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Proper Pseudocompact Subgroups of Pseudocompact Abelian Groups

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ABSTRACT: We prove among other things that if G is a pseudocompact Abelian topological group such that $|G| > \mathfrak{c}$ or $\omega_1 \leq w(G) \leq \mathfrak{c}$ then G has a proper dense pseudocompact subgroup.

INTRODUCTION

All topological spaces considered here (in particular, all topological groups) are assumed to be completely regular, Hausdorff spaces, i.e., Tychonoff spaces. A space X is *pseudocompact* if every real-valued continuous function on X is bounded.

It has been known for some time that there exist countably compact topological groups G of uncountable weight having no countably compact proper dense subgroup [2, 3.3]. This suggests the question whether every pseudocompact topological group of uncountable weight has a pseudocompact proper dense subgroup. It seems that this question was first considered in Comfort and Robertson [2], where it is shown that the answer is in the affirmative provided that the group under consideration is Abelian and zero-dimensional in the sense that it has a base consisting of open and closed sets. In [3] it was subsequently shown that the answer is in the affirmative if the group G under consideration is Abelian and connected, and satisfies one of the cardinal inequalities $wG \leq \mathfrak{c}$ or $|G| \geq (wG)^{\omega}$. Here wG denotes the weight of G. The basic unsolved problem in [3] is whether every dense pseudocompact subgroup G of $\mathbb{T}^{\mathfrak{c}^+}$ such that $|G| = \mathfrak{c}$ has a proper dense pseudocompact subgroup. Here $\mathbb{T}$ denotes the circle group.

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Several natural variations of the above problem arose while we tried to solve it. For example, whether every topological group has a dense proper subgroup. This is trivially false of course; one "correct" version of this question turned out to be whether every totally bounded Abelian group G such that $\omega \leq wG \leq |G|$ has a proper dense subgroup — see [4] for a discussion of this problem. Here a topological group G is said to be *totally bounded* (by some authors: *precompact*) if for every nonempty open subset $U \subseteq G$, G is covered by a finite number of translates of U. Every pseudocompact group is totally bounded and has a proper dense subgroup [4, Theorem 4.2].

If a topological group G has a proper dense subgroup H then it trivially has two disjoint dense subsets, namely, H and any coset xH with $x \notin H$. So a topological group with a proper dense subgroup is *resolvable*, i.e., contains two disjoint dense subsets. A space that is not resolvable is called *irresolvable*. These concepts are due to Hewitt [8] who proved that every dense in itself locally compact space is resolvable and also presented the first examples of dense in itself irresolvable spaces. So the most naive form of the original question is the question whether every dense in itself topological group is resolvable. Initially this seems to be of no interest because topological groups are very nice spaces and irresolvable spaces are very pathological. So it seems extremely unlikely that there are irresolvable topological groups. There is, however, a striking consistent example of an irresolvable Abelian group due to Malyhin [10]. He constructed a dense in itself topological group G whose topology is *maximal*, i.e., G cannot be given a stronger dense in itself topology. The first and the third authors of the present paper will discuss the question which Abelian groups are resolvable in every dense in itself group topology in [5]. Due to our interest in pseudocompact groups, and because we know that they are resolvable, it is natural to ask whether an example such as Malyhin's can be totally bounded. Our first result shows that it cannot be.

THEOREM 1.1: Every infinite totally bounded Abelian group is resolvable.

See Proof of Theorem 1.1 for details.

A pseudocompact group with a proper pseudocompact subgroup is resolvable in a strong sense, namely, it contains two disjoint *pseudocompact* dense subsets: again consider the subgroup and one of its cosets. So if one wants to solve the original problem, one should at least be able to solve the obvious question of whether every pseudocompact group of uncountable weight has two disjoint pseudocompact dense *subsets*. Our next result shows that in the Abelian case such sets can always be found.

THEOREM 1.2: Every pseudocompact Abelian group of uncountable weight has two complementary dense pseudocompact subsets.

See Proof of Theorem 1.2 for details.

Now back to the original problem: does every pseudocompact group have a proper dense pseudocompact subgroup? Here is our main result.

