof. Fix $m \in \mathbb{N}$. Let $(A_k)_k$ be a sequence in $C(\Gamma)$ converging to an element $A \in C(\Gamma)$. Put $F = \{x_i : x_i \in g(A)\}$ and $F_k = \{x_i : x_i \in g(A_k)\}$ $(k \in \mathbb{N})$, respectively. Since $\{x_1, \ldots, x_N\}$ is finite, we may assume without loss of generality that $F_1 = F_k$ for every $k \in \mathbb{N}$. By continuity of the function g, it clearly follows that $F_1 \subseteq F$. In addition, by continuity of the functions s_i, the only way we can get into trouble with the continuity of $\widehat{S}^m$ is if there exist points in $F \setminus F_1$. So assume that there exists $x_i \in F \setminus F_1$. Then $\varrho(x_i, A_k) > \delta$ for every k, which implies that $\varrho(x_i, A) \geq \delta$, i.e., $\varrho(x_i, A) = \delta$ because $x_i \in g(A)$. But then $s_i(A) = 0$ so that x_i adds nothing to $\widehat{S}^m(A)$.

That $\lim_{m\to\infty} \widehat{S}^m = g$ is clear. ■

Define $\widehat{S}_m : C(\Gamma) \to 2^{\Gamma \times S}$ $(m \in \mathbb{N})$ by

$$\widehat{S}_m(A) = (g(A) \times \{\mathbf{0}\}) \cup \bigcup_{i=1}^{N} \bigcup_{n=1}^{\infty} g(A) \times \{2^{-2n} 2^{-(m-1)} s_i(A) t_i\}\,.$$

4.3. Lemma. *The functions $\widehat{S}_m$ $(m \in \mathbb{N})$ are continuous. Moreover*, $\lim_{m\to\infty} \widehat{S}_m = g$. ■

Define $\widehat{T}_m : C(\Gamma) \to 2^{\Gamma \times S}$ by

$$\widehat{T}_m(A) = \widehat{S}^m(A) \cup \widehat{S}_m(A) \qquad (A \in C(\Gamma))\,.$$

4.4. Lemma. (1) *The functions $\widehat{T}_m$ $(m \in \mathbb{N})$ are continuous*;

(2) *for every $A \in C(\Gamma)$, $\widehat{T}_m(A)$ is connected*;

(3) $\lim_{m\to\infty} \widehat{T}_m = g$.

Proof. (1) and (3) follow from Lemmas 4.2 and 4.3. In addition, (2) is clear. ■

For every $m \in \mathbb{N}$ we define $T_m : C(\Gamma) \to C(D(\Gamma))$ by

$$T_m(A) = \bigcup e[\widehat{T}_m(A)] \qquad (A \in C(\Gamma)).$$

4.5. Lemma. (1) *The functions* T_m $(m \in \mathbb{N})$ *are continuous*;
(2) *for every* $A \in C(\Gamma)$, $T_m(A)$ *is connected*;
(3) $\lim_{m\to\infty} T_m = g$.

Proof. Since the union operator in hyperspaces is continuous ([14, Proposition 5.3.6]), this follows immediately from Lemma 4.4. ∎

We now come to the main result in this section.

4.6. Theorem. *If* $n \geq 3$ *then* $L([-1,1]^n)$ *is contained in a* σZ*-set of* $C([-1,1]^n)$.

Proof. Fix $n \geq 3$ and let d denote the euclidean metric on $[-1,1]^n$. Consider the collection

$$\mathcal{B} = \{C \in C([-1,1]^n) : C \text{ can be covered by finitely many subcontinua of diameter } \leq \tfrac{1}{3}\operatorname{diam}(C)\}.$$

In §3 it was shown that $L([-1,1]^n) \subseteq \mathcal{B}$ and that $\mathcal{B}$ is σ-compact. We will show that $\mathcal{B}$ is a σZ-set. So let $\varepsilon > 0$. Our task is to produce a continuous function $f : C([-1,1]^n) \to C([-1,1]^n) \setminus \mathcal{B}$ such that $\widehat{d}_{\mathrm{H}}(f, 1_{C([-1,1]^n)}) < \varepsilon$.

