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DISJOINT EMBEDDINGS OF COMPACTA

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Abstract. Let X be a separable and metrizable space containing uncountably many pairwise disjoint copies of the compactum K. We discuss the question whether X must contain $K \times 2^{\omega}$.

§1. *Introduction. All spaces are separable and metrizable.* Notation and terminology generally follows Moschovakis [**9**] on descriptive set theory, games, *etc.*, and Kuratowski [**4**] on matters of general topology. It is an old and well-known result of Suslin [**12**] that if X is analytic (*i.e.*, a continuous image of the irrationals ω^{ω}) and uncountable, then X contains a copy of the Cantor set 2^{ω}. In other words: if X is analytic and contains uncountably many pairwise disjoint copies of the space $K = \{p\}$, then X contains a copy of $K \times 2^{\omega}$. This rephrasing leads naturally to the question whether a similar result holds for other compact spaces. Obviously, we can take for K any compact zero-dimensional space, since $K \times 2^{\omega}$ is homeomorphic to 2^{ω} for such K. In [**1**], van Douwen shows that if X is completely metrizable, then the result holds for an arbitrary compact K.

In this paper, we first give a short and easy proof of van Douwen's theorem, using function spaces. These function spaces are subsequently used to define a game which is used to extend van Douwen's result to spaces X in certain more general classes Γ, assuming the determinacy of games with payoff set in Γ. As a particular case, we obtain the following: if Det $(\mathbf{\Pi}_1^1)$ then every co-analytic space containing uncountably many pairwise disjoint copies of the compactum K contains a copy of $K \times 2^{\omega}$. In fact, Det $(\mathbf{\Pi}_1^1)$ is too strong a hypothesis here: it suffices to assume the existence of a Cantor set in every uncountable $\mathbf{\Pi}_1^1$-set. Finally, we give an example of a σ-compact space containing uncountably many pairwise disjoint copies of the circle S^1 but not

$S^1 \times 2^\omega$, granting the existence of an uncountable coanalytic space *without* perfect subsets.

§2. *Embeddings in complete spaces.* In [**1**], van Douwen showed that if X is completely metrizable and contains uncountably many pairwise disjoint copies of the compactum K, then X contains $K \times 2^\omega$. His proof is a rather technical direct approximation of $K \times 2^\omega$ by the given embeddings of K in X. The idea of our proof is basically to reduce everything to the well-known case $K = \{p\}$ by considering the space $I(K, X)$ of embeddings of K in X.

Let K now be a fixed compact space, and (X, d) an arbitrary space. Put

$$C(K, X) = \{f: f \text{ is a continuous mapping } K \to X\},$$

and for $f, g: K \to X$ define $\hat{d}(f, g) = \max_{x \in K} d(f(x), g(x))$. Then $(C(K, X), \hat{d})$ is a separable metric space; furthermore, if d is a complete metric then so is $\hat{d}$. Let $C_\varepsilon(K, X)$ be the open subspace

$$\{f \in C(K, X): \text{for all } x \in X, \operatorname{diam} f^{-1}(x) < \varepsilon\}$$

of $C(K, X)$; clearly, $\bigcap_{n \in \mathbb{N}} C_{1/n}(K, X) = \{f \in C(K, X): f \text{ is an embedding}\}$, so

$$I(K, X) = \{f \in C(K, X): f \text{ is an embedding}\}$$

is a $\mathbf{G}_\delta$ in $C(K, X)$. As a consequence, $I(K, X)$ is complete if X is complete. (For details on these matters see van Mill [**8**, Chapter 1.3]).

2.1. LEMMA. *Let Y be an analytic subspace of $I(K, X)$ containing an uncountable subset F of mappings with pairwise disjoint images. Then X contains a copy of $K \times 2^\omega$.*

Proof. First note that if $f, g \in F$ then we can find $\varepsilon(f, g) > 0$ such that if $\hat{d}(f, \tilde{f}) < \varepsilon(f, g)$ and $\hat{d}(g, \tilde{g}) < \varepsilon(f, g)$ then $\tilde{f}[K] \cap \tilde{g}[K] = \varnothing$.

Let $\varphi: \omega^\omega \to Y$ be onto, and put

$$G = \omega^\omega - \bigcup\{U: U \text{ is open in } \omega^\omega, \varphi[U] \cap F \text{ is countable}\}.$$

Since F is uncountable G is non-empty, and G is complete since it is a $\mathbf{G}_\delta$ in ω^ω. It is easy to verify that if V is open and non-empty in G then $\varphi[V] \cap F$ is uncountable. Fix a complete metric on G.