THEOREM 1.3: Let G be a pseudocompact Abelian group such that $|G| > \mathfrak{c}$ or $\omega_1 \leq wG \leq \mathfrak{c}$. Then G has a proper dense pseudocompact subgroup.

See Proof of Theorem 1.3 for details.

So for Abelian groups the only remaining case is that of groups with cardinality $\mathfrak{c}$ and weight bigger than $\mathfrak{c}$. The basic unsolved problem of [3] remains unsolved. For emphasis, we state it here again.

QUESTION 1.4: Does every dense pseudocompact subgroup G of $\mathbb{T}^{\mathfrak{c}^+}$ such that $|G| = \mathfrak{c}$ have a proper dense pseudocompact subgroup?

It is natural to ask why the cardinal $\mathfrak{c}$ in Theorem 1.3 surfaces, and not for example the cardinal ω_{1081}. This will be explained in the Proof of Theorem 1.3.

Dealing with Abelian groups we find it convenient to use additive notation. The identity element of an Abelian group is denoted by 0. If G is a group with identity element e and $n \geq 0$ then $t_n(G) = \{x \in G\colon (\exists\, m \leq n)\ (x^m = e)\}$. The *order* of an element $x \in G$ is the smallest integer n for which $x^n = e$ if such an integer exists; otherwise the order of x is ∞. So for $n \geq 0$, $t_n(G)$ is the subset of G consisting of all elements with order at most n. Finally, if $n \geq 0$ then $G^{(n)} = \{p \in G : (\exists\, x \in G)(nx = p)\}$. The continuous image of a pseudocompact space is pseudocompact, so if G is pseudocompact then so is $G^{(n)}$ for every $n \geq 0$.

PROOF OF THEOREM 1.1

In order to prove our theorem, we first derive some preliminary results.

LEMMA 2.1: Let G be a topological group. Then G is resolvable if and only if one of its subgroups is resolvable.

Proof: We only need to prove that if G has a resolvable subgroup H then G is resolvable itself. So let A and B be two disjoint dense subsets of the subgroup H of G. There is a subset S of G such that:

(1) $SH = G$,

(2) if $x,y \in S$ are distinct then $xH \cap yH = \varnothing$.

It is clear that SA and SB are disjoint subsets of G. We claim that they are also dense. We need only verify that SA is dense. To this end, let V be a nonempty open subset of G. Without loss of generality we may assume that V is of the form xU, where U is a neighborhood of the identity element e of G and $x \in G$. There exist $s \in S$ and $h \in H$ such that $x = sh$. Then hU intersects H since it contains h. There consequently exists an element $a \in hU \cap A$. Then $sa \in shU \cap SA = xU \cap SA$, as required. □

COROLLARY 2.2: Let G be a topological group. If G has a subgroup that is not closed, then G is resolvable.

Proof: Let H be a subgroup of G that is not closed. Then $\overline{H}$ has a proper dense subgroup and hence is resolvable. Now apply Lemma 2.1. □

LEMMA 2.3: Let G be a dense in itself topological group which is algebraically isomorphic to $\mathbb{Z}$. Then G is resolvable.

Proof: We identify G and $\mathbb{Z}$. The disjoint dense sets are the positive integers P (excluding 0), and the negative integers N, respectively. Assume that N is not dense and fix a nonempty open subset $U \subseteq P \cup \{0\}$. Let u denote the minimum of

U. Then $U_0 = U - u$ is a neighborhood of 0 which misses the negative integers. There is a neighborhood V of 0 such that $V \cup -V \subseteq U_0$. But since U_0 misses the negative integers, this implies that $V = \{0\}$, i.e., G is discrete, a contradiction. This shows that P is dense, and since $N = -P$ it also follows that N is dense. □

LEMMA 2.4: Let G be totally bounded topological group for which there exists a sequence $\langle F_n \rangle_{n<\omega}$ with the following properties:

(1) $\forall\, n$, F_n is a subgroup of G.

(2) $\forall\, n$, F_n is properly contained in F_{n+1}.

Then G is resolvable.

