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BAIRE 1 FUNCTIONS WHICH ARE NOT COUNTABLE UNIONS OF CONTINUOUS FUNCTIONS

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1. Introduction

A real-valued function f on a space X is *countably continuous* provided that X can be partitioned into countably many sets $E_1, E_2, \dots$ such that for every i, the restriction $f \restriction E_i$ is continuous. Adjan and Novikov [1] constructed (answering a question of Lusin, cf. also Keldyš [6]) an upper semicontinuous function on $[0,1]$ that is not countably continuous (we discuss their construction in Lemma 4.1 and Comment 6.1(B) below). A similar construction was used also by Sierpiński [10] (who did not address Lusin's question directly, but the solution is implicit in his reasoning). We thank the referee for pointing out this fact to us.

Jackson and Mauldin [5] proved recently, using some notions from recursion theory, that Lebesgue measure λ considered on the space of nonempty closed subsets of the unit interval is not countably continuous (being upper semicontinuous). They conjectured [5, Questions 5 and 6] that in the Banach spaces of bounded Baire 1 functions and of bounded derivatives, respectively, the countably continuous functions form meager sets.

In this note we prove these conjectures. We also establish a universal property of the map λ on the space of nonempty closed subsets of the unit interval, which gives in particular a direct proof of the result of Jackson and Mauldin mentioned above.

2. Terminology

As usual, $\mathbf{I}$ denotes the interval $[0,1]$ and Q the infinite product $\mathbf{I}^\infty$. By a *space* we mean a metrizable topological space. If $X = \prod_{n=1}^\infty X_n$ is an infinite

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product of spaces then for every $x \in X$ and $n \in \mathbf{N}$ the n-th coordinate of x is denoted by x_n.

Let X be a compact space. The collection of all nonempty closed subsets of X is denoted by $\mathcal{K}(X)$. It can be topologized as follows. Let d be an arbitrary admissible metric for X. If $A \subseteqq X$ and $\varepsilon > 0$ then $U_\varepsilon(A)$ denotes the open ε-ball of radius ε about A. The formula

$$d_H(A,B) = \inf\left\{\varepsilon \colon A \subseteqq U_\varepsilon(B) \text{ and } B \subseteqq U_\varepsilon(A)\right\}$$

defines a metric on $\mathcal{K}(X)$, the so-called *Hausdorff metric*, and $\mathcal{K}(X)$ endowed with the topology derived from this metric is called the *hyperspace of* X. One can show that the topology of $\mathcal{K}(X)$ is independent of the choice of the admissible metric d. Also, $\mathcal{K}(X)$ is a compact space. For details, see Engelking [4] and [9, §4.7].

Let X and (Y,d) be spaces. For functions $f,g\colon X \to Y$ we define their *distance* $\hat{d}(f,g) \in [0,\infty]$ as follows:

$$\hat{d}(f,g) = \sup\left\{d\big(f(x),g(x)\big) \colon x \in X\right\}.$$

Let X be a space. A function $f\colon X \to \mathbf{R}$ is called *lower (upper) semicontinuous* if for every $r \in \mathbf{R}$ the set $f^{-1}(r,\infty)$ (the set $f^{-1}(-\infty,r)$) is open. It is clear that a function $f\colon X \to \mathbf{R}$ is continuous if and only if it is both lower and upper semicontinuous. We will use the well-known fact that for every lower (upper) semicontinuous function f on X there exists a sequence $\{f_i\}_i$ of continuous real-valued functions on X such that for every $x \in X$ we have $f_i(x) \nearrow f(x)$ ($f_i(x) \searrow f(x)$). We will also use the fact that the functions $\inf\colon Q \to \mathbf{I}$ and $\sup\colon Q \to \mathbf{I}$ defined by

$$\inf(x) = \inf\{x_n \colon n \in \mathbf{N}\}$$

and

$$\sup(x) = \sup\{x_n \colon n \in \mathbf{N}\}$$

are upper semicontinuous and lower semicontinuous, respectively. For details and references concerning these facts, see Engelking [4, pp. 61–62].