By Curtis and Schori [7] there is a finite connected graph $\Gamma \subseteq [-1,1]^n$ and a map $\xi : C([-1,1]^n) \to C(\Gamma)$ such that $\widehat{d}_{\mathrm{H}}(\xi, 1_{C([-1,1]^n)}) < \frac{1}{4}\varepsilon$. By pushing Γ into $(-1,1)^n$ if necessary, we may assume that $\Gamma \subseteq (-1,1)^n$.

Let ϱ be a convex metric on Γ (for example the path-length metric).

Claim 1. *There exists* $\delta > 0$ *such that if* $x, y \in \Gamma$ *and* $\varrho(x,y) \leq \delta$ *then* $d(x,y) \leq \frac{1}{8}\varepsilon$.

Let $\{x_1,\ldots,x_N\} \subseteq \Gamma$ be a $\frac{1}{2}\delta$-net with respect to the metric ϱ, i.e., for every $x \in \Gamma$ there exists $i \leq N$ with $\varrho(x,x_i) \leq \frac{1}{2}\delta$. Because Γ has finitely many vertices only, we may assume without loss of generality that every vertex of Γ belongs to $\{x_1,\ldots,x_N\}$. We triangulate Γ in such a way that $\{x_1,\ldots,x_N\} \subseteq \Gamma$ is its new vertex set.

We have now specified Γ, ϱ, δ and $\{x_1,\ldots,x_N\}$.

Assume that $D(\Gamma)$, g and T_m $(m \in \mathbb{N})$ are constructed as above. Since $n \geq 3$ and $\dim(D(\Gamma)) = 1$, we can approximate the projection $\pi : D(\Gamma) \to \Gamma$ arbitrarily closely by an embedding ([14, Remark 4.4.5]). So we may assume that $D(\Gamma) \subseteq [-1,1]^n$ and that Γ corresponds to the subspace $\Gamma \times \{\mathbf{0}\}$ of $\Gamma \times S$.

Claim 2. $\widehat{d}_{\mathrm{H}}(1_{C(\Gamma)}, g) < \frac{1}{4}\varepsilon$.

Since g sends every $A \in C(\Gamma)$ onto its closed δ ϱ-ball, this follows directly from Claim 1.

By Lemma 4.5 we may pick $M \in \mathbb{N}$ so large that

$$\forall m \geq M \ [\widehat{d}_{\mathrm{H}}(g, T_m) < \tfrac{1}{4}\varepsilon]\,. \tag{2}$$

This implies by Claim 2 that for all $m \geq M$ we have $\widehat{d}_{\mathrm{H}}(1_{C([-1,1]^n)}, T_m \circ \xi) < \varepsilon$. The approximation of $1_{C([-1,1]^n)}$ that we are looking for will be $T_m \circ \xi$ for some large $m \geq M$. So at this stage we already know that our map is ε-close to the identity. Since its image must also miss $\mathcal{B}$, all there remains to prove is that for some large m the image of T_m misses $\mathcal{B}$.

CLAIM 3. *Fix $\eta > 0$. There exists $\gamma \in (0,1]$ such that for all $\gamma_0, \gamma_1 \in [0,\gamma]$, all $i \leq N$ and all $x, y \in \Gamma$ with $d(x,y) \geq \eta$ and $\widehat{x} \in e((x, \gamma_0 t_i))$ and $\widehat{y} \in e((y, \gamma_1 t_i))$ we have*

$$d(\widehat{x}, \widehat{y}) \geq \tfrac{7}{8} d(x,y)\,.$$

Suppose that such a γ does not exist. Then there exist sequences $(p_n)_n$ and $(q_n)_n$ in $(0,1]$ with $p_n, q_n \leq 1/n$ for every n, an $i(n) \leq N$ and a pair a_n, b_n in Γ such that

(1) $\forall n \in \mathbb{N}\,[d(a_n, b_n) \geq \eta]$;

(2) for some $\widehat{a}_n \in e((a_n, p_n t_{i(n)}))$ and $\widehat{b}_n \in e((b_n, q_n t_{i(n)}))$,

$$d(\widehat{a}_n, \widehat{b}_n) < \tfrac{7}{8} d(a_n, b_n)\,.$$

We may assume without loss of generality that $i(n) = i(1)$ for every n and that $(\{a_n, b_n\})_n$ converges to $\{a, b\}$. Observe that $d(a,b) \geq \eta$. We easily arrive at a contradiction because the sequence $(\{\widehat{a}_n, \widehat{b}_n\})_n$ converges to $\{a, b\}$ (here we use the continuity of e and the fact that e is the identity on Γ), which implies that

$$d(a,b) = \lim_{n\to\infty} d(\widehat{a}_n, \widehat{b}_n) \leq \tfrac{7}{8} \lim_{n\to\infty} d(a_n, b_n) = \tfrac{7}{8} d(a,b)\,.$$

But this contradicts $d(a,b) \geq \eta > 0$.