Proof: We will prove that $H = \bigcup_{n<\omega} F_n$ is resolvable. The desired result then follows from Lemma 2.1. Observe that H is totally bounded because G is.

Put $E = \bigcup_{n \text{ even}} F_{n+1} \setminus F_n$ and $O = F_0 \cup \bigcup_{n \text{ odd}} F_{n+1} \setminus F_n$, respectively. We claim that both E and O are dense in H. To this end, suppose that there is a nonempty open subset $U \subseteq H$ that misses E. There is a finite $F \subseteq H$ such that $FU = H$. There is $N \in \mathbb{N}$ such that $F \subseteq F_N$. Without loss of generality assume that N is odd. Now take a point $x \in F_{N+1} \setminus F_N$. There exists $y \in F_N$ such that $x \in yU$, say $x = yu$ for certain $u \in U$. Then $u = y^{-1}x \in F_{N+1} \setminus F_N$, which is a contradiction because $U \cap E = \emptyset$. This proves that E is dense, and similarly it follows that O is dense. □

Proof of Theorem 1.1: Let G be an infinite totally bounded Abelian topological group. First observe that a subgroup of a totally bounded group is totally bounded, and that every infinite totally bounded group is dense in itself. Therefore, in case G contains an isomorphic copy of $\mathbb{Z}$ we are done because of Lemmas 2.3 and 2.1. We may consequently assume that G does not contain an isomorphic copy of $\mathbb{Z}$, i.e., G is a torsion group. For every $n \geq 0$ define

$$F_n = \{x \in G : x^{n!} = e\}.$$

Observe that every F_n is a subgroup of G (here we use that G is Abelian), that $\bigcup_{n<\omega} F_n = G$, and that for every n, $F_n \subseteq F_{n+1}$. We distinguish between two subcases. If there are infinitely many n for which $F_{n+1} \neq F_n$, then we are done because of Lemma 2.4. If this is not the case, then for some n, $F_n = G$, so G is a torsion group of bounded order. But this means that for every finite subset $F \subseteq G$, the subgroup generated by F is again finite (here we again use that G is Abelian). Since G is infinite, it is therefore a triviality to construct a strictly increasing chain of subgroups of G. So we are again in the situation where we can apply Lemma 2.4. □

REMARK 2.6: We do not know whether Theorem 1.1 is also true in the non-Abelian case. But we do know that a counterexample is rather peculiar. To see this, let G be an infinite totally bounded group which is irresolvable. As in the proof of Theorem 1.1 it follows that G is a torsion group. A group H is called *locally finite* if for every finite subset A of H the subgroup of H generated by A is also finite. Clearly, an infinite locally finite group contains a strictly increasing chain of finite subgroups. So Lemma 2.4 implies that G is not locally finite. Since infinite subgroups of irresolvable totally bounded groups are also irresolvable by Lemma 2.1, this implies that G contains an irresolvable (countably infinite) subgroup G_0 which is generated by a finite set. Assume that G_0 is of bounded order. Then so is its Weil completion $\overline{G}_0$. It was announced by Zel'manov at the International Algebra Conference in Algebra honoring A. I. Mal'cev, held in Novosibirsk in 1989, that every compact torsion group of bounded order is locally finite. Since subgroups of locally finite groups are locally finite, this implies that G_0 is locally finite and we know that this is not the case. So G_0 is not of bounded order. In conclusion, the counterexample G (if it exists) must contain a torsion group G_0 of unbounded order which is generated by a finite set. Unfortunately (for us) such groups exist, see [15] for details. (We are grateful to James D. Reid for this reference.)

PROOF OF THEOREM 1.2

A subset A of a topological space X is said to be G_δ-dense in X if A intersects every nonempty G_δ-subset of X. Every G_δ-dense subset of a space is dense, but not conversely.

A dense pseudocompact subspace is necessarily G_δ-dense, but the converse fails. (If D is an uncountable discrete space then D is G_δ-dense in its one point compactification.) But in the realm of topological groups these concepts are equivalent, as the next result makes clear.