We finish this section by establishing the following easy results which are probably well-known.

2.1. THEOREM. *Let $r \in [0,1)$. In addition, let X be a compact space and let $f\colon X \to [0,r]$ be upper semicontinuous. Then there is an embedding $e\colon X \to Q$ such that for each $x \in X$ we have*

$$\inf\big(e(x)\big) = f(x).$$

PROOF. Write $\mathbf{N}$ as the union of two disjoint infinite sets, say E_1 and E_2. Since Q is universal for separable metrizable spaces ([9, Theorem 1.4.18]), there is an embedding $\xi\colon X \to [r,1]^{E_1}$. Since f is upper semicontinuous there is a sequence $\{f_i\}_{i\in E_2}$ of continuous functions from X to $[0,r]$ such that for every $x \in X$ we have $\{f_i(x)\}_{i\in E_2} \searrow f(x)$. Now define $e\colon X \to Q$ by

$$e(x)_i = \begin{cases} \xi(x)_i & (i \in E_1), \\ f_i(x) & (i \in E_2). \end{cases}$$

Then e is clearly as required. □

We conclude that in a sense the pair $(Q, \inf)$ is "universal" for upper semicontinuous functions. Similarly one derives that the pair $(Q, \sup)$ is "universal" for lower semicontinuous functions.

2.2. THEOREM. *Let $r \in (0,1]$. In addition, let X be a compact space and let $f\colon X \to [r,1]$ be lower semicontinuous. Then there is an embedding $e\colon X \to Q$ such that for each $x \in X$ we have*

$$\sup\big(e(x)\big) = f(x).$$

3. A universal property of Lebesgue measure

In this section we formulate and prove that the pair $\big(\mathcal{K}([-1,1]),\lambda\big)$ is "universal" for upper semicontinuous functions. In §6.1 we will present several "explicit" examples of upper semicontinuous functions that are not countably continuous. In view of Theorem 3.1 below this implies that λ is not countably continuous.

3.1. THEOREM. *Let X be a compact space and let $f\colon X \to \mathbf{I}$ be upper semicontinuous. Then there is a topological embedding $e\colon X \to \mathcal{K}\big([-1,1]\big)$ such that for every $x \in X$ we have*

$$\lambda\big(e(x)\big) = f(x).$$

PROOF. We will construct a function $\alpha\colon X \to \mathcal{K}\big([-1,0]\big)$ and a function $\beta\colon X \to \mathcal{K}\big([0,1]\big)$. The desired embedding e will then be defined by the formula $e(x) = \alpha(x) \cup \beta(x)$ $(x \in X)$.

Claim 1. There is an embedding $\alpha\colon X \to \mathcal{K}\big([-1,0]\big)$ such that for every $x \in X$ we have $\lambda\big(\alpha(x)\big) = 0$.

This is easy. Pick points a_n and b_n in $[-1,0]$ such that

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < \cdots \nearrow 0.$$

Let $\hat{Q} = \prod_{n=1}^{\infty}[a_n, b_n]$. Define an embedding $\varphi\colon \hat{Q} \to \mathcal{K}([-1,0])$ by $\varphi(x) = \{0\} \cup \{x_n : n \in \mathbf{N}\}$. Clearly, $\lambda(\varphi(x)) = 0$ for every $x \in \hat{Q}$. The desired result now easily follows because $\hat{Q} \approx Q$ is universal for separable metrizable spaces ([9, Theorem 1.4.18]).

We now come to the interesting part of the proof.

Claim 2. There is a continuous function $\beta\colon X \to \mathcal{K}(\mathbf{I})$ such that for every $x \in X$ we have $\lambda(\beta(x)) = f(x)$.

