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DENSE EXTREMALLY DISCONNECTED SUBSPACES

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ABSTRACT. We prove that every compact Basically Disconnected space of π-weight ω_1 has a dense Extremally Disconnected subspace. In Boolean algebraic terms: every σ-complete Boolean algebra B with density ω_1 carries an ultrafilter which generates an ultrafilter in the completion of B. The statement that every compact Basically Disconnected space of weight $\mathfrak{c}$ has a dense Extremally Disconnected subspace is shown to be equivalent to **CH**.

1. INTRODUCTION

Three closely related classes of zero-dimensional spaces are the *Extremally Disconnected* (the clopen algebra is complete), abbreviated ED, the *Basically Disconnected* (the clopen algebra is σ-complete), abbreviated BD, and the P-spaces (the clopen algebra is closed under countable unions and intersections). The question of when certain spaces have dense P-spaces has been extensively studied. We are looking for dense Extremally Disconnected subspaces. As can be seen from the discussion below, for zero-dimensional spaces this is the same as finding ultrafilters on Boolean algebras which generate ultrafilters on the completion.

A point in a space is a *λ-point* (for a cardinal λ) if there are λ disjoint open sets each with the point in the closure. A space is ED if and only if no point of the space is a 2-point. Call a point which is not a 2-point an ED-point.

A point $p \in \beta X \backslash X$ is called a *remote point* of X if p is not in the closure of any nowhere dense subset of X. Close connections have been established between remote points, ED spaces, and 2-points. For example, Woods [Woo71] established that the set of remote points of a space X embeds homeomorphically (and canonically) into the (ED) Gleason space of βX. More detailed connections between remote points and 2-points are explored in van Douwen's paper [vD81]. In particular, every remote point of X is an ED-point of βX. The converse is false because if X is ED then βX is ED at every point. It is easily seen that a space has a dense ED subspace if and only if it has a dense set of points which are not 2-points.

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In the first section we prove that every compact BD space of π-weight ω_1 has a dense ED subspace. This is somewhat surprising because the usual way to construct ED-points, i.e., via remote points, does not work here; indeed, it follows trivially from Dow [Dow83] that if **CH** fails there is a BD space with π-weight ω_1 which has no remote points. We also construct an example of a compact BD space with weight $\max\{\omega_2, \mathfrak{c}\}$ and π-weight ω_2 in which every point is a 2-point. Therefore the statement that every compact BD space of weight $\mathfrak{c}$ has a dense ED subspace is equivalent to **CH**. It has already been shown that the BD space consisting of the uniform ultrafilters on ω_1 (which has weight 2^{ω_1}) does not have a dense ED space (see Balcar and Simon [BaSi82]).

2. Compact BD spaces of π-weight $\aleph_1$ have dense ED subspaces

Recall that a *π-base* for a space X is a collection $\mathscr{B}$ of nonempty open subsets of X such that every nonempty open subset of X contains a member of $\mathscr{B}$. The π-weight, $\pi(X)$, of X is the smallest cardinality of a π-base for X. In the proof of the following result we will make use of ideas in Chae and Smith [CS80] and van Douwen [vD81].

Theorem 2.1. *Every compact BD space with $\pi(X) \leq \omega_1$ has a dense ED subspace.*

Proof. Let X be a compact BD space with π-base $\mathscr{B} = \{B_\alpha : \alpha < \omega_1\}$. We may assume without loss of generality that $\mathscr{B}$ consists of clopen sets. Let $\mathscr{W}$ be the family of all nonempty open subsets of X. For each $W \in \mathscr{W}$ put

$$H(W) = \{\alpha < \omega_1 : B_\alpha \subset W \text{ or } B_\alpha \cap \overline{W} = \varnothing\}.$$

Note that $\overline{W} - W$ is nowhere dense, for $W \in \mathscr{W}$, hence $\{B_\alpha : \alpha \in H(W)\}$ is again a π-base for X. For $W \in \mathscr{W}$ set $\mu(W) < \omega_1$ to be a nonzero ordinal with the property that

$$(\forall \beta < \mu(W))(\exists \alpha < \mu(W)) B_\alpha \subseteq B_\beta \text{ and } \alpha \in H(W).$$