Inductively, we construct for each $s \in 2^{<\omega}$ a non-empty open subset U_s of G such that

(i) if $s \subseteq t$ then $U_s \supseteq \bar{U}_t$;
(ii) $\operatorname{diam}(U_s) < 2^{-|s|}$;
(iii) if $|s| = |t|$, $s \neq t$, $f \in \varphi[U_s]$, $g \in \varphi[U_t]$ then $f[K] \cap g[K] = \varnothing$.

Here $|s| = n$ if $s = (s_0, \ldots, s_{n-1})$. Take $U_{\langle\rangle} = G$. If U_s has been constructed then $\varphi[U_s] \cap F$ is uncountable, so we can choose $f \neq g$ in $\varphi[U_s] \cap F$, say $f = \varphi(x_0)$, $g = \varphi(x_1)$ with $x_i \in U_s$. Now let $U_{s^\frown i}$ be an open neighbourhood of x_i of diameter less than $2^{-|s|-1}$, such that $\bar{U}_{s^\frown i} \subseteq U_s$, and such that furthermore $\varphi[U_{s^\frown i}]$ has diameter less that $\varepsilon(f, g)$ in Y. This completes the induction.

For each $z \in 2^\omega$, let $\psi(z)$ be the unique point in $\bigcap_{n<\omega} U_{z|n}$ (by (i), (ii) and completeness of the metric on G). We claim that $\varphi \circ \psi[2^\omega] = C$ is a copy of 2^ω in Y consisting of mappings with pairwise disjoint images. Indeed, $\varphi \circ \psi$ is clearly continuous, and if z and w are distinct elements of 2^ω then for some $|s| = |t|$, $s \neq t$ we have $\psi(z) \in U_s$, $\psi(w) \in U_t$ whence $\varphi \circ \psi(z)[K] \cap \varphi \circ \psi(w)[K] = \varnothing$ by (iii). Finally, define $j: K \times C \to X$ by $j(x, f) = f(x)$. Then j is continuous, and if $(x, f) \neq (y, g)$ then $f(x) \neq g(y)$ either because $f[K] \cap g[K] = \varnothing$ (if $f \neq g$) or because f is an embedding (if $f = g$ and $x \neq y$). Thus, j is the required embedding.

Remark. The construction of the Cantor set in the above proof could be replaced by an application of the boundedness principle (see Mauldin and Schlee [7]).

Van Douwen's result follows immediately.

2.2. THEOREM. *Let X be completely metrizable, and suppose that X contains uncountably many pairwise disjoint copies of the compact space K. Then X contains a copy of $K \times 2^\omega$.*

Proof. Apply Lemma 2.1 to $Y = I(K, X)$.

§3. *Embeddings in more general spaces.* Throughout this section, Γ will denote a class of subspaces of various complete spaces, with the following two properties:

(1) Γ is closed under continuous preimages, and
(2) Γ is closed under universal quantification.

Examples are the classes $\mathbf{\Pi}^1_n$ and the class of all projective sets. More precisely, (1) says that if X and Y are complete spaces, $f: X \to Y$ is continuous, and $Z \subseteq Y$ is in Γ, then $f^{-1}[Z] \in \Gamma$; in particular if $\Gamma \neq \varnothing$ then Γ contains all complete spaces. Note that (2) is equivalent to saying that the point class $\check{\Gamma}$ consisting of all complements of spaces in Γ is closed under projection.

We also continue to use the terminology of function spaces that was introduced in Section 2.

3.1. LEMMA. *If $X \in \Gamma$ then $I(K, X) \in \Gamma$ for each compact space K.*

Proof. Let Y be a complete space containing X, and define $\varphi: K \times I(K, Y) \to Y$ by $\varphi(p, f) = f(p)$. Then φ is continuous and

$$f \in I(K, X) \iff (\forall p \in K)((p, f) \in \varphi^{-1}[X]).$$

Thus, by properties (1) and (2) of Γ, $I(K, X) \in \Gamma$ (noting that $I(K, X) = I(K, Y) - \pi_2 \varphi^{-1}[Y - X]$ we could have used the equivalent form of (2)).

The main result of this section is the following theorem.

3.2. THEOREM. *If* Det (Γ), *and* $X\in\Gamma$ *contains uncountably many pairwise disjoint copies of the compact space* K, *then* X *contains a copy of* $K\times 2^{\omega}$.