Since f is upper semicontinuous we may pick a sequence $\{f_i\}_i$ of continuous functions from X to $\mathbf{I}$ such that for every $x \in X$, $f_i(x) \searrow f(x)$. Define $\xi_1\colon X \to \mathcal{K}(\mathbf{I})$ by $\xi_1(x) = [0, f_1(x)]$. Then ξ_1 is clearly a continuous function and has the property that $\lambda(\xi_1(x)) = f_1(x)$ for every $x \in X$. Define $\xi_2\colon X \to \mathcal{K}(\mathbf{I})$ as follows:

$$\xi_2(x) = \left[0, \frac{1}{2}f_2(x)\right] \cup \left[\frac{1}{2}f_1(x), \frac{1}{2}f_1(x) + \frac{1}{2}f_2(x)\right].$$

Then ξ_2 is clearly a continuous function. Observe the following:

(1) If $x \in X$ then the intervals $\left[0, \frac{1}{2}f_2(x)\right]$ and $\left[\frac{1}{2}f_1(x), \frac{1}{2}f_1(x) + \frac{1}{2}f_2(x)\right]$ overlap in at most one point because $f_2(x) \leqq f_1(x)$, so that

$$\lambda(\xi_2(x)) = \frac{1}{2}f_2(x) + \frac{1}{2}f_2(x) = f_2(x).$$

(2) If $x \in X$ then $\xi_2(x) \subseteqq \xi_1(x)$. (Again because $f_2(x) \leqq f_1(x)$.)
(3) If $x \in X$ then

$$d_H(\xi_1(x), \xi_2(x)) = \frac{1}{2}(f_1(x) - f_2(x)) \leqq \frac{1}{2}.$$

(Here d is the euclidean metric on $\mathbf{I}$.)

We now continue in the obvious way and obtain a sequence of continuous functions $\xi_n\colon X \to \mathcal{K}(\mathbf{I})$ having the following properties:

(1) For every $x \in X$, $\xi_1(x) \supseteqq \xi_2(x) \supseteqq \cdots$.
(2) For every $x \in X$ and $n \in \mathbf{N}$, $\lambda(\xi_n(x)) = f_n(x)$.
(3) For every $n \in \mathbf{N}$, $\hat{d}_H(\xi_n, \xi_{n+1}) \leqq 2^{-n}$.

We conclude that the sequence $(\xi_n)_n$ is Cauchy and that the formula

$$\beta(x) = \lim_{n\to\infty} \xi_n(x) = \bigcap_{n=1}^{\infty} \xi_n(x)$$

defines a continuous function from X to $\mathcal{K}(\mathbf{I})$. Also,

$$\lambda(\beta(x)) = \inf\{\lambda(\xi_n(x)) : n \in \mathbf{N}\} = f(x)$$

for every $x \in X$. This completes the construction of β.

As announced, we now define $e: X \to \mathcal{K}([-1,1])$ by

$$e(x) = \alpha(x) \cup \beta(x) \qquad (x \in X).$$

Then e is clearly as required. $\square$

4. Typical bounded Baire 1 functions are not countably continuous

Before we explicitly formulate and prove the result indicated in the title of this section, we shall introduce some terminology which will allow us to apply the original idea of Sierpiński, Adjan and Novikov in the more general situation that we are dealing with.

Let $k \in \mathbf{N}$. $\Sigma(k)$ denotes the collection of all strings $\sigma = (i_1, \dots, i_p)$, where every i_j is a natural number $\leqq k$ and $p \leqq k$; the *length* of σ is p and the empty string which has length 0 is denoted by $\emptyset$. For convenience, put $\Sigma = \bigcup_{k=1}^{\infty} \Sigma(k)$. If $\sigma = (i_1, \dots, i_p) \in \Sigma$ and $i \in \mathbf{N}$ then $\sigma^\frown i$ denotes the string $(i_1, \dots, i_p, i)$.

Let X be a space. Given a compact set $C \subsetneqq X$, we fix a countable basis $B_1(C), B_2(C), \dots$ for the open sets in C with $\lim_{i\to\infty} \operatorname{diam} B_i(C) = 0$.