The following closing off argument shows that there is such an ordinal $\mu(W)$:

$$\mu(W, 1) = \min H(W),$$
$$\mu(W, m+1) = \min\{\alpha < \omega_1 : [\forall \beta \leq \mu(W, m)][\exists \xi \in H(W) \cap \alpha][B_\xi \subseteq B_\beta]\}.$$

Then $\mu(W) = \sup\{\mu(W, m) : 1 \leq m < \omega\}$ has the desired property.

For each $W \in \mathscr{W}$ let $U_W = \bigcup\{B_\alpha : \alpha \in H(W) \cap \mu(W)\}$, note that U_W is an open F_σ.

Fact 2.2. If $\mathscr{E} \in [\mathscr{W}]^{<\omega}$ then there is an $\alpha < \max\{\mu(W) : W \in \mathscr{E}\}$ such that $B_\alpha \subseteq \bigcap_{W \in \mathscr{E}} U_W$.

We will prove the fact by induction on $n = |\mathscr{E}|$. For $n = 1$, it is obvious from the definition of $\mu(W)$. So assume the fact to be true for n, and consider arbitrary $W_1, \ldots, W_{n+1} \in \mathscr{W}$. We may assume that for all $i \leq n+1$ we have $\mu(W_i) \leq \mu(W_{n+1})$. By our inductive hypothesis, there is an $\alpha < \max\{\mu(W_i) : 1 \leq i \leq n\}$ such that $B_\alpha \subseteq \bigcap_{i=1}^n U_{W_i}$. Since $\alpha < \mu(W_{n+1})$ there is $\beta < \mu(W_{n+1})$ with $B_\beta \subseteq B_\alpha$ and $\beta \in H(W_{n+1})$. Then $B_\beta \subseteq \bigcap_{i=1}^{n+1} U_{W_i}$ and $\beta < \max\{\mu(W_i) : i \leq n+1\}$, so we are done.

We conclude that in particular the collection $\{U_W : W \in \mathscr{W}\}$ has the finite intersection property. So by compactness of X we may pick a point x in $\bigcap_{W \in \mathscr{W}} \overline{U}_W$.

Fact 2.3. x is not a 2-point.

To the contrary, assume that x is a 2-point. Then there is an open W of X such that

$$x \in \overline{W} \quad \text{and} \quad x \in \overline{X - \overline{W}}.$$

Put $A = U_W \cap W$ and $B = U_W - \overline{W}$, respectively. Observe that both these sets are open F_σ's since $A = \bigcup\{B_\beta : \beta < \mu(W) \text{ and } B_\beta \subset W\}$ and $B = \bigcup\{B_\beta : \beta < \mu(W) \text{ and } B_\beta \cap \overline{W} = \varnothing\}$. Now A is disjoint from the open set $X - \overline{W}$, hence so is $\overline{A}$. Since $\overline{A}$ is clopen it is also disjoint from $\overline{X - \overline{W}}$. Therefore x cannot be a member of $\overline{A}$. Similarly $\overline{B}$ is disjoint from $\overline{W}$, hence x is not a member of $\overline{B}$. However this contradicts that $x \in \overline{U_W} \subset \overline{A} \cup \overline{B}$.

We conclude that X is somewhere ED. But the same reasoning can be applied to every nonempty clopen subspace of X. So it follows that X contains a dense ED subspace. □

Observe that the above proof can be generalized. If X is a compact zero-dimensional space of π-weight κ such that the union of fewer than κ clopen sets has clopen closure, then X has a dense ED subspace.