CLAIM 4. *There exists $\mu > 0$ such that for all $A \in C(\Gamma)$, $\operatorname{diam}(g(A)) \geq \mu$.*

For every $A \in C(\Gamma)$, $g(A)$ has nonempty interior in Γ and consequently contains more than one point because Γ is connected. By compactness of $C(\Gamma)$ and continuity of g this yields $0 < \min\{\operatorname{diam}(g(A)) : A \in C(\Gamma)\}$.

Let μ be such as in Claim 4, and let γ be such as in Claim 3 for $\eta = \frac{1}{2}\mu$. Pick $m_0 \geq M$ so large that $2^{-m_0} < \gamma$. Since $\operatorname{diam}(g(A)) \geq \mu > 0$ for every $A \in C(\Gamma)$, by Lemma 4.5 we may pick $m \geq m_0$ so large that

$$\forall A \in C(\Gamma)\ [\operatorname{diam}(T_m(A)) \leq \tfrac{8}{7}\operatorname{diam}(g(A))]\,. \tag{3}$$

Now consider the function T_m.

CLAIM 5. *If $A \in C(\Gamma)$ then $T_m(A)$ cannot be covered by finitely many subcontinua of diameter at most $\frac{1}{3}\operatorname{diam}(T_m(A))$.*

Fix $A \in C(\Gamma)$. We first claim that there exists $i \le N$ such that $s_i(A) > 0$ and $x_i \in g(A)$. Since $x_1, \ldots, x_N$ is a $\frac{1}{2}\delta$-net there exists $i \le N$ such that $\varrho(x_i, A) \le \frac{1}{2}\delta$. For this i clearly $s_i(A) > 0$ and $x_i \in g(A)$.

Take a point $x \in g(A)$ such that $d(x, x_i) = \max\{d(y, x_i) : y \in g(A)\}$. Observe that $d(x, x_i) \ge \frac{1}{2}\operatorname{diam}(g(A)) \ge \eta$. For every n let

$$\widehat{p}_n = (x, 2^{-2n}2^{-(m-1)}s_i(A)t_i)\,.$$

Put $K = \{x_i\} \times [\mathbf{0}, 2^{-(m-1)}s_i(A)t_i]$. We claim that for every n

$$d\Big(e(\widehat{p}_n), \bigcup e[K]\Big) > \tfrac{1}{3}\operatorname{diam}(T_m(A))\,.$$

To see this, pick arbitrary $p_n \in e(\widehat{p}_n)$ and $q \in \bigcup e[K]$. Then by Claim 3 and (3) we have

$$\begin{aligned} d(p_n, q) &\ge \tfrac{7}{8}d(x_i, x) \ge \tfrac{7}{8}\cdot\tfrac{1}{2}\operatorname{diam}(g(A)) \\ &\ge \tfrac{7}{8}\cdot\tfrac{1}{2}\cdot\tfrac{7}{8}\operatorname{diam}(T_m(A)) > \tfrac{1}{3}\operatorname{diam}(T_m(A))\,. \end{aligned}$$

So a subcontinuum of $T_m(A)$ of diameter $\le \frac{1}{3}\operatorname{diam}(T_m(A))$ that intersects $e(\widehat{p}_n)$ misses $\bigcup e[K]$.