Proof. Fix a complete space Y containing X, and a countable basis $\mathscr{B}$ for $C(K, Y)$. In basically the same way as in Section 2 we will construct a Cantor set in $I(K, Y)$ of mappings with pairwise disjoint images. However, here we will in addition use a winning strategy in a certain game to get the mappings to have images that are in fact inside X.

We define the infinite two-person game **G** as follows. Two players alternate moves in the usual way. On move n, Player I plays a pair (U_n^0, U_n^1) of elements of $\mathscr{B}$ and Player II plays $i_n\in 2=\{0, 1\}$.

$$\left(\mathbf{G}\quad \begin{array}{lllllll} \mathrm{I}\ (U_0^0, U_0^1) & & (U_1^0, U_1^1) & & (U_2^0, U_2^1) & & \dots \\ \mathrm{II} & i_0 & & i_1 & & i_2 & \dots \end{array}\right)$$

Player I must play U_n^m's satisfying the following conditions (otherwise he loses):

(a) $I_n^0\cap I_n^1=\varnothing$, where $I_n^m=\bigcup\{f[K]: f\in U_n^m\}$;

(b) diam $(U_n^m)<\dfrac{1}{n}$;

(c) $(U_{n+1}^m)^-\subseteq U_n^{i_n}$.

Assume Player I has played so that (a)–(c) hold. Then $\bigcap_{n<\omega} U_n^{i_n}=\bigcap_{n<\omega}(U_n^{i_n})^-$ contains exactly one point $f\in C(K, Y)$; call f the *outcome* of the game. Player I wins the round of the game, if, and only if, the outcome is in $I(K, X)$.

CLAIM 1. **G** *is determined.*

Indeed, fix an enumeration $\{V_i: i\in\omega\}$ of the basis $\mathscr{B}$. This enumeration makes **G** equivalent to a game $\hat{\mathbf{G}}$ in which, on each move, Player I plays two elements of ω and Player II plays one element of 2 (*equivalent* meaning that if either player has a winning strategy in the one game, that same player has a winning strategy in the other).

$$\left(\hat{\mathbf{G}}\quad \begin{array}{lllllll} \mathrm{I}\ (x_0, y_0) & & (x_1, y_1) & & (x_2, y_2) & & \dots \\ \mathrm{II} & i_0 & & i_1 & & i_2 & \dots \end{array}\right)$$

After ω moves in a round of $\hat{\mathbf{G}}$, the two players have played $(x, y, z)\in\omega^{\omega}\times\omega^{\omega}\times 2^{\omega}$. Let $W\subseteq\omega^{\omega}\times\omega^{\omega}\times 2^{\omega}$ be the subspace consisting of all rounds of the game $\hat{\mathbf{G}}$ in which Player I wins. We are assuming Det (Γ), so to complete the proof of the claim all that must be shown is that W is in Γ. Let F be the subspace of $\omega^{\omega}\times\omega^{\omega}\times 2^{\omega}$ consisting of all rounds of $\hat{\mathbf{G}}$ corresponding to rounds in **G** where Player I has played following (a)–(c); then F is closed. Define $\psi: F\to C(K, Y)$ to be the function which sends a round of the game $\hat{\mathbf{G}}$ to the outcome of the corresponding game **G**. That is,

$$\{\psi(x, y, z)\}=\bigcap\{V_{x(n)}: n\in\omega, z(n)=0\}\cap\bigcap\{V_{y(n)}: n\in\omega, z(n)=1\}.$$

Clearly, ψ is continuous, and both F and $C(K, Y)$ are complete. By Lemma 3.1, $I(K, X) \in \Gamma$, so since Γ is closed under continuous preimages, $W = \psi^{-1}[I(K, X)] \in \Gamma$.

CLAIM 2. *Player* II *does not have a winning strategy for* **G**.

Indeed, let us assume, towards a contradiction, that Player II has a winning strategy τ for **G**. Consider a *position*

$$p = \langle (U_0^0, U_0^1), i_0, \ldots, (U_n^0, U_n^1), i_n \rangle$$

in the game in which it is Player I's turn to play (or $p = \langle\ \rangle$, and the game is about to begin). For any such position p and any $f \in C(K, Y)$, we say that τ *rejects f at p* if the following three conditions hold:

(i) p is consistent with the strategy τ;

(ii) $f \in U_n^{i_n}$;

(iii) for every move (U_{n+1}^0, U_{n+1}^1) for Player I at position p which is legal (*i.e.*, satisfies (a)–(c)), τ calls for Player II to respond with an $i_{n+1} \in 2$ such that $f \notin U_{n+1}^{i_{n+1}}$.