Corollary 2.4. *If* **CH** *holds then every compact BD space of weight* $\mathfrak{c}$ *contains a dense ED subspace.*

Two questions naturally arise. Is **CH** (or its full strength) needed to prove Corollary 2.4? Can the hypothesis $\pi(X) \leq \omega_1$ be (consistently) weakened in Theorem 2.1? In fact, only the parenthetical questions are open. Indeed, Balcar and Simon have shown that $U(\omega_1)$ does not have a dense ED subspace [BaSi82]. This space is BD and has π-weight at most 2^{ω_1}. In the next section we prove that both 2.1 and 2.4 are best possible in the sense that the statement in Corollary 2.4 is equivalent to **CH**.

3. A small BD space with no dense ED subspace

In this section a σ-complete Boolean algebra $\mathfrak{U}$ of cardinality $\omega_2 \cdot \mathfrak{c}$ is constructed whose Stone space is shown to have the property that every point is a 2-point, hence there is no dense ED subspace.

For every space X, let $RO(X)$ denote the complete Boolean algebra of regular open subsets of X. If $\mathscr{A} \subseteq RO(X)$ then its supremum in $RO(X)$ is $\operatorname{int}\operatorname{cl}(\bigcup\mathscr{A})$. For more information on $RO(X)$ we refer the reader to Porter and Woods [PW88].

Notation 3.1. For an $s \in {}^{<\omega}\omega_2$, let $[s] = \{x \in {}^{\omega}\omega_2 : s \subset x\}$. In addition, put

$$\mathfrak{S} = \{[s] : s \in {}^{<\omega}\omega_2\}.$$

For convenience we treat members of ${}^{<\omega}\omega_2$ as ordered sequences of ordinals. In particular, $s^\frown\gamma$ denotes the obvious extension of s.

We endow ω_2 with the discrete topology and ${}^{\omega}\omega_2$ with the product topology, i.e., the topology having $\mathfrak{S}$ as a basis. Observe that if $[s]$ and $[t]$ are in $\mathfrak{S}$ and if $[s] \cap [t] \neq \varnothing$ then $[s] \subseteq [t]$ or $[t] \subseteq [s]$.

Definition 3.2. Let $U \in RO({}^{\omega}\omega_2)$. Then $\mathscr{A}_U$ is the collection of all countable sets $A_U \subseteq \omega_2$ such that for any $s \in {}^{<\omega}\omega_2$, either $U \cap [s^\frown\gamma]$ is empty for all $\gamma \notin A_U$, or $[s^\frown\gamma] \subseteq U$ for all $\gamma \notin A_U$. Let $\mathfrak{U}$ be the set of all $U \in RO({}^{\omega}\omega_2)$ for which $\mathscr{A}_U \neq \varnothing$.

Proposition 3.3. *$\mathfrak{U}$ is a σ-complete subalgebra of $RO({}^{\omega}\omega_2)$ that contains $\mathfrak{S}$.*

Proof. It is immediate that $\mathfrak{U}$ is closed under complements (in $RO({}^{\omega}\omega_2)$). Suppose that $\{U_n : n \in \omega\} \subset \mathfrak{U}$ and let $U = \bigvee_{n\in\omega} U_n = \operatorname{int}\operatorname{cl}(\bigcup_{n\in\omega} U_n)$. For each n, let A_{U_n} witness that U_n is in $\mathfrak{U}$ and let $A = \bigcup_n A_{U_n}$. Fix any $s \in {}^{<\omega}\omega_2$ and let $\gamma \notin A$. Suppose that $U \cap [s^\frown\gamma] \neq \varnothing$. Then there is an n such that $U_n \cap [s^\frown\gamma] \neq \varnothing$. It follows that $[s^\frown\delta] \subset U_n$ for all $\delta \notin A_{U_n}$. Therefore $[s^\frown\delta] \subset U$ for all $\delta \notin A$, hence $U \in \mathfrak{U}$.

It is routine to check that $\mathfrak{U} \supseteq \mathfrak{S}$. □

It will be useful to find an explicit description of the elements of $\mathfrak{U}$.

Remark 3.4. Let us emphasize that if $U \in \mathfrak{U}$ and $s \in {}^{<\omega}\omega_2$ and if for some $\gamma_0 \notin A_U \in \mathscr{A}_U$, $[s^\frown\gamma_0] \cap U \neq \varnothing$ then $[s^\frown\gamma] \subseteq U$ for every $\gamma \notin A_U$.