Put $\alpha = 2^{-(m-1)}s_i(A)$. Observe that

$$T_m(A) \setminus \bigcup e[K] \subseteq \bigcup_{n=1}^{\infty} \bigcup e[\Gamma \times \{2^{-2n}\alpha t_i\}] \cup \bigcup_{j \ne i} \Gamma \times L_j\,.$$

So by Proposition 4.1, if C is a subcontinuum of $T_m(A) \setminus \bigcup e[K]$ then either $C \subseteq \Gamma$ or for some unique n, $C \subseteq \bigcup e[\Gamma \times \{2^{-2n}\alpha t_i\}]$. Now simply observe that there are infinitely many $\widehat{p}_n$ so that there are infinitely many subcontinua of $T_m(A)$ of diameter at most $\frac{1}{3}\operatorname{diam}(T_m(A))$ needed to cover $T_m(A)$. ∎

5. $L([-1,1]^n)$ is strongly $F_{\sigma\delta}$-universal. The aim of this section is to prove that $L([-1,1]^n)$ is strongly $F_{\sigma\delta}$-universal in $C([-1,1]^n)$, provided that $n \ge 2$. The strategy of the proof is roughly speaking the following. First we use Theorem 2.6 to approximate a continuum $C \subseteq [-1,1]^n$ arbitrarily closely by a finite set F. Then we add straight line segments to F to make it connected. Moreover, to each point of F we add small sets of the form used in the proof of Theorem 3.1. These sets are needed to make sure that some but not all of the approximations that we construct are locally connected. Then we add to each point of F a closed half-ball. This ball is added for technical reasons: it allows us later to establish rather easily that our approximation is an embedding.

5.1. Theorem. *If $n \ge 2$ then $L([-1,1]^n)$ is strongly $F_{\sigma\delta}$-universal in $C([-1,1]^n)$.*

Proof. Let $A \subseteq Q$ be an $F_{\sigma\delta}$-subset, let $f : Q \to C([-1,1]^n)$ be an embedding that restricts to a Z-embedding of some compact subset $K \subseteq Q$

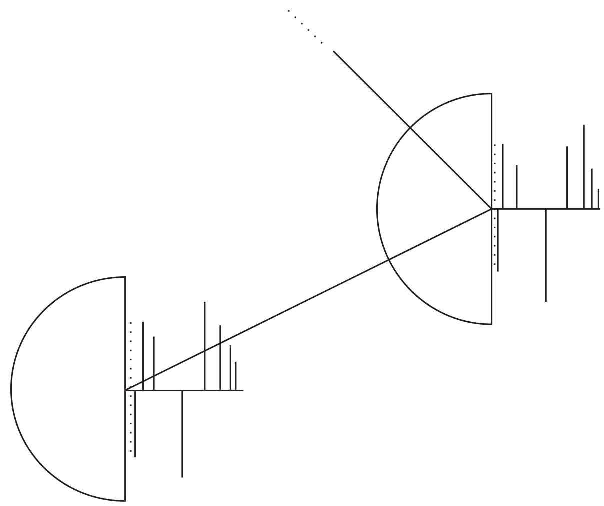

Fig. 3

and let $\varepsilon > 0$. Define $\mu : Q \to [0, \frac{1}{9}\varepsilon]$ by

$$\mu(x) = \tfrac{1}{9}\min\{\varepsilon, d_{\mathrm{H}}(f(x), f[K])\}\,.$$

By Theorem 2.6 there is a homotopy $H : 2^{[-1,1]^n} \times I \to 2^{[-1,1]^n}$ such that

(1) $H_0 = 1_{2^{[-1,1]^n}}$;
(2) for every $t \in (0,1]$, $H_t[2^{[-1,1]^n}] \subseteq \mathcal{F}([-1,1]^n)$.

It is clear that we may additionally assume that

(3) for every $t \in I$, $\widehat{d}_{\mathrm{H}}(H_t, 1_{2^{[-1,1]^n}}) \le 2t$;
(4) for every $t \in (0,1]$,

$$H[2^{[-1,1]^n} \times [t,1]] \subseteq \mathcal{F}([-1+t, 1-t]^n)\,.$$

Let $g : Q \to Q$ be an embedding such that $g^{-1}[\widehat{c}_0] = A$ (Theorem 2.5). For every $x \in Q$ let $\widehat{x} \in Q$ be defined as follows:

$$\widehat{x} = (x_1, \underbrace{x_1, x_2}, \underbrace{x_1, x_2, x_3}, \underbrace{x_1, x_2, x_3, x_4}, \ldots)\,.$$