First note that for any $f \in I(K, X)$, there exists some position p such that τ rejects f at p. For if not, Player I could play against τ in such a way that the outcome of the game is f. That means that there is a round of the game played according to τ in which Player I wins, contrary to the assumption that τ is a winning strategy. Second, note that for any $f, g \in C(K, Y)$, if f and g have disjoint images, then they cannot both be rejected by τ at the same position. For suppose they are both rejected at $p = \langle (U_0^0, U_0^1), i_0, \ldots, (U_n^0, U_n^1), i_n \rangle$. By (ii), $f, g \in U_n^{i_n}$. Since f and g have disjoint images, there exist U_{n+1}^0 and U_{n+1}^1 in $\mathscr{B}$, satisfying (a)–(c), and such that $f \in U_{n+1}^0$ and $g \in U_{n+1}^1$. If Player I plays this (U_{n+1}^0, U_{n+1}^1) at p, then τ calls for Player II to respond with either $i_{n+1} = 0$ or $i_{n+1} = 1$. In the former case f is not rejected by τ at p, and in the latter case g is not. By hypothesis, however, there is an uncountable subset F of $I(K, X)$ consisting of mappings with pairwise disjoint images. Any member of F is rejected by τ at some position. There are only countably many positions. Hence two distinct members of F must be rejected by τ at the same position, and we have a contradiction.

By the two claims, Player I has a winning strategy σ for the game **G**. For $z \in 2^\omega$, let f_z denote the outcome of the game when Player II plays so that $z = (i_0, i_1, i_2, \ldots)$ and Player I follows σ. Since σ is a *winning* strategy, $f_z \in I(K, X)$, and by (a), if $z \neq w$ then $f_z[Z] \cap f_w[K] = \varnothing$. Clearly the function $z \mapsto f_z$ is continuous. Thus, $I(K, X)$ contains a copy of 2^ω consisting of mappings with pairwise disjoint images. This implies as in Lemma 2.1 that X contains a copy of $K \times 2^\omega$.

Remark. Although the class of complete spaces does not satisfy condition (2) imposed upon Γ in this section, van Douwen's theorem can nonetheless be seen as a special case of Theorem 3.2. Indeed, in this case the game in the proof is closed, so since closed games are determined, the above proof is in ZFC.

§4. *Embeddings in coanalytic spaces.* In this section we will show that in the special case where $\Gamma = \mathbf{\Pi}^1_1$ (the class of all coanalytic spaces) the assumption Det (Γ) in Theorem 3.2 can be weakened to the following hypothesis.

($*$) For any $a \subseteq \omega$ there are only countably many subsets of ω in $L(a)$. $L(a)$ is the smallest (class) model of ZFC containing all ordinals and containing a. ZFC + ($*$) is equiconsistent with ZFC + there exists an inaccessible cardinal, see Solovay [**11**]. The hypothesis ($*$) is a consequence of Det $(\mathbf{\Pi}^1_1)$ but is known to be much weaker. For information on the connections between the models $L(a)$ and descriptive set theory, see Jech [**2**], Mansfield and Weitkamp [**6**], and Moschovakis [**9**].

We start out by stating two lemmas.

4.1. LEMMA (Shoenfield; see Moschovakis [**9**, 8F.8]). *Let $P \subseteq \omega^\omega$ be any $\mathbf{\Pi}^1_1$-set. There exists a tree T on $\omega \times \omega_1$ such that the following conditions hold*:

(a) $P = \{w \in \omega^\omega$: *there exists* $\varphi \in (\omega_1)^\omega$ *such that* (w, φ) *is an infinite branch through* $T\}$;

(b) *There exists $b \subseteq \omega$ such that T is in the model $L(b)$.*

The next lemma we will state concerns closed games. Let A be any discrete space. If X is closed in A^ω then we can put

$$A^\omega - X = \bigcup\{[s]: s \in \mathscr{S}\}$$

for some set $\mathscr{S}$ of finite sequences from A, where $[s]$ is the basic open subset of A^ω determined by s; conversely, every such $\mathscr{S}$ determines a closed subset of A^ω. Since Player II wins a round of the closed game $\mathbf{G}_A(X)$, if, and only if, after some finite number of moves the sequence produced (by both players) is in $\mathscr{S}$, we can identify closed games with pairs $(A, \mathscr{S})$ of a set A and a set $\mathscr{S}$ of finite sequences from A. Furthermore, let us remark that closed games are determined.

4.2. LEMMA. *If M is a transitive model of* ZFC *and $(A, \mathscr{S}) \in M$ is a closed game, then one of the players has a winning strategy $\tau \in M$ for this game.*

For a proof of Lemma 4.2 we refer to Kechris and Moschovakis [**3**, p. 40].