Let $k \in \mathbf{N}$. A *k-system* $\mathcal{S}(k)$ in X consists of:

(1) a collection of Cantor subsets $\{C(\sigma): \sigma \in \Sigma(k)\}$ of X,

(2) a collection of Cantor subsets $\{D(\sigma): \sigma \in \Sigma(k)\}$ of X,

(3) a sequence $\{\varepsilon(\sigma): \sigma \in \Sigma(k)\}$ of positive numbers,

such that the following conditions are satisfied:

(i) $C(\emptyset) = D(\emptyset)$;

(ii) $\forall \sigma, \sigma^\frown i, \sigma^\frown j \in \Sigma(k)$:

(a) $C(\sigma^\frown i) \subsetneqq D(\sigma^\frown i) \subsetneqq B_i(C(\sigma))$;

(b) $C(\sigma^\frown i)$ has empty interior and $D(\sigma^\frown i)$ is clopen relative to $C(\sigma)$;

(c) if $i \neq j$ then $C(\sigma^\frown i) \cap C(\sigma^\frown j) = \emptyset$.

We say that a $(k+1)$-system $\mathcal{S}(k+1)$ *extends* a k-system $\mathcal{S}(k)$ if the objects in $\mathcal{S}(k+1)$ associated with the strings in $\Sigma(k)$ coincide with the corresponding objects in $\mathcal{S}(k)$.

We say that a function $f: X \to \mathbf{R}$ is *compatible* with a k-system $\mathcal{S}(k)$ in X if for any string $\sigma \in \Sigma(k)$,

$$(*) \qquad \sup\{f(x): x \in D(\sigma) \setminus C(\sigma)\} + \varepsilon(\sigma) < \inf\{f(x): x \in C(\sigma)\}.$$

For such an f we put

$$\eta(f) = \min\Big\{\inf\{f(x): x \in C(\sigma)\} - \varepsilon(\sigma) -$$

$$- \sup\{f(x): x \in D(\sigma) \setminus C(\sigma)\}: \sigma \in \Sigma(k)\Big\}.$$

We call a function $f: X \to \mathbf{R}$ of *Sierpiński–Adjan–Novikov type*, if there exists a sequence of k-systems $\mathcal{S}(1), \mathcal{S}(2), \ldots, \mathcal{S}(k), \ldots$ such that for all $k \in \mathbf{N}$,

(1) $\mathcal{S}(k+1)$ extends $\mathcal{S}(k)$;

(2) f is compatible with $\mathcal{S}(k)$.

Observe that if $Y \subseteqq X$ and $f: X \to \mathbf{R}$ has the property that $f \restriction Y: Y \to \mathbf{R}$ is of Sierpiński–Adjan–Novikov type then so is f.

The following lemma is implicit in Adjan and Novikov [1]. Since their paper is in Russian we include a proof for the convenience of the reader.

4.1. LEMMA. *If $f: X \to \mathbf{R}$ is of Sierpiński–Adjan–Novikov type then it is not countably continuous.*

PROOF. Let us fix a sequence $\mathcal{S}(1), \mathcal{S}(2), \ldots, \mathcal{S}(k), \ldots$ of k-systems compatible with f such that for every k the system $\mathcal{S}(k+1)$ extends $\mathcal{S}(k)$. Define

$$E = \bigcap_{p=1}^{\infty} \bigcup \{C(\sigma): \sigma \text{ has length } p\}.$$

Write E as $E_1 \cup E_2 \cup \cdots$. We claim that for some $p \in \mathbf{N}$ and $\sigma \in \Sigma$ the set

$$(**) \qquad E_p \cap C(\sigma) \text{ is dense in } C(\sigma).$$

Otherwise (using (ii)(a)) we could choose inductively numbers $i_1, i_2, \ldots$ such that for every $p \in \mathbf{N}$, $E_p \cap C(i_1, \ldots, i_p) = \emptyset$. But then the non-empty set $\bigcap_{p=1}^{\infty} C(i_1, \ldots, i_p)$ is contained in $E \setminus \bigcup_{i=1}^{\infty} E_i$, which is a contradiction.