Define $T : Q \to C([-1,1]^2)$ by the formula

$$\begin{aligned} T(x) = {} & (\{0\} \times [-1,1]) \cup ([0,1] \times \{0\}) \\ & \cup \bigcup_{n=1}^{\infty} \{\tfrac{1}{2n}\} \times \begin{cases} [0, g(x)_n] & (g(x)_n \ge 0), \\ [g(x)_n, 0] & (g(x)_n \le 0), \end{cases} \\ & \cup \bigcup_{n=1}^{\infty} \{1 - \tfrac{1}{3n}\} \times \begin{cases} [0, \frac{1}{n}\widehat{x}_n] & (\widehat{x}_n \ge 0), \\ [\frac{1}{n}\widehat{x}_n, 0] & (\widehat{x}_n \le 0). \end{cases} \end{aligned}$$

Just as in the proof of Theorem 3.1 it follows that $T(x) \in L([-1,1]^2)$ iff $x \in A$. Observe that $(0,0) \in T(x)$ for all $x \in Q$.

For all $x, y \in [-1,1]^n$ let $\overline{xy}$ denote the straight line segment in $[-1,1]^n$ connecting x and y. In addition, for $x, y \in [-1,1]^n$ and $r \in [0,\infty)$ let

$$\ell(x,y,r) = \{p \in \overline{xy} : d(p,\{x,y\}) \le r\}\,.$$

If $x \in [-1,1]^n$ and $\delta \ge 0$ then

$$\widehat{B}_l(x,\delta) = \{p \in [-1,1]^n : \|x-p\| \le \delta \text{ and } p_1 \le x_1\}\,.$$

For every $x \in Q$ let $F(x) = H(f(x), 2\mu(x))$. Then if $\mu(x) > 0$, $F(x)$ is a finite approximation of the continuum $f(x)$. Now define $h : Q \to C([-1,1]^n)$ as follows:

$$\begin{aligned} h(x) = {} & \bigcup_{a,b \in F(x)} \ell(a,b,4\mu(x)) \\ & \cup \bigcup_{a \in F(x)} a + \mu(x)(T(x) \times \underbrace{\{0\} \times \ldots \times \{0\}}_{n-2 \text{ times}}) \\ & \cup \bigcup_{a \in F(x)} \widehat{B}_l(a,\mu(x))\,. \end{aligned}$$

Claim 1. *h is well-defined, continuous and $h|K = f|K$. Moreover, for every $x \in Q$, $d_{\mathrm{H}}(f(x),h(x)) \le \frac{8}{9}\min\{\varepsilon, d(f(x), f[K])\}$.*

• Let $x \in Q$. Then by (4), $F(x) \subseteq [-1+2\mu(x), 1-2\mu(x)]^n$, which implies that

$$\bigcup_{a,b \in F(x)} \ell(a,b,4\mu(x)) \subseteq [-1+2\mu(x), 1-2\mu(x)]^n\,,$$

$$\bigcup_{a \in F(x)} \widehat{B}_l(a,\mu(x)) \subseteq [-1+\mu(x), 1-\mu(x)]^n\,.$$

Since

$$\mu(x)(T(x) \times \underbrace{\{0\} \times \ldots \times \{0\}}_{n-2 \text{ times}}) \subseteq [0,\mu(x)] \times [-\mu(x),\mu(x)] \times \underbrace{\{0\} \times \ldots \times \{0\}}_{n-2 \text{ times}}$$

we therefore conclude that $h(x) \subseteq [-1+\mu(x), 1-\mu(x)]^n$.

• If $\mu(x) > 0$ then $h(x)$ is compact and nonempty, being a finite union of compact nonempty sets. If $\mu(x) = 0$ then $h(x) = f(x)$, which is also compact and nonempty. So for every $x \in Q$, $h(x) \in 2^{[-1,1]^n}$.

• We claim that $h(x)$ is connected. Observe that it suffices to show that

$$P = \bigcup_{a,b \in F(x)} \ell(a,b,4\mu(x))$$

is connected. Suppose that P is not connected. Then we can write P as $U \cup V$, where U and V are disjoint nonempty open subsets of P. Put $F = U \cap F(x)$ and $G = V \cap F(x)$, respectively. Then both F and G are nonempty. Since by (3) we have $d_{\mathrm{H}}(f(x), F \cup G) \le 4\mu(x)$, it follows that $f(x) \subseteq \widehat{B}(F, 4\mu(x)) \cup \widehat{B}(G, 4\mu(x))$. The connectedness of $f(x)$ and the fact that both F and G are nonempty now imply that

$$\widehat{B}(F, 4\mu(x)) \cap \widehat{B}(G, 4\mu(x)) \neq \emptyset\,.$$

So there exist $a \in F$ and $b \in G$ such that $d(a,b) \le 8\mu(x)$. But then $\frac{1}{2}(a+b)$ is contained in U as well as in V, which is a contradiction.