We are now ready to prove our theorem.

4.3. THEOREM. *If ($*$) holds, and $X \in \mathbf{\Pi}^1_1$ contains uncountably many pairwise disjoint copies of the compact space K, then X contains a copy of $K \times 2^\omega$.*

Proof. Fix a complete space Y containing X, and a countable basis $\{V_i: i \in \omega\}$ for $C(K, Y)$. Let $\psi: \omega^\omega \to C(K, Y)$ be a continuous surjection and let $P = \psi^{-1}[I(K, X)]$. Since $\Gamma = \mathbf{\Pi}^1_1$ satisfies the conditions of Section 3 we have $I(K, X) \in \mathbf{\Pi}^1_1$ by Lemma 3.1, and hence P is also $\mathbf{\Pi}^1_1$; let the tree T on $\omega \times \omega_1$ and $b \subseteq \omega$ be obtained from Lemma 4.1 applied to this particular P.

We now define yet another game, $\mathbf{G}'$. On move n, Player I plays a countable ordinal $\varphi(n)$ and three integers $w(n)$, $x(n)$ and $y(n)$, and Player II plays $z(n) \in 2$.

Let $U_n^0 = V_{x(n)}$, $U_n^1 = V_{y(n)}$, and $i_n = z(n)$. Player I wins the round of $\mathbf{G}'$, if, and only if, the following five conditions are satisfied: firstly, the conditions of the game $\mathbf{G}$ (or $\hat{\mathbf{G}}$),

(a) $I_n^0 \cap I_n^1 = \varnothing$, where $I_n^m = \bigcup\{f[K]: f \in U_n^m\}$,

(b) $\operatorname{diam}(U_n^m) < \dfrac{1}{n}$, and

(c) $(U_{n+1}^m)^- \subseteq U_n^{i_n}$,

which imply the existence of an outcome of the game, but also

(d) $(w, \varphi) \in \omega^\omega \times (\omega_1)^\omega$ is an infinite branch through T;

(e) $\psi(w)$ is the outcome of the game.

Note that $\mathbf{G}'$ has the same plays as $\hat{\mathbf{G}}$, x, y and z, plus additional plays ψ and w for Player I; conditions (d) and (e) imply that the outcome of the game is in $I(K, X)$. So what the above definition really means is that Player I wins $\mathbf{G}'$, if, and only if, he wins $\hat{\mathbf{G}}$ and, *in addition*, plays a φ and w that witness his winning $\hat{\mathbf{G}}$. Also, note that if Player I has a winning strategy σ for $\mathbf{G}'$, he clearly also has a winning strategy for $\hat{\mathbf{G}}$: play according to σ just ignoring φ and w. In this case, we can complete the proof of the theorem as in Theorem 3.2. So all that remains to be proved is that $\mathbf{G}'$ is determined and that Player II does not have a winning strategy.

Claim 1. *$\mathbf{G}'$ is a closed game and is therefore determined.*

Indeed, let $(\varphi^k, w^k, x^k, y^k, z^k)_k$ be a sequence of winning rounds for Player I, and suppose the sequence converges to (φ, w, x, y, z). Since (a)–(d) clearly only depend on initial segments, it suffices to prove (e). We will show that if $z(n)=0$ then $\psi(w) \in \bar{V}_{x(n)}$; the proof that $\psi(w) \in \bar{V}_{y(n)}$ if $z(n)=1$ is similar. So assume that $z(n)=0$, and choose an integer N such that if $k \geqslant N$ then $z^k(n)=0$ and $x^k(n)=x(n)$. Then $\psi(w^k) \in V_{x^k(n)} = V_{x(n)}$ for all $k \geqslant N$. Since the sequence $\psi(w^k)$ converges to $\psi(w)$, we get $\psi(w) \in \bar{V}_{x(n)}$ as required.

Claim 2. *Player* II *does not have a winning strategy for* $\mathbf{G}'$.