THEOREM 3.1: (Comfort and Ross [6]) Let G be a topological group. Then the following statements are equivalent:

(1) G is pseudocompact.
(2) βG can be given the structure of a topological group having G as a subgroup.
(3) G is a G_δ-dense subgroup of some compact group.
(4) If G is a subgroup of a compact group H then G is G_δ-dense in its closure in H.

So if G is a compact group and H is a dense subgroup of G then H is pseudocompact if and only if H is G_δ-dense in G. This particular consequence of Theorem 3.1 was recently generalized as follows.

THEOREM 3.2: (Hernández and Sanchis [7]) A dense subspace A of a compact group G is pseudocompact if and only if A is G_δ-dense in G.

COROLLARY 3.3: Let G be a pseudocompact topological group and let $A \subseteq G$. If A is G_δ-dense in G then A is pseudocompact.

Proof: By Theorem 3.1, βG is a topological group in which G is G_δ-dense. Obviously, A is G_δ-dense in βG. Now apply Theorem 3.2. □

REMARK 3.4: (Added July, 1993.) The referee has pointed out that statements stronger than Theorem 3.2 and Corollary 3.3 were implicit in the literature some years before the appearance of [7]. For example, Ščepin [13, Section 6] has shown that every (locally) compact group is regularly κ-metrizable, hence is perfectly κ-normal [12] in the sense that each of its regular-closed subsets is the zero-set of a continuous, real-valued function. (The same conclusion was established earlier by Ross and Stromberg [11].) And according to Tkačenko [14], every G_δ-dense subspace of a perfectly κ-normal space is C-embedded. (The same conclusion is immediate from earlier work of Blair [1, 1.1 and 5.1].) It is then clear that a space A as in 3.2 and 3.3 is indeed pseudocompact, and for the following strong reason: A is C-embedded in a certain pseudocompact space.

If G is a topological group then $\Lambda(G)$ denotes the collection of all closed, normal, G_δ-subgroups of G.

Let X be a space. The *G_δ-topology* on X is the topology generated by the G_δ-subsets of X. By X^δ we mean X endowed with its G_δ-topology. Observe that $A \subseteq X^\delta$ is dense if and only if $A \subseteq X$ is G_δ-dense. If G is a topological group then so is G^δ, i.e., the algebraic operations on G are also continuous in the G_δ-topology on G. Every G_δ of G containing the identity contains also an element of $\Lambda(G)$ (see [9, (8.7)]). This useful fact will be used several times without explicit reference in the remaining part of this paper.

LEMMA 3.5: Let G be a pseudocompact group of uncountable weight. Then G^δ is dense in itself.

Proof: Suppose that G^δ is not dense in itself, i.e., is discrete. Then $\{e\}$ is open in G^δ and hence $\{e\}$ is a G_δ in G and hence, by pseudocompactness of G and [3, Lemma 2.5], is a G_δ in βG. But βG is a topological group by Theorem 3.1, and hence has countable weight because its identity element is a G_δ. This shows that G has countable weight, a contradiction. □

Proof of Theorem 1.2: Let G be a pseudocompact Abelian group. Then G^δ is an Abelian topological group, which is dense in itself by Lemma 3.5. If G^δ has a subgroup that is not closed, then it is resolvable by Corollary 2.2. So we may assume without loss of generality that every subgroup of G^δ is closed. Let H be the subgroup of G consisting of those elements whose order is a power of 2. Then $H \subseteq G^\delta$ is a subgroup, hence is closed. We distinguish between two subcases. We first assume that H has nonempty interior in G^δ, hence is clopen. Then for some $N \in \Lambda(G)$ we have $N \subseteq H$. Then N as a subgroup of G is a pseudocompact [3, Theorem 2.7(d)] torsion group. Then N is of bounded order by [2, Lemma 7.4], hence is zero-dimensional by [2, Lemma 7.1]. Hence G has a proper dense pseudocompact subgroup by Proposition 4.10 below. So we are done in this case. Assume therefore that H has empty interior in G^δ. Then $\tilde{G}^\delta = G^\delta/H$ is a group having no points of order 2. By [5, Theorem 5.2] every Abelian topological group with only finitely many elements of order 2 is resolvable. So let A and B be complementary dense subsets of $\tilde{G}^\delta$. Since the natural quotient map $\varphi\colon G^\delta \to G^\delta/H$ is open [9, Theorem 5.17], it easily follows that both $\varphi^{-1}[A]$ and $\varphi^{-1}[B]$ are complementary dense subsets of G^δ. Then A and B are complementary G_δ-dense subsets of G, and hence they are pseudocompact by Corollary 3.3.