With p and σ as in $(**)$, choose any $x_0 \in E_p \cap C(\sigma)$. By the definition of E there exists $i \in \mathbf{N}$ with $x_0 \in C(\sigma^\frown i)$. By (ii)(b) and $(*)$ we can find a sequence $x_n \in \big(E_p \cap D(\sigma^\frown i)\big) \setminus C(\sigma^\frown i)$ converging to x_0. But then

$$f(x_0) > f(x_n) + \varepsilon(\sigma^\frown i)$$

for all n, demonstrating that $f \restriction E_p$ is not continuous at x_0. $\square$

4.2. REMARK. An inspection of the proof of Lemma 4.1 shows that condition $(*)$ above is much more than we need. It suffices for example if for every $\sigma \in \Sigma(k)$, $k \in \mathbf{N}$, there is a relatively open set $G(\sigma) \subseteqq D(\sigma) \setminus C(\sigma)$ such that $C(\sigma) \subseteqq \overline{G(\sigma)}$ while moreover

$$(*') \qquad \sup\{f(x): x \in G(\sigma)\} + \varepsilon(\sigma) < \inf\{f(x): x \in C(\sigma)\}.$$

By abuse of terminology we call functions satisfying such conditions also of Sierpiński–Adjan–Novikov type. The point is that the precise condition is not so important, as long as it is strong enough for the arguments in the proof of Lemma 4.1 to work. For the time being the definition of Sierpiński–Adjan–Novikov type is the one with the above condition $(*)$. We will warn the reader when it is time for a change.

A function $f: \mathbf{I} \to \mathbf{R}$ is of *first Baire class* if it is the pointwise limit of a sequence of continuous functions. The set $B_1(\mathbf{I})$ consists of all *bounded* functions of the first Baire class and is endowed with the supremum norm. It is well-known that with this norm, $B_1(\mathbf{I})$ is a (non-separable) Banach space.

4.3. THEOREM. *$B_1(\mathbf{I})$ contains a dense G_δ-subset consisting of functions of Sierpiński–Adjan–Novikov type.*

Consequently, by Lemma 4.1 we obtain the following corollary.

4.4. COROLLARY. *The set of all countably continuous functions in $B_1(\mathbf{I})$ is meager.*

Before presenting the proof of Theorem 4.3 we derive the following preliminary results.

4.5. LEMMA. *Let $\mathcal{S}(k)$ be a k-system. Then the set*

$$\{f \in B_1(\mathbf{I}): f \text{ is compatible with } \mathcal{S}(k)\}$$

is open in $B_1(\mathbf{I})$.

PROOF. Let $\mathcal{S}(k) = \langle C(\sigma), D(\sigma), \varepsilon(\sigma)\rangle_{\sigma\in\Sigma(k)}$. In addition, let f be compatible with $\mathcal{S}(k)$. It is easy to verify that if $g \in B_1(\mathbf{I})$ and $\|f-g\| < \eta(f)/3$ then g is compatible with $\mathcal{S}(k)$. □

4.6. LEMMA. *Let $K \subseteqq \mathbf{I}$ be a Cantor set, $u \in B_1(\mathbf{I})$ and $C_1 \subseteqq K$ a Cantor set with empty interior in K. Then if U is a nonempty open subset of K and $\delta > 0$ then there are a Cantor set $C \subseteqq U \setminus C_1$ having empty interior in K, a clopen neigborhood D of C in K and a nonempty open subset $W \subseteqq \subseteqq \{v \in B_1(\mathbf{I}): \|u-v\| < \delta\}$ such that for all $w \in W$:*

$$\sup\{w(x): x \in D \setminus C\} + \frac{\delta}{5} < \inf\{w(x): x \in C\}.$$