As in the proof of Theorem 3.2, we assume that Player II does have a winning strategy and derive a contradiction. Note that only condition (d) of this closed game involves ordinals. Conditions (a)–(c) and (e) of this closed game are defined by four sets of finite sequences of *integers*, so we can clearly find an $a \subseteq \omega$ such that these four sets, plus b, are all in $L(a)$. Then $T \in L(b) \subseteq L(a)$, so the closed game $\mathbf{G}'$ is in $L(a)$. By Lemma 4.2, there is a winning strategy τ for Player II such that τ is in $L(a)$. Consider a position

$$p = \langle(\varphi(0), w(0), x(0), y(0)), z(0), \ldots, (\varphi(n), w(n), x(n), y(n)), z(n), (\varphi(n+1), w(n+1))\rangle$$

in the game $\mathbf{G}'$ in which Player I is half-way through his $(n+1)$st move, that is, Player I has played $\varphi(n+1)$ and $w(n+1)$ but has not yet played $x(n+1)$ or $y(n+1)$, and such that p satisfies conditions (a)–(c) (but not necessarily (d) or (e)). For any such position p and any $f \in C(K, Y)$, we say that τ *rejects f at p* if the following three conditions hold:

(i) p is consistent with the strategy τ;

(ii) $f \in U_n^{z(n)}$;

(iii) for every play $(x(n+1), y(n+1))$ for Player I which completes his $(n+1)$st move and which satisfies (a)–(c), τ calls for Player II to respond with a $z(n+1) \in 2$ such that $f \notin U_{n+1}^{z(n+1)}$.

As in Theorem 3.2, for any $f \in I(K, X)$, there exists some position p such that τ rejects f at p. To see this, choose a φ and w such that $\psi(w) = f$ and (w, φ) is a branch through T. If no such position p existed, Player I could play against τ, always playing this fixed φ and w, and playing in such a way that the outcome of the game is f. That means that Player I can win $\mathbf{G}'$ playing against τ. Also as in Theorem 3.2, for any $f, g \in C(K, Y)$, if f and g have disjoint images, then they cannot both be rejected by τ at the same position p. But in this game $\mathbf{G}'$, there are uncountably many positions. For p a position, define

$$R(p) = \{f \in C(K, Y) : \tau \text{ rejects } f \text{ at } p\}.$$

We claim that $\{R(p) : p \text{ is a position}\}$ is countable. Assuming this claim we get a contradiction as in Theorem 3.2.

We say that a position p *has type k* if V_k is the last open set of p; formally,

$$p = \langle(\varphi(0), w(0), x(0), y(0)), z(0), \ldots, \\ (\varphi(n), w(n), x(n), y(n)), z(n), (\varphi(n+1), w(n+1))\rangle$$

and either $z(n) = 0$ and $x(n) = k$ or $z(n) = 1$ and $y(n) = k$. Fix k. It will suffice to show that $\{R(p) : p \text{ is a position of type } k\}$ is countable. For any position p of type k, define $r(p) \subseteq \omega$ as follows:

$r(p) = \{i \in \omega :$ there exists $j \in \omega$ such that (i, j) [or (j, i)] is a play for Player I at p which completes Player I's move and which satisfies (a)–(c), and if Player I plays (i, j) [resp. (j, i)] at p then τ calls for Player II to respond by playing 0 [resp. 1]$\}$.

For p a position of type k, $R(p) = V_k - (\bigcup_{i \in r(p)} V_i)$. So it will suffice to show that $R = \{r(p) : p \text{ is a position of type } k\}$ is countable. Since τ is in $L(a)$ and (a)–(c) are defined by sets in $L(a)$, $r(p)$ can be defined, as above, inside the model $L(a)$, and this definition is absolute. Hence $r(p)$ is in $L(a)$. That is, R is a collection of subsets of ω, and $R \subseteq L(a)$. By $(*)$, there are only countably many subsets of ω in $L(a)$, so R is countable.

§5. *Embeddings in σ-compact spaces.* The assumption $(*)$ used in Section 4 is also a necessary condition for the truth of "for each compact K and each $X \in \mathbf{\Pi}_1^1$, if X contains uncountably many pairwise disjoint copies of K, then X contains a copy of $K \times 2^\omega$":

5.1. LEMMA (Solovay [10, Theorem 1]). *The hypothesis $(*)$ is equivalent to the hypothesis that every uncountable coanalytic space contains a copy of 2^ω.*

Thus, if $(*)$ fails, the lemma gives us for $K=\{p\}$ a coanalytic space containing uncountably many pairwise disjoint copies of K but no copy of $K\times 2^{\omega}$, and we cannot hope for a full extension of van Douwen's theorem to $\mathbf{\Pi}_1^1$ in ZFC. We do however have the following partial extension.