PROOF OF THEOREM 1.3

In this section we present the proof of our main result.

Tools

Let G be a pseudocompact group. If $N \in \Lambda(G)$ then G/N is a compact metrizable space [2, 6.1], and, therefore, has cardinality at most $\mathfrak{c}$. We conclude from this that if $N \in \Lambda(G)$ then N has at most $\mathfrak{c}$-many cosets. This means, in particular, that if $A \subseteq N$ is a G_δ-dense subgroup in N then the union of at most $\mathfrak{c}$-many cosets of A is G_δ-dense in G. If $B \subseteq G$ has cardinality at most $\mathfrak{c}$ then the subgroup generated by B has cardinality at most $\mathfrak{c}$, so there is even a subgroup E of G of cardinality at most $\mathfrak{c}$ such that $A + E$ is G_δ-dense in G. Combining this observation with [3, Lemma 2.13(b),(c)], we obtain the following key lemma.

LEMMA 4.1:

(a) Let G be a pseudocompact Abelian group such that $G = \bigcup_{n=1}^{\infty} A_n$, with each A_n a subgroup of G. Then there are $N \in \Lambda(G)$ and $n \in \mathbb{N}$ such that $A_n \cap N$ is G_δ-dense in N.

(b) Let G be a pseudocompact Abelian group, let $N \in \Lambda(G)$, and assume that $D \subseteq N$ is G_δ-dense. Then there is a subgroup E of G such that

(1) $|E| \leq \mathfrak{c}$, and

(2) $D + E$ is G_δ-dense in G.

The strategy of the proof of Theorem 1.3 will be roughly speaking to write G as the union of a carefully selected countable subfamily of subgroups, i.e., the A_n's. Having established that, Lemma 4.1 gives us the desired pseudocompact proper dense subgroup, i.e., $(A_n \cap N) + E$. We try to keep $(A_n \cap N) + E$ proper by identifying a "direction" in G of cardinality $\mathfrak{c}^+$ which is "orthogonal" to $A_n \cap N$. Then $\mathfrak{c}$-many cosets of $A_n \cap N$ cannot cover G.

We also will use the method of proof in [2]. It is shown there that every zero-dimensional pseudocompact Abelian group G with $wG > \omega$ has a proper dense pseudocompact subgroup. By applying the method of proof in that paper, the following two results can be derived. Although these results at first glance seem to be rather technical, they are precisely what we need later on.

PROPOSITION 4.2: Let G be a pseudocompact Abelian topological group. Assume that for some $n \in \mathbb{N}$,

$$w(G/\overline{G^{(n)}}) > \omega.$$

Then G has a proper dense pseudocompact subgroup.

Proof: Observe that $H = G/\overline{G^{(n)}}$ is a pseudocompact torsion group (in fact, each element of H has order at most n). By the proof of [2, Theorem 7.3] there is prime number p such that

$$w(H/\overline{H^{(p)}}) > \omega;$$

here, $\overline{H^{(p)}}$ denotes the closure of $H^{(p)}$ in H. Put $H_0 = H/\overline{H^{(p)}}$. Then H_0 is a pseudocompact, Abelian, elementary p-group of uncountable weight. So by [2, Theorem 5.8] it has a proper dense pseudocompact subgroup E. We assume without loss of generality, enlarging E to a maximal proper subgroup of H_0 if necessary, that $|H_0/E| = p$. Let $\varphi\colon H \to H_0$ and $\psi\colon G \to H$ be the natural projections. We set

$$F = (\varphi \circ \psi)^{-1}(E).$$

Then F is a subgroup of G such that $|G/F| = p$. We claim that F is pseudocompact, and dense in G. If F is not dense then from $|G/F| = p$ it follows that F is closed (here we use that p is prime), and hence open. But then $(\varphi \circ \psi)(F) = E$ is open and closed in H_0 because φ and ψ are both quotient maps (these maps are even open). This is a contradiction.