If $U \in \mathfrak{U}$ then we put $B_U = \bigcap \mathscr{A}_U$. We will show that $B_U \in \mathscr{A}_U$ (see Corollary 3.10). Define a function $\varphi_U : {}^{<\omega}(B_U) \to 2$ as follows:

$$\varphi_U(s) = 1 \Leftrightarrow (\forall\gamma \notin B_U)([s^\frown\gamma] \subseteq U).$$

Proposition 3.5. *Let $U \in \mathfrak{U}$. Then*

$$\bigcup\{[s^\frown\gamma] : s \in {}^{<\omega}(B_U),\ \gamma \notin B_U,\ \textit{and } \varphi_U(s) = 1\}$$

is dense in U.

Proof. Suppose that $V = \bigcup\{[s^\frown\gamma] : s \in {}^{<\omega}(B_U),\ \gamma \notin B_U, \text{ and } \varphi_U(s) = 1\}$ is not dense in U. Pick a nonempty $[t] \in \mathfrak{S}$ such that $[t] \subseteq U \backslash V$. We may assume without loss of generality that the last element of t is not in B_U (because if it is, then we can replace t by $t^\frown\gamma$ for an arbitrary $\gamma \notin B_U$). Let γ be the first member of t that does not belong to B_U. Write t in the form $t = t_0{}^\frown\gamma^\frown t_1$. Observe that $t_0 \in {}^{<\omega}(B_U)$.

Claim 3.6. $\varphi_U(t_0) = 1$.

Pick an arbitrary $\delta \notin B_U$. There exist elements A_U, $\widehat{A_U} \in \mathscr{A}_U$ such that $\delta \notin A_U$ and $\gamma \notin \widehat{A_U}$. Since $\varnothing \neq [t] \subseteq [t_0{}^\frown\gamma] \cap U$, it follows by Remark 3.4 that for cocountably many ξ we have $[t_0{}^\frown\xi] \subseteq U$. Now if $[t_0{}^\frown\delta]$ were not contained in U then it would follow again by Remark 3.4 that for cocountably many η we have $[t_0{}^\frown\eta] \cap U = \varnothing$; this is clearly impossible. So we conclude that $[t_0{}^\frown\delta] \subseteq U$.

Since $\varphi_U(t_0) = 1$ and $\gamma \notin B_U$ it now follows that $[t_0{}^\frown\gamma] \subseteq V$. But this clearly contradicts the fact that $[t]$ and V do not intersect because $[t] \subseteq [t_0{}^\frown\gamma]$. □

Corollary 3.7. *Let $U \in \mathfrak{U}$ and $s \in {}^{<\omega}(B_U)$ and let $B = B_U$. Then*

$$\varphi_U(s) = 0 \Leftrightarrow (\forall\gamma \notin B)([s^\frown\gamma] \cap U = \varnothing).$$

Proof. Suppose that $\varphi_U(s) = 0$. Pick an arbitrary γ not in B, and assume that $[s^\frown\gamma] \cap U \neq \varnothing$. By Proposition 3.5 there exists $t \in {}^{<\omega}B$ with $\varphi_U(t) = 1$ and $\gamma_1 \notin B$ such that $[s^\frown\gamma] \cap [t^\frown\gamma_1] \neq \varnothing$. Consequently, $[s^\frown\gamma] \subseteq [t^\frown\gamma_1]$ or $[t^\frown\gamma_1] \subseteq [s^\frown\gamma]$. Since both s and t belong to ${}^{<\omega}B$ and both γ and γ_1 do not belong to B, it follows that $s = t$ and $\gamma = \gamma_1$. But this is a contradiction because $\varphi_U(s) = 0$ and $\varphi_U(t) = 1$. □

To every $U \in \mathfrak{U}$ we assigned a function φ_U. We now aim to show that this function completely determines U.