5.2. THEOREM. *Let X be coanalytic, and assume that X contains more than ω_1 pairwise disjoint copies of the compact space K. Then X contains a copy of $K\times 2^{\omega}$.*

Proof. Since X is coanalytic so is $I(K, X)$ by Lemma 3.1. By Luzin and Sierpiński [**5**] we can write $I(K, X)=\bigcup_{\alpha<\omega_1} B_\alpha$ for some family $\{B_\alpha : \alpha<\omega_1\}$ of absolute Borel sets, and some B_α must contain an uncountable subset F of mappings with pairwise disjoint images. Now apply Lemma 2.1.

Theorem 5.2 can be generalized to other classes; for example, using a result of Martin (see Moschovakis [**9**, 8H.13]): if X is $\mathbf{\Pi}_2^1$ (in particular if X is analytic) and X contains more than ω_2 pairwise disjoint copies of K, then X contains a copy of $K\times 2^{\omega}$ if Det $(\mathbf{\Pi}_1^1)$. Note that in Theorem 3.2 we used Det $(\mathbf{\Pi}_2^1)$ for $\mathbf{\Pi}_2^1$-sets X.

We will now show that failure of $(*)$ even precludes an extension of the full van Douwen's theorem to the class of σ-compact spaces.

5.3. THEOREM. *Let K be a compact connected space having more than one point, and containing no proper copies of itself. If $(*)$ does not hold then there exists a σ-compact space X containing uncountably many pairwise disjoint copies of K but no copies of $K\times 2^{\omega}$.*

Proof. Let $\varphi: K\to\omega^{\omega}$ be a bijection such that if A is closed in ω^{ω} then $\varphi^{-1}[A]$ is a $\mathbf{G}_\delta$ in K (in fact, we can take φ a so-called $(1, 1)$-homeomorphism, see Kuratowski [**4**, §37]). Fix an uncountable coanalytic subset E of 2^{ω} containing no Cantor sets. Then $Y=2^{\omega}-E$ is analytic, so there exists a continuous surjection $\psi:\omega^{\omega}\to Y$. Let $j: Y\to 2^{\omega}$ be the embedding, and put $f= j\circ\psi\circ\varphi: K\to 2^{\omega}$. Now if A is closed in 2^{ω}, then $f^{-1}[A]=\varphi^{-1}\psi^{-1}[A\cap Y]$. Since $A\cap Y$ is closed in Y and ψ is continuous, $\psi^{-1}[A\cap Y]$ is closed in ω^{ω} whence $\varphi^{-1}\psi^{-1}[A\cap Y]$ is a $\mathbf{G}_\delta$ in K. By Kuratowski [**4**, §31] the graph G of f is a $\mathbf{G}_\delta$ in $K\times 2^{\omega}$. We claim that $X=(K\times 2^{\omega})-G$ is as required.

Clearly, X is σ-compact. Furthermore, note that since $K\times\{e\}\subseteq X$ for each $e\in E$, we have that X contains uncountably many pairwise disjoint copies of K. Now suppose that X contains a copy of $K\times 2^{\omega}$, and let $i: K\times 2^{\omega}\to X$ be an embedding. Since K is connected, each $i[K\times\{p\}]$ is contained in some $X\cap(K\times\{y_p\})$, and because K does not contain any proper copies of itself, we actually have $i[K\times\{p\}]=K\times\{y_p\}\subset X$. Note that $y_p\notin Y$: indeed, f maps K onto Y, and if $f(x)=y_p$, then $(x, y_p)\in(K\times\{y_p\})-X$. Thus, $y_p\in E$. Also note that if $p\neq q$ then $y_p\neq y_q$ since the sets $i[K\times\{p\})$ are pairwise disjoint. Thus, $C=\{y_p : p\in 2^{\omega}\}$ is uncountable, and since it is the projection of $i[K\times\{2^{\omega}\}]$ onto the second coordinate, it is compact. We conclude that $C \subseteq E$ contains a copy of 2^ω, a contradiction.

5.4. COROLLARY. *If* $(*)$ *does not hold, then there exists a* σ*-compact space containing uncountably many pairwise disjoint copies of the circle* S^1 *but no copies of* $S^1 \times 2^\omega$.

Theorem 5.3 does not give an example for as simple a space as the unit interval; in fact we cannot get a σ-compact example for $[0, 1]$ or any n-cube, as was noted by K. Kunen (use a Baire category argument and Theorem 2.2). However, we can get an absolute $\mathbf{F}_{\sigma\delta}$-example in these cases, as the following theorem shows.