• It is clear that h is continuous.

• Fix $x \in Q$. It is clear that $d_{\mathrm{H}}(f(x), h(x)) \le 8\mu(x)$ from which it follows that $d_{\mathrm{H}}(f(x), h(x)) \le \frac{8}{9} \min\{\varepsilon, d_{\mathrm{H}}(f(x), f[K])\}$. So we are done because this inequality implies that $h|K = f|K$.

Claim 2. *h is injective.*

First observe that from Claim 1 and the fact that f is an embedding it follows that

$$h[Q \setminus K] \cap h[K] = \emptyset\,. \tag{5}$$

Now fix $x, y \in Q$ such that $h(x) = h(y)$. Our task is to show that $x = y$. If both x and y belong to K then since $h|K = f|K$ and f is an embedding, it is trivial that $x = y$. If e.g. $x \notin K$ and $y \in K$ then from (5) it follows that $h(x) \neq h(y)$. So without loss of generality we may assume that $x, y \in Q \setminus K$. So both $\mu(x)$ and $\mu(y)$ are positive.

We will first prove that $\mu(x) = \mu(y)$. Assume the contrary, e.g. that $\mu(x) < \mu(y)$. There exists a point $\underline{m} = (m_1, \ldots, m_n) \in h(x)$ such that

$$\forall p \in h(x)\ [p_1 \le m_1]\,.$$

Then for every $p \in F(x)$ we have $p_1 \le m_1 - \mu(x)$. Moreover, the point

$$q = (m_1 - \mu(x), m_2, \ldots, m_n) \in F(x)\,.$$

Since $\widehat{B}_l(q, \mu(x)) \subseteq h(x)$ and $\mu(y) > \mu(x) > 0$, this implies that

$$\dim(h(x) \cap ((m_1 - \mu(y), 1] \times [-1,1]^{n-1})) = n \ge 2\,.$$

On the other hand, we have

$$\forall p \in F(y)\ [p_1 \le m_1 - \mu(y)].$$

This implies that

$$h(y) \cap ((m_1 - \mu(y), 1] \times [-1,1]^{n-1})$$

is a countable union of 1-dimensional compact sets, and therefore is 1-dimensional ([14, Theorem 4.3.7]). This contradiction establishes that $\mu(x) = \mu(y)$.

Again consider the point $\underline{m} \in h(x)$. Since $\mu(x) = \mu(y)$,

$$m^* = (m_1 - \mu(x), m_2, \ldots, m_n) \in F(x) \cap F(y)\,.$$

Since $F(x)$ and $F(y)$ are finite, m_1 is maximal, and the intervals $[0, \frac{1}{n}\widehat{x}_n]$ have length at most $1/n$ ($n \in \mathbb{N}$) there are a neighborhood U of $\underline{m}$ and a $\xi \in (0,1]$ such that

$$\begin{aligned} U \cap h(x) &= m^* + \mu(x)((T(x) \cap ([\xi,1] \times [-1,1]) \times \underbrace{\{0\} \times \ldots \times \{0\}}_{n-2 \text{ times}})) \\ &= m^* + \mu(y)((T(y) \cap ([\xi,1] \times [-1,1]) \times \underbrace{\{0\} \times \ldots \times \{0\}}_{n-2 \text{ times}}))\,. \end{aligned}$$

Since the coordinates of x appear infinitely often in the coordinates of $\widehat{x}$ (at pregiven places), and the same is true for y, it now easily follows that $x = y$.

Claim 3. $h^{-1}[L([-1,1]^n)] \setminus K = A \setminus K = h^{-1}[L((-1,1)^n)] \setminus K$.