PROOF. Since u is of the first Baire class, there is a point $p \in V = U \setminus C_1$ at which $u \restriction K$ is continuous ([2, Theorem 8.3.1]). Let $D \subseteqq V$ be a clopen neighborhood of p in K such that

$$|u(x) - u(p)| < \frac{\delta}{5} \qquad (x \in D).$$

Let $C \subseteqq D$ be a Cantor set containing p and having empty interior in K. Define $v \in B_1(\mathbf{I})$ as follows:

$$v(x) = \begin{cases} u(x) & (x \notin C), \\ u(p) + \frac{3}{5}\delta & (x \in C). \end{cases}$$

Clearly $v \in B_1(\mathbf{I})$ and $\|u - v\| < \delta$. Also,

$$\inf \{v(x) : x \in C\} - \sup \{v(x) : x \in D \setminus C\} =$$

$$= u(p) + \frac{3}{5}\delta - \sup \{v(x) : x \in D \setminus C\} \geqq \frac{2}{5}\delta.$$

So if W is a sufficiently small neighborhood of v then for every $w \in W$, $\inf \{w(x) : x \in C\} - \sup \{w(x) : x \in D \setminus C\} > \frac{\delta}{5}$. □

By a repeated application of Lemma 4.6 one obtains:

4.7. COROLLARY. *Let $f \in B_1(\mathbf{I})$ be compatible with the k-system $\mathcal{S}(k)$. Then for any $\alpha > 0$ one can extend $\mathcal{S}(k)$ to a $k+1$-system $\mathcal{S}(k+1)$ and one can find a function $g \in B_1(\mathbf{I})$ in the α-ball about f such that g is compatible with $\mathcal{S}(k+1)$.*

We are now in a position to present the proof of Theorem 4.3.

4.8. PROOF OF THEOREM 4.3. Let $\mathcal{U}_1$ be a family consisting of pairwise disjoint nonempty open subsets of $B_1(\mathbf{I})$ such that

(1) $\forall U \in \mathcal{U}_1$: $\operatorname{diam}(U) < 2^{-1}$,

(2) $\bigcup \mathcal{U}_1$ is dense in $B_1(\mathbf{I})$.

For every $U \in \mathcal{U}_1$ pick an arbitrary element $f_U^1 \in U$. Then every f_U^1 is compatible with the 0-system. So by applying Corollary 4.7 we find for every $U \in \mathcal{U}_1$ a 1-system $\mathcal{S}_U^1$ and a function $g_U^1 \in U$ compatible with $\mathcal{S}_U^1$. By Lemma 4.5, for every $U \in \mathcal{U}_1$ we may pick an open neighborhood $V_U \subseteqq U$ of g_U^1 such that every function in V_U is compatible with $\mathcal{S}_U^1$. Without loss of generality we may assume that every V_U has diameter less than 2^{-2}. For every $U \in \mathcal{U}$ enlarge $\{V_U\}$ to a pairwise disjoint family $\mathcal{V}_U$ consisting of nonempty open subsets of U of diameter less than 2^{-2} and dense union. Let $\mathcal{U}_2$ denote the collection $\bigcup_{U \in \mathcal{U}_1} \mathcal{V}_U$. Observe that there are two types of sets in $\mathcal{U}_2$. Now we repeat the same procedure. The sets in $\mathcal{U}_2$ that are "compatible" with a 1-system are being replaced by smaller sets that are "compatible" with a 2-system that extends the 1-system. Next, the sets that are "compatible" with the 0-system are being replaced by smaller sets that are "compatible" with a 1-system. Finally, we add sets that are compatible with the 0-system in order to get a family $\mathcal{U}_3$ with dense union. Then we again repeat the same procedure but now at three levels. At the end of the construction each function in the dense G_δ-set $\bigcap_{n=1}^{\infty} \bigcup \mathcal{U}_n$ is of Sierpiński–Adjan–Novikov type. □

5. Typical bounded derivatives are not countably continuous

The approach in this note provides also an answer to another question in Jackson and Mauldin [5].

5.1. THEOREM. *In the Banach space of bounded derivatives on* **I** *the countably continuous functions form a meager set.*

Let us indicate which modifications in the proof of Theorem 4.3 are necessary to obtain this result. Our terminology and facts from differentiation theory are all taken from Bruckner [3].