Lemma 3.8. *If U, $V \in \mathfrak{U}$, and $U \neq V$ then $\varphi_U \neq \varphi_V$.*

Proof. We may assume that the domains of φ_U and φ_V agree. Since $U \neq V$ by Proposition 3.5 we may also assume without loss of generality that there exists $s \in {}^{<\omega}(B_U)$ with $\varphi_U(s) = 1$ and $\gamma \notin B_U$ such that $[s^\frown\gamma]$ intersects the complement of V. But since the domains of φ_U and φ_V agree, this implies that $\varphi_V(s) = 0$ because otherwise $[s^\frown\gamma] \subseteq V$. □

Since for each countable B there are $\mathfrak{c}$ functions from B into 2 and there are $\omega_2^\omega = \mathfrak{c}\cdot\omega_2$ countable subsets of ω_2, this lemma implies that $|\mathfrak{U}| \leq \mathfrak{c}\cdot\omega_2$.

Proposition 3.9. *If $B \subset \omega_2$ is countable and $\varphi: {}^{<\omega}B \to 2$ is any function, then*

$$U = \operatorname{int}\operatorname{cl}\left(\bigcup\{[s^\frown\gamma]: s \in {}^{<\omega}B,\ \gamma \notin B,\ \text{and } \varphi(s) = 1\}\right)$$

is in $\mathfrak{U}$ and $B \in \mathscr{A}_U$.

Proof. We prove that $B \in \mathscr{A}_U$. To this end, pick any arbitrary $t \in \mathfrak{S}$. Assume that there exists $\gamma \notin B$ such that $[t^\frown\gamma] \cap U \neq \varnothing$. Pick $s \in {}^{\omega}B$ with $\varphi(s) = 1$ and an element $\gamma_1 \notin B$ such that $[t^\frown\gamma] \cap [s^\frown\gamma_1] \neq \varnothing$. Then $[t^\frown\gamma] \subseteq [s^\frown\gamma_1]$ or $[s^\frown\gamma_1] \subseteq [t^\frown\gamma]$. If $t^\frown\gamma$ is an initial sequence of $s^\frown\gamma_1$ then since $\gamma \notin B$ it follows that $t = s$ and $\gamma = \gamma_1$. But then $[t^\frown\delta] \subseteq U$ for every $\delta \notin B$. Suppose therefore that $s^\frown\gamma_1$ is an initial sequence of $t^\frown\gamma$. We may assume without loss of generality that γ_1 comes before γ because otherwise we are again in the situation that $s = t$. So $s^\frown\gamma_1$ is an initial sequence of t, which implies that *every* extension of t is contained in $[s^\frown\gamma_1] \subseteq U$. □

Corollary 3.10. *Let $U \in \mathfrak{U}$. Then $B_U \in \mathscr{A}_U$.*

Proof. This follows easily from Propositions 3.5 and 3.9. □

Although we do not need it, let us remark that $\mathfrak{U}$ is the smallest σ-complete subalgebra of $RO({}^{\omega}\omega_2)$ that contains $\mathfrak{S}$.

We have shown above that the cardinality of $\mathfrak{U}$ is no more than $\omega_2\cdot\mathfrak{c}$. It also contains $\mathfrak{S}$ which has cardinality ω_2. Since the cardinality of $\mathfrak{U}$ is an ω-power by the result of Comfort and Hager [CH72] it follows that its cardinality is $\omega_2\cdot\mathfrak{c}$. This also follows easily from Proposition 3.9.

We will proceed to prove that each point of $S(\mathfrak{U})$ is a 2-point.

Definition 3.11. For each $\alpha < \omega_2$, let E be the set of even ordinals in ω_2 and $O = \omega_2 \backslash E$, and define

$$W_\alpha = \bigcup\{[s]: (\exists n)(s(n) \in E\backslash\alpha \wedge s \restriction n \in {}^{n}\alpha)\}$$

and

$$V_\alpha = \bigcup\{[s]: (\exists n)(s(n) \in O\backslash\alpha \wedge s \restriction n \in {}^{n}\alpha)\}.$$

We now come to the main result in this section.