In two steps we will prove that F is pseudocompact. First define $F_0 = \varphi^{-1}(E)$. Note that $F_0 \supseteq \ker(\varphi) = \overline{H^{(p)}}$ and consequently, $F_0/\overline{H^{(p)}}$ and E are topologically isomorphic. So F_0 is pseudocompact by [2, Theorem 6.3] because $\overline{H^{(p)}}$ is pseudocompact. By repeating the same procedure replacing E by F_0, it now easily follows that F is pseudocompact. □

Let G be a pseudocompact Abelian group, let $n \geq 0$ and let $N \in \Lambda(G)$. Then $N^{(n)} \subseteq N = \overline{N}$ and so $\overline{N^{(n)}} \subseteq N$. So it makes sense to consider the quotient group $N/\overline{N^{(n)}}$.

PROPOSITION 4.3: Let G be a pseudocompact Abelian group such that for some $N \in \Lambda(G)$ and some prime number p, $w(N/\overline{N^{(p)}}) > \omega$ (the closure is taken in G). Then G has a proper dense pseudocompact subgroup.

Proof: In this proof $\overline{G}$ denotes the Weil completion of G. In our case $\overline{G} = \beta G$ because G is pseudocompact (Theorem 3.1). If $A \subseteq G$ then $\overline{A}$ will denote the closure of A in $\overline{G}$.

Observe that by [9, Theorem 5.35],

$$H = \overline{G}/\overline{G}^{(p)} = \frac{\overline{G}/\overline{N}^{(p)}}{\overline{G}^{(p)}/\overline{N}^{(p)}}.$$

Also, by the proof of [2, Theorem 7.3],

$$w(\overline{G}/\overline{N}^{(p)}) \geq w(\overline{N}/\overline{N}^{(p)}) > \omega.$$

Notice that $w(\overline{G}^{(p)}/\overline{N}^{(p)}) = \omega$ because $\overline{N} \in \Lambda(\overline{G})$ by [3, Theorem 2.7(c)] and the map $x \mapsto px$ from $\overline{G}$ onto $\overline{G}^{(p)}$ is open. This clearly implies that $w(H) > \omega$ for if $w(H) = \omega$, (1) would imply that $w(\overline{G}/\overline{N}^{(p)}) = \omega$, proving that $\{0\}$ is a G_δ-subset of $\overline{G}/\overline{N}^{(p)}$ which violates (2). Now again apply the proof of [2, Theorem 7.3] to conclude that G has a proper dense pseudocompact subgroup. □

Groups of Cardinality Greater Than $\mathfrak{c}$

The following result is the main new tool in our proof.

PROPOSITION 4.4: Let G be a pseudocompact Abelian topological group. Suppose that there exist subgroups V_n, $n \in \mathbb{N}$, of G such that

(1) $V_1 \subseteq V_2 \subseteq \cdots$.
(2) $\forall\, n,m \in \mathbb{N}\ \exists\, k > m\colon |V_k/\{x \in V_k\colon nx \in V_m\}| > \mathfrak{c}$.

Then G contains a proper dense pseudocompact subgroup.

Proof: Put $V = \bigcup_{n=1}^{\infty} V_n$. Let A be a subgroup of G which is maximal with respect to the property that $A \cap V = \{0\}$. For all n,m put

$$A_n{}^m = \{x \in G\colon nx \in A + V_m\}.$$

Then clearly, every $A_n{}^m$ is a subgroup of G. We first claim that $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_n{}^m = G$. This is easy. First observe that for every m, $A \subseteq A_1{}^m$. Take an arbitrary $x \notin A$. Then by the maximality of A we have $\langle\langle A \cup \{x\}\rangle\rangle \cap V \neq \{0\}$, so there are $n \geq 0$ and $a \in A$ such that

$$nx + a = h \in V \setminus \{0\}.$$

Then since $h \neq 0$ and $A \cap V = \{0\}$ we have $n \neq 0$. Pick m such that $h \in V_m$. Then $x \in A_n{}^m$, as required.