(A) We use here the definition of Sierpiński–Adjan–Novikov function with condition $(*)$ in §4 replaced by condition $(*')$ in Remark 4.2.
(B) We construct the Cantor sets $C(\sigma)$ in such a way that additionally each nonempty relatively open set in $C(\sigma)$ has positive Lebesgue measure.
(C) Because of (B), we can define the subsequent Cantor sets $C(\sigma)$ and the relatively open sets $G(\sigma)$ so that there exists an approximately continuous function $h\colon \mathbf{I} \to \mathbf{I}$ (hence a derivative by [3, Ch. II, Theorem 5.5(a)]) such that $h(x) \geqq \frac{3}{5}$ on $C(\sigma)$ and $h(x) = 0$ on $G(\sigma)$. The jump in condition $(*')$ can then be created by using the function $u + \delta \cdot h$ instead of v, where δ and v are as in Lemma 4.6.

Only (C) needs some additional justification. To this end, let C be a Cantor set in **I** such that nonempty relatively open sets in C have positive Lebesgue measure. Let $K \subseteqq C$ be a Cantor set of positive Lebesgue measure such that $G = C \setminus K$ is dense in C, and let E be the set of Lebesgue density one points of K ([3, Ch. II, Theorem 5.1]). Removing a set of measure 0 if necessary, we can assume that E is as in [3, Ch. II, Theorem 6.5]; let $f\colon \mathbf{I} \to \mathbf{I}$ be the function described in that theorem. For every n, let $E_n = \{x \in E\colon f(x) \geqq \frac{1}{n}\}$ and pick n such that E_n has positive Lebesgue measure. There is a Cantor set $L \subseteqq E_n$ having the property that all its nonempty relatively open subsets have positive measure. Then $\frac{1}{n} \leqq f(x) \leqq 1$ on L and $f(x) = 0$ on G. Finally, set $h = \ell \circ f$, where $\ell\colon I \to \mathbf{I}$ is a continuous function with $\ell(0) = 0$ and $\ell[\frac{1}{n}, 1] \subseteqq [\frac{3}{5}, 1]$. Then h is approximately continuous by [3, Ch. II, Theorem 5.4].

6. Comments

6.1. *Explicit examples of functions that are not countably continuous.* We present here two explicit examples of first Baire class functions that are not countably continuous. Each, combined with Theorem 3.1, (re)proves the result of Jackson and Mauldin quoted in the introduction.

(A) Let $C \subseteqq \mathbf{I}$ be the Cantor set. Since C is canonically homeomorphic to $\{0,1\}^\infty$ it follows that C is canonically homeomorphic to C^∞. The continuous function $\xi: C \times C \to [-1, 1]$ defined by $\xi(x, y) = x - y$ is easily seen to be surjective. Consequently, there is an explicit map from C^2 onto $[-1, 1]$. By taking the infinite product of this map, we conclude that there is an explicit map from C^∞ onto Q. Consequently, there is an explicit map from C onto Q, say f. (This is well-known of course.)

Define the functions $\ell, u: Q \to \mathbf{I}$ by

$$\ell(x) = \min f^{-1}(x) \qquad (x \in Q)$$

and

$$u(x) = \max f^{-1}(x) \qquad (x \in Q),$$

respectively.

6.1.1. THEOREM. *ℓ is lower semicontinuous and u is upper semicontinuous. Moreover, ℓ and u are not countably continuous.*

PROOF. We will prove that ℓ is lower semicontinuous. The proof that u is upper semicontinuous is similar and is left to the reader. To this end, let $r \in \mathbf{R}$ and $x \in \ell^{-1}(r, \infty)$. Then $\ell(x) > r$ and so $f^{-1}(x) \subseteqq (r, \infty)$. By compactness of $\mathbf{I}$ we have that the function f is closed. Consequently, there exists a neighborhood V of x in Q such that $f^{-1}[V] \subseteqq (r, \infty)$. Now for every $y \in V$ we have $\ell(y) > r$ which proves that $V \subseteqq \ell^{-1}(r, \infty)$. We conclude that $\ell^{-1}(r, \infty)$ is open.