Theorem 3.12. *Every point of the Stone space of $\mathfrak{U}$ is a 2-point.*

Proof. Let p be an arbitrary ultrafilter on $\mathfrak{U}$ and assume that it is not a 2-point. Let $S(\mathfrak{U})$ denote the Stone space of $\mathfrak{U}$. Observe that for every α both W_α and V_α are unions of elements of $\mathfrak{U}$; moreover, $W_\alpha \cap V_\alpha = \varnothing$. So the sets W_α and V_α correspond in a natural way to disjoint open subsets, say W_α^* and V_α^*, of $S(\mathfrak{U})$. Observe that p is in the closure of W_α^* if and only if $U \cap W_\alpha \neq \varnothing$ for every $U \in p$. Similarly for V_α^*.

So, since p is not a 2-point, there is, for each $\alpha < \omega_2$, a $U_\alpha \in p$ such that U_α is disjoint from either W_α or V_α. By definition of $\mathfrak{U}$ there is, for each $\alpha \in \omega_2$, $A_\alpha \in \mathscr{A}_{U_\alpha}$. Choose an increasing sequence, $\{\lambda_\xi : \xi \in \omega_1\}$, so that $A_{\lambda_\xi} \subset \lambda_{\xi+1}$ for each $\xi < \omega_1$. Let λ_{ω_1} be the supremum of the sequence $\{\lambda_\xi : \xi \in \omega_1\}$. Choose $\alpha < \omega_1$ so that $A_{\lambda_{\omega_1}} \cap \lambda_{\omega_1} \subseteq \lambda_\alpha$. Let $s \in {}^{<\omega}\omega_2$ be such that $[s]$ is contained in $U_{\lambda_\alpha} \cap U_{\lambda_{\omega_1}}$. Since, without loss of generality, $U_{\lambda_\alpha} \cap W_{\lambda_\alpha}$ is empty, there is a minimum k such that $s(k) \geq \lambda_\alpha$; else any extension of s by a sufficiently large even ordinal witnesses that U_{λ_α} meets W_α.

We will show that $s(k) \in A_{\lambda_\alpha} \cap A_{\lambda_{\omega_1}}$, which is a contradiction since $A_{\lambda_\alpha} \cap A_{\lambda_{\omega_1}} \subseteq \lambda_{\omega_1} \cap A_{\lambda_{\omega_1}} \subseteq \lambda_\alpha$, and yet $s(k) \geq \lambda_\alpha$.

We first prove by contradiction that $s(k) \in A_{\lambda_\alpha}$. Observe that $[s \restriction k] \cap U_{\lambda_\alpha} \neq \varnothing$. Apply Remark 3.4 to see that for all but countably many γ, $[(s \restriction k)^\frown\gamma]$ is contained in U_{λ_α}. But now since $s \restriction k \in {}^{<\omega}\lambda_\alpha$ it follows that some of these clopen sets are contained in W_{λ_α}.

The proof that $s(k) \in A_{\lambda_{\omega_1}}$ is identical. □

REFERENCES

[BaSi82] B. Balcar and P. Simon, *Strong decomposability of ultrafilters* I, Studies in Logic, vol. 108, North-Holland, Amsterdam, 1980, pp. 1–10.

[CS80] S. B. Chae and J. H. Smith, *Remote points and G-spaces*, Topology Appl. **11** (1980), 243–246.

[CH72] W.W. Comfort and T. Hager, *Cardinality of κ-complete Boolean algebras*, Pacific J. Math. **28** (1972), 541–545.

[vD81] E. K. van Douwen, *Remote points*, Dissertationes Math. **188** (1981), 1–45.

[Dow83] A. Dow, *Products without remote points*, Topology Appl. **15** (1983), 239–246.

[PW88] J. Porter and R. G. Woods, *Extensions and absolutes of Hausdorff spaces*, Springer-Verlag, New York, Heidelberg, and Berlin, 1988.

[Woo71] R. G. Woods, *Homeomorphic sets of remote points*, Canad. J. Math. **23** (1971), 495–502.

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