5.5. THEOREM. *Let* K *be a compact connected space having more than one point and containing no nowhere dense copies of itself. If* $(*)$ *does not hold then there exists an absolute* $\mathbf{F}_{\sigma\delta}$*-space* X *containing uncountably many pairwise disjoint copies of* K *but no copies of* $K \times 2^\omega$; *if* K *in fact contains no proper copies of itself, then* X *can be taken to be* σ*-compact.*

Proof. Let $\mathscr{B}$ be a countable basis for K. Then for each $B \in \mathscr{B}$, $\bar{B}$ is uncountable and complete, so there exist bijections $\varphi_B : \bar{B} \to \omega^\omega$ such that if A is closed in ω^ω then $\varphi^{-1}[A]$ is a $\mathbf{G}_\delta$ in $\bar{B}$ (again we use Kuratowski [4, §37]). Fix an uncountable coanalytic subset E of 2^ω containing no Cantor sets, and a continuous surjection $\psi : \omega^\omega \to Y = 2^\omega - E$. Let $j : Y \to 2^\omega$ be the embedding, and put $f_B = j \circ \psi \circ \varphi_B : \bar{B} \to 2^\omega$. As before, the graph G_B of f_B is a $\mathbf{G}_\delta$ in $K \times 2^\omega$ whence $X_B = (K \times 2^\omega) - G_B$ is σ-compact. We claim that $X = \bigcap_{B \in \mathscr{B}} X_B$ is as required.

Clearly, X is an absolute $\mathbf{F}_{\sigma\delta}$. Furthermore, note that since $K \times \{e\} \subseteq X$ for each $e \in E$, we have that X contains uncountably many pairwise disjoint copies of K. Now suppose that X contains a copy of $K \times 2^\omega$, and let $i : K \times 2^\omega \to X$ be an embedding. First note that for each $y \in Y$, $\bar{B} \times \{y\} \not\subseteq X$: indeed, f_B maps $\bar{B}$ onto Y, and if $f_B(x) = y$, then $(x, y) \in (\bar{B} \times \{y\}) - X$. Now since K is connected, each $i[K \times \{p\}]$ is contained in some $X \cap (K \times \{y_p\})$, and because $i[K \times \{p\}]$ is not nowhere dense in $K \times \{y_p\}$ it must contain some $\bar{B} \times \{y_p\}$. By the above remark, $y_p \notin Y$, whence $y_p \in E$. Let $C = \{y_p : p \in 2^\omega\}$. Since C is the projection of $i[K \times \{2^\omega\}]$ onto the second coordinate, it is compact. Also, C is uncountable: the copies $i[K \times \{p\}]$ are pairwise disjoint and have non-empty interior, so for each $e \in E$, the set $\{p \in 2^\omega : y_p = e\}$ is countable. Thus, $C \subseteq E$ contains a copy of 2^ω, a contradiction.

The following further generalization is possible: let K be connected, not containing uncountably many pairwise disjoint copies of itself, and admitting a continuous mapping g onto the unit interval such that no fibre contains a copy of K; then we obtain an example which again is an absolute $\mathbf{F}_{\sigma\delta}$-space. In fact, the example is just $(g \times \mathrm{id})^{-1}[X]$, where X is the example for the unit interval constructed as in the proof of the theorem.

It would be interesting to know more precisely for what compacta K we can build examples as above, and also what descriptive complexity can be achieved. Minimal restrictions on K are that K is not zero-dimensional (as noted in Section 1), and that K does not contain uncountably many pairwise disjoint copies of itself (apply van Douwen's theorem to just one copy of K in X). As far as descriptive complexity of examples is concerned, note that if K is connected and does not contain uncountably many pairwise disjoint copies of itself, and E is an uncountable subset of 2^ω which does not contain any copy of 2^ω, then $X = K \times E$ does not contain a copy of $K \times 2^\omega$; thus, if $(*)$ does not hold we can find a coanalytic X for such K.

We formulate one more corollary.

COROLLARY 5.6. *The following are equivalent.*

(a) *Every σ-compact space X containing uncountably many pairwise disjoint copies of S^1 contains a copy of $S^1 \times 2^\omega$.*

(b) *For every coanalytic space X and every compact space K, if X contains uncountably many pairwise disjoint copies of K then X contains a copy of $K \times 2^\omega$.*

(c) *Every uncountable coanalytic space contains a copy of 2^ω.*

(d) *For any $a \subseteq \omega$ there are only countably many subsets of ω in $L(a)$.*

Proof. The implication from (b) to (a) is trivial, and the equivalence of (c) and (d) is Lemma 5.1. That (b) follows from (d) is Theorem 4.3, and (a) implies (c) is Corollary 5.4.

For the class of analytic spaces there does not seem to be any statement similar to Corollary 5.6; it would be particularly interesting to know the exact strength of part (b) for "analytic".

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