We will next prove that ℓ is not countably continuous. The proof that u is not countably continuous is similar and is left to the reader. To this end, assume that $Q = E_1 \cup E_2 \cup \cdots$. Since Q is not the union of countably many zero-dimensional subspaces ([9, Corollary 4.8.5]) and every finite-dimensional separable metrizable space is the union of finitely many zero-dimensional subspaces ([9, Corollary 4.4.8]), it follows that for some i, $\dim E_i = \infty$. We claim that $\ell \restriction E_i$ is not continuous. Observe that the composition

$$f \circ \ell \restriction E_i$$

is the identity on E_i and that f is continuous. But then if $\ell \restriction E_i$ were continuous this would imply that $\ell \restriction E_i: E_i \to \ell[E_i]$ is a topological homeomorphism which is impossible because E_i is infinite-dimensional and every nonempty subspace of C is zero-dimensional. □

6.1.2. COROLLARY. *$\sup: Q \to \mathbf{I}$ is lower semicontinuous but not countably continuous. In addition, $\inf: Q \to \mathbf{I}$ is upper semicontinuous but not countably continuous.*

PROOF. The function $\frac{1}{2}\ell + \frac{1}{2}: Q \to [\frac{1}{2}, 1]$ is lower semicontinuous but not countably continuous (Theorem 6.1.1). The result for sup now easily follows from Theorem 2.2. The result for inf can be proved analogously. □

6.1.3. QUESTION. Is there a homeomorphism $\alpha: \mathcal{K}(\mathbf{I}) \to Q$ such that for every $A \in \mathcal{K}(\mathbf{I})$ we have

$$\lambda(A) = \inf\left(\alpha(A)\right),$$

i.e., are the pairs $\left(\mathcal{K}(\mathbf{I}), \lambda\right)$ and $(Q, \inf)$ topologically equivalent?

(B) The second example is a reformulation of the original construction of Adjan and Novikov. Again, let $C \subseteqq \mathbf{I}$ be the Cantor set and let $D = \{d_1, d_2, \dots\}$ be a countable dense set in C. Define $\phi: C \to \mathbf{I}$ by the formula

$$\phi(x) = \begin{cases} 0 & (x \in C \setminus D), \\ \frac{1}{i} & (x = d_i) \end{cases}$$

and let $f: C \times C \times \cdots \to \mathbf{I}$ be defined by

$$f(x_1, x_2, \dots) = \sum_{i=1}^{\infty} 2^{-i} \phi(x_1) \cdots \phi(x_i).$$

The reasoning of Adjan and Novikov that was reproduced by us in the proof of Lemma 4.1 shows that f is not countably continuous. It is easily seen that f is upper semicontinuous.

Notice that one can identify $C \times C \times C \times \cdots$ with C in $\mathbf{I}$ which, as can easily be seen, provides a corresponding example defined on $\mathbf{I}$.

6.2. *Zero-dimensional spaces.* In the special case of zero-dimensional spaces it is possible to derive Theorem 6.1.1 from well-known selection theorems. To see this, let X be a compact zero-dimensional space and let $f: X \to \mathbf{I}$ be upper semicontinuous. Put

$$G = \left\{(x, A) \in X \times \mathcal{K}(\mathbf{I}) \colon f(x) = \lambda(A)\right\}.$$

Then G is a G_δ-subset of $X \times \mathcal{K}(\mathbf{I})$, and hence is completely metrizable. From the upper semicontinuity of the function f one readily concludes that the multifunction F which assigns to each $x \in X$ the vertical section of G at x is lower semicontinuous. There exists a continuous selection β for F by a selection theorem of Kuratowski and Ryll-Nardzewski [7] or Michael [8]. This function is what was needed in Claim 2 of the proof of Theorem 3.1.

Let us finally notice that the second function considered in §6.1 is defined on a zero-dimensional compact space.

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