By Lemma 4.1 there exist m,n and a subgroup $E \subseteq G$ with $|E| \leq \mathfrak{c}$ such that the group $\tilde{A} = A_n{}^m + E$ is G_δ-dense in G. We claim that $\tilde{A}$ is proper.

Let $k > m$ be such that $|V_k/\{x \in V_k\colon nx \in V_m\}| > \mathfrak{c}$, and let $\pi\colon V_k \to V_k/\{x \in V_k\colon nx \in V_m\}$ be the natural homomoprhism. We prove that $\tilde{A}$ is proper by proving that $\pi[\tilde{A} \cap V_k]$ is a proper subgroup of $V_k/\{x \in V_k\colon nx \in V_m\}$. To this end, fix $e \in E$ and pick two elements $x,y \in A_n{}^m$ such that both $x + e$ and $y + e$ both belong to V_k. Then $x - y \in V_k$. Also, $nx,ny \in A + V_m$, so $n(x - y) \in (A + V_m) \cap V_k = V_m$. So $x - y$ is in the kernel of π, which implies that

$$\pi(x + e) = \pi(x + (y - x) + e) = \pi(y + e).$$

We conclude that $|\pi[(A_n{}^m + \{e\}) \cap V_k]| \leq 1$, and so, because $e \in E$ was arbitrary, that $|\pi[\tilde{A} \cap V_k]| \leq \mathfrak{c}$. □

COROLLARY 4.5: Let G be a pseudocompact Abelian topological group containing an isomorphic copy of $\oplus_{\alpha<\mathfrak{c}^+} \mathbb{Z}_\alpha$. Then G has a proper dense pseudocompact subgroup.

COROLLARY 4.6: Let G be a pseudocompact Abelian topological group such that $|G/t(G)| > \mathfrak{c}$. Then G has a proper dense pseudocompact subgroup.

Proof: One easily constructs an isomorphic copy of $\oplus_{\alpha<\mathfrak{c}^+} \mathbb{Z}_\alpha$ in G. □

So we now have to deal with groups G for which $|G/t(G)| \leq \mathfrak{c}$.

PROPOSITION 4.7: Let G be a pseudocompact, Abelian group having the following property: there exists $N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}\colon |t_{(nN)^2}(G)/t_{nN}(G)| > \mathfrak{c}$. Then G has a proper dense pseudocompact subgroup.

Proof: Put $\xi_1 = N$ and $\xi_{n+1} = \xi_n^2!$. Then since $N > 1$ we have $n < \xi_n < \xi_{n+1}$ for all n. We first claim that for all $m,n \in \mathbb{N}$, $n \cdot \xi_m \mid \xi_{nm+1}$. Indeed, for all n,m,

$$n \cdot \xi_m < \xi_n \cdot \xi_m \leq \xi_{nm}^2,$$

so $n \cdot \xi_m \mid \xi_{nm+1}$.

For every n put $V_n = t_{\xi_n}(G)$. Observe that $V_1 \subseteq V_2 \subseteq \cdots$ because $\xi_1 | \xi_2 | \cdots$. Take $m,n \in \mathbb{N}$. Then

$$\{x \in V_{nm+2} : nx \in V_m\} = \{x \in V_{nm+2} : n \cdot \xi_m \cdot x = 0\}$$

$$\subseteq V_{nm+1}.$$

Since ξ^2_{nm+1} divides ξ_{nm+2} this implies that

$$|V_{nm+2}/\{x \in V_{nm+2} : nx \in V_m\}| \geq |V_{nm+2}/V_{nm+1}| > \mathfrak{c}.$$

So now apply Proposition 4.4. □

We are now ready to complete the proof of the first part of Theorem 1.3.

THEOREM 4.8: Let G be a pseudocompact, Abelian topological group with $|G| > \mathfrak{c}$. Then G has a proper dense pseudocompact subgroup.