Observe that by construction $h[Q \setminus K] \subseteq 2^{(-1,1)^n}$. It is clear that a finite union of locally connected continua is locally connected. Also, recall that T was constructed such that $T(x) \in L([-1,1]^2)$ iff $x \in A$. So if $x \in A \setminus K$ then $h(x)$ is locally connected. Now assume that $x \notin A \cup K$. Then $\mu(x) > 0$. We first assume that there exists $s > 0$ such that $T(x)$ is not locally connected at any point of the segment $\{0\} \times (0, s)$.

Let $\widehat{m} = (m_1, \ldots, m_n)$ be a point in $F(x)$ such that

$$\forall p \in F(x)\ [p_1 \leq m_1]\,.$$

Without loss of generality assume that

$$\forall p \in F(x)\ [p_1 = m_1 \Rightarrow p_2 \leq m_2]\,.$$

Then clearly $h(x)$ is not locally connected at some point of the segment

$$Z = \{m_1\} \times (m_2, m_2 + \mu(x)s) \times \{m_3\} \times \ldots \times \{m_n\}\,.$$

To see this, use the fact that $F(x)$ is finite and that the only possibility for a point of Z to be a point of local connectedness of $h(x)$ is if it lies on a horizontal segment in $h(x)$ with left endpoint in $F(x)$.

In case there exists $s < 0$ such that $T(x)$ is not locally connected at any point of the segment $\{0\} \times (0, s)$, proceed analogously.

Claim 4. *h is a Z-embedding.*

Since $h[K] = f[K]$ is a Z-set, it suffices to show that if $Y \subseteq Q \setminus K$ is compact, then $h[Y]$ is a Z-set. But this is clear because each continuum in $h[Y]$ contains a free arc, so the expansion homotopy $A \mapsto B_s(A) = \{p \in$

$[-1,1]^n : d(p,A) \le s\}$ maps $C([-1,1]^n)$ into the complement of $h[Y]$, for every positive s.

This completes the proof of the theorem. ∎

6. The main result and comments. A combination of Theorems 3.1, 4.6 and 5.1 gives a proof of the following:

6.1. THEOREM. *If $n \ge 3$ then $L([-1,1]^n)$ is an $F_{\sigma\delta}$-absorber in $C([-1,1]^n)$.* ∎

Fix $n \ge 3$. It is clear that $\{A \in C([-1,1]^n) : A \cap \partial[-1,1]^n \ne \emptyset\}$ is a Z-set in $C([-1,1]^n)$. Since an $F_{\sigma\delta}$-absorber in Q minus a Z-set in Q is an $F_{\sigma\delta}$-absorber (Baars, Gladdines and van Mill [1, Theorem 9.3]), it follows that the set of all Peano continua in $[-1,1]^n$ that miss the boundary also forms an $F_{\sigma\delta}$-absorber in $C([-1,1]^n)$. An application of Corollary 2.4 yields our main result.

6.2. THEOREM. *If $n \ge 3$ then $L(\mathbb{R}^n)$ is homeomorphic to B^∞.* ∎

6.3. Remark. As is clear from Theorem 3.1(1), our main result is false for $n = 1$. This leaves open the question whether it can be proved for $n = 2$. We do not know the answer to this question. Observe that by Theorems 3.1 and 5.1, the only thing left to prove is that $L([-1,1]^2)$ is contained in a σZ-set of $C([-1,1]^2)$. An inspection of our proof of Theorem 4.6 will show that our methods essentially do not apply to this case. It is known, however, as was brought to our attention recently by Cauty, that the space of all *arcs* in the plane *is* homeomorphic to B^∞ (see Cauty [5]).

It is natural to ask whether $L(Q)$ and $L(s)$ are homeomorphic to B^∞. Observe that we nowhere used the fact that we were dealing with a *finite-dimensional* cube. So in fact we also proved that $L(Q)$ is an $F_{\sigma\delta}$-absorber in Q. The question remains what can be said about $L(s)$. Note that the above trick for $\mathbb{R}^n$ does not apply here because B is not compact. But in Theorem 5.1 we made our approximations in such a way that they miss the boundary of the cube under consideration; see Claim 3 in the proof of Theorem 5.1. So our proof also shows that $L(s)$ is an $F_{\sigma\delta}$-absorber in $C(Q)$ and consequently that $L(s)$ and B^∞ are homeomorphic.

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Received 7 April 1992