Q & A in General Topology, Vol. 13 (1995)

Remark on products of 1-dimensional compacta

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Abstract: Let $n \geq 2$. We present two elementary proofs of Borsuk's Theorem that the n-sphere S^n cannot be embedded in the product of at most n 1-dimensional spaces.

AMS Subject Classification: 54C25, 54F45.

Key words & phrases: Embedding, n-sphere, product of 1-dimensional spaces.

Karol Borsuk [1] proved that for $n \geq 2$ the n-sphere S^n can not be embedded in a product $X_1 \times \ldots \times X_n$ of 1-dimensional compacta.

Borsuk used some homological notions in his proof. The aim of this note is to provide two different proofs of his theorem in the realm of point-set topology.

Recall that if (X, d) is a metric space and $\varepsilon > 0$ then a function $f: X \to Y$ is called an ε-map provided that for every $y \in Y$ the fiber $f^{-1}(y)$ has diameter less than ε.

The first proof. Let S^n be the unit sphere in the $(n+1)$-euclidean space, and let B^{n+1} be the ball bounded by S^n, $n \geq 2$.

Our aim is to prove a stronger result than the one of Borsuk. We will in fact show that for $n \geq 2$ there is no 1-map $f = (f_1, \ldots, f_n): S^n \longrightarrow X_1 \times \ldots \times X_n$, the X_i being 1-dimensional compacta.

Striving for a contradiction, assume that f is such a map. There is $\varepsilon > 0$ such that $\operatorname{diam} f^{-1}(A) < 1$ for all A with diameter $< \varepsilon$ (the metric in the product is the maximum of the coordinate metrics). For every $i \leq n$, let $u_i: X_i \longrightarrow L_i$ be an ε-map into a graph L_i (cf. [2, Theorem V.10]) and let $g = (u_1 \circ f_1, \ldots, u_n \circ f_n): S^n \longrightarrow L_1 \times \ldots \times L_n$. Then g is a 1-map and so $T = g(S^n)$ can be covered by nonempty open sets $V_1, \ldots, V_m$ with $\operatorname{diam} g^{-1}(V_i) < 1$. Let $\varphi_1, \ldots, \varphi_m$ be the partition of unity subordinated to this cover, $c_i \in g^{-1}(V_i)$, and let $h: T \longrightarrow B^{n+1}$ be defined by $h(x) = \sum_i \varphi_i(x) \cdot c_i$. The map $h \circ g: S^n \longrightarrow B^{n+1}$ has the property that for every $x \in S^n$, $\| x - h \circ g(x) \| < 1$. Hence by compactness of S^n there exists $\gamma < 1$ such that for every $x \in S^n$, $\| x - h \circ g(x) \| \leq \gamma$. So for the projection from the center $r: B^{n+1} \setminus \{0\} \longrightarrow S^n$, the composition $k = r \circ h \circ g: S^n \longrightarrow S^n$ satisfies $\| x - k(x) \| \leq 1 + \gamma < 2$ for every $x \in S^n$. Therefore k is homotopic to the identity, and there is no extension of k to a map $B^{n+1} \longrightarrow S^n$.

On the other hand, $\dim(L_1 \times \ldots \times L_n) \leq n$, hence $r \circ h: T \longrightarrow S^n$ can be extended over $L_1 \times \ldots \times L_n$, cf. [2, Theorem VI.4], and S^n being simply connected, g can be extended over B^{n+1} (more specifically, each $u_i \circ f_i: S^n \longrightarrow L_i$ can be lifted to the universal cover of L_i, which is a tree, cf. [3, VI.7], and then extended over B^{n+1}). This provides the forbidden extension for k, a contradiction.

The second proof. Assume that $h = (h_1, h_2): S^n \longrightarrow X \times Y$ is an embedding, where X, Y are compacta and $\dim X \leq 1$. We shall show that $\dim Y \geq n$.

Let A be a 0-dimensional compactum in X separating $h_1(S^n)$ (if $h_1(S^n)$ is a singleton, there is nothing to prove). Then $h_1^{-1}(A)$ separates S^n, and there exists a component C of $h_1^{-1}(A)$ which also separates S^n, cf. [2, Theorem VI.11]. By [2, Theorem VI.12] there is a map $f: C \longrightarrow S^{n-1}$ which can not be extended over S^n. Since $h_1(C)$ is a connected subset of A, hence a singleton, h_2 embeds C onto a compactum $D \subset Y$, and let $u = (h_2 \mid C)^{-1}: D \longrightarrow C$ be the inverse map. Because the map $f = (f \circ u) \circ (h_2 \mid C): C \longrightarrow S^{n-1}$ does not extend over S^n, there is no extension of the map $f \circ u: D \longrightarrow S^{n-1}$ over Y. But this implies $\dim Y \geq n$, cf. [2, Theorem VI.4].

References

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Received October 3, 1994