Proof: We may assume by Corollary 4.6 that $|G/t(G)| < \mathfrak{c}$, which implies that $|t(G)| > \mathfrak{c}$ because $|G| > \mathfrak{c}$. There exists $N \in \mathbb{N}$ such that $t_N(G)$ has cardinality greater than $\mathfrak{c}$. Now let n be a multiple of N.

First assume that $|t_{n^2}(G)/t(G)| \leq \mathfrak{c}$. Then there is a subset $K \subseteq t_{n^2}(G)$ of cardinality at most $\mathfrak{c}$ such that $K + t_n(G) = t_{n^2}(G)$. Pick an arbitrary point $g \in G$, and assume that $ng \in t_n(G)$. Then $n^2 g = 0$, so $g \in t_{n^2}(G)$. There consequently exist $k \in K$ and $y \in t_n(G)$ such that $g = k + y$. But then $ng = nk \in nK$. Since $|K| \leq \mathfrak{c}$, we conclude that $|G^{(n)} \cap t_n(G)| \leq \mathfrak{c}$. By Proposition 4.2, we may assume without loss of generality that $\overline{G^{(n)}}$ is a G_δ-subgroup of G. Moreover, $G^{(n)}$ is pseudocompact. As before, find a subgroup E of G of cardinality at most $\mathfrak{c}$ such that $G^{(n)} + E$ is G_δ-dense in G. Now since $|G^{(n)} \cap t_n(G)| \leq \mathfrak{c}$ and $|t_n(G)| > \mathfrak{c}$, it follows that $t_n(G)$ cannot be covered by $\mathfrak{c}$-many cosets of $G^{(n)}$ or fewer. So we conclude that $G^{(n)} + E$ is proper.

We may therefore without loss of generality assume that for every multiple n of N we have $|t_{n^2}(G)/t(G)| > \mathfrak{c}$. So we are in a position to apply Proposition 4.7, to get a proper dense pseudocompact subgroup of G. □

Groups of Weight at Most $\mathfrak{c}$

We now turn our attention to the second part of Theorem 1.3.

THEOREM 4.9: Let G be a pseudocompact Abelian topological group with $\omega_1 \leq w(G) \leq \mathfrak{c}$. Assume, moreover, that $\forall N \in \Lambda(G)$: $|N/tN| \geq \mathfrak{c}$. Then G has a proper dense pseudocompact subgroup.

Proof: This is a direct consequence of the proof of [3, Theorem 4.2]. □

PROPOSITION 4.10: Let G be a pseudocompact Abelian topological group of uncountable weight such that for some $N \in \Lambda(G)$, N is zero-dimensional. Then G has a proper dense pseudocompact subgroup.

Proof: By [3, Theorem 2.7(d),(e)], N is pseudocompact and $w(N) > \omega$. By the proof of [2, Theorem 7.3] there is prime number p such that

$$w(N/\overline{N^{(p)}}) > \omega.$$

Now apply Proposition 4.3. □

We are now ready to present a proof of the second part of Theorem 1.3.

THEOREM 4.11: Let G be a pseudocompact Abelian group with $\omega_1 \leq w(G) \leq \mathfrak{c}$. Then G has a proper dense pseudocompact subgroup.

Proof: By Proposition 4.10, we may assume that no $N \in \Lambda(G)$ is zero-dimensional. We claim that for every $N \in \Lambda(G)$ we have $|N/tN| \geq \mathfrak{c}$. Indeed, fix $N \in \Lambda(G)$. Then since N is Abelian and not zero-dimensional, there is a continuous character $\chi: N \rightarrow \mathbb{T}$ such that $\chi[N]$ is uncountable. Since N is pseudocompact [3, Theorem 2.7(d),(e)], this means that $\chi[N]$ is an uncountable closed subgroup of $\mathbb{T}$, hence is $\mathbb{T}$. But since $t\mathbb{T}$ is countable, and $|\mathbb{T}| = \mathfrak{c}$ we obviously have $|\mathbb{T}/t\mathbb{T}| = \mathfrak{c}$ and in turn that $|N/tN| \geq \mathfrak{c}$. Now apply Theorem 4.9. □

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