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Is $\omega^* - \{u\}$ Absolutely Countably Compact?

JAN VAN MILL[a] AND JERRY E. VAUGHAN[b]

[a]*Department of Mathematics*
Vrije Universiteit Amsterdam
Amsterdam, The Netherlands
and
[b]*Department of Mathematical Sciences*
University of North Carolina at Greensboro
Greensboro, North Carolina 27412

ABSTRACT: We construct an ultrafilter $u \in \omega^* = \beta\omega - \omega$ such that the subspace $\omega^* - \{u\}$ is not absolutely countably compact, and we show that under the continuum hypothesis, for every $u \in \beta\omega - \omega$, the subspace $\omega^* - \{u\}$ is not absolutely countably compact.

INTRODUCTION

We consider the following concept.

DEFINITION 1.1: (M.V. Matveev [3]) A space X is called *absolutely countably compact (acc)* provided for every open cover $\mathcal{U}$ of X and every dense $D \subseteq X$, there exists a finite set $F \subseteq D$ such that

$$\mathrm{St}(F, \mathcal{U}) = \bigcup\{U \in \mathcal{U}: U \cap F \neq \varnothing\} = X.$$

Matveev proved (among other things) that

$$\text{compact} \Rightarrow \text{acc} \Rightarrow \text{countably compact}$$

and that neither arrow can be reversed. It is well-known that removing one point from ω^*, the remainder of the Čech-Stone compactification of the integers, results in a countably compact subspace. Thus it is natural to ask the question: Are the spaces $\omega^* - \{u\}$ acc for all $u \in \omega^*$? In this paper we prove the following two results:

THEOREM 1.2: There exists $u \in \omega^*$ such that the subspace $\omega^* - \{u\}$ is not acc.

THEOREM 1.3: [CH] For every $u \in \omega^*$, the subspace $\omega^* - \{u\}$ is not acc.

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Still open is the question of whether the statement in Theorem 1.3 is a theorem of ZFC.

2. SOME LEMMAS

We begin with deriving a few general lemmas that will be important later. Let X be a space with dense subset D. We say that a closed subset $T \subseteq X$ *avoids* D if there is a family $\mathcal{U}$ of open subsets of X such that the following conditions are satisfied:

(1) $\bigcap \mathcal{U} = T$,
(2) $|\mathcal{U}| = |D|$, and
(3) for every $d \in D$ we have $|\{U \in \mathcal{U}: d \in U\}| < |\mathcal{U}|$.

Observe that if $T \subseteq X$ avoids D, then $T \cap D = \emptyset$.

Let κ be an infinite cardinal. A subset P of a space X is called a P_κ-*set* if the intersection of fewer than κ neighborhoods of P is again a neighborhood of P. A P-*set* is a P_{ω_1}-set and a *P-point* is a P-set singleton. We omit the simple proof of the following lemma.

LEMMA 2.1: Suppose that X is a compact space with weight κ. If $D \subseteq X$ is a dense subset of cardinality κ and if $T \subseteq X - D$ is a closed P_κ-set then T avoids D.

We now formulate and prove our main tool for recognizing spaces that are not acc. Recall that if X is a space and if $p \in X$ then $\chi(p, X)$ is the *character* of p in X, i.e., the smallest cardinality of a neighborhood base of p.

LEMMA 2.2: Let X be a compact T_2-space. If $D \subseteq X$ is dense, $T \subseteq X$ avoids D and $p \in T$ is such that $\chi(p, T) = |D|$, then $X - \{p\}$ is not acc.

Proof: Let $\mathcal{U}$ be a family of open neighborhoods of T that witnesses the fact that T avoids D. Let $\mathcal{V}$ be a neighborhood base for p with $|\mathcal{V}| = |D|$. List $\mathcal{V}$ as $\mathcal{V} = \{V_U: U \in \mathcal{U}\}$ and put

$$\mathcal{W} = \{X - T\} \cup \{U - \overline{V}_U: U \in \mathcal{U}\}.$$

Then $\mathcal{W}$ is clearly an open cover of $X - \{p\}$. Pick an arbitrary finite $F \subseteq D$. The family

$$\mathcal{U}_F = \{U \in \mathcal{U} : F \cap U \neq \emptyset\}$$

has cardinality less than $|D|$. Pick an arbitrary point

$$x \in \bigcap \{V_U \cap T: U \in \mathcal{U}_F\} - \{p\}.$$

Observe that such a point exists because T is compact and character and pseudocharacter agree in compact spaces. We claim that $x \notin \mathrm{St}(F, \mathcal{W})$. To this end, pick an arbitrary element $W \in \mathcal{W}$ that intersects F. Since $x \in T$, we may clearly assume that W is of the form $U - \overline{V}_U$ for certain $U \in \mathcal{U}$. Then $U \in \mathcal{U}_F$ since W meets F. But since $x \in V_U$ we have $x \notin W = U - \overline{V}_U$. ❑

Let X be compact, $D \subseteq X$ be dense and $p \in X - D$. Lemma 2.2 suggests the natural question of whether $X - \{p\}$ is not acc provided that $\{p\}$ avoids D. But this is not true. Because countably compact spaces of countable tightness are acc, Matveev [3, Theorem 1.8], it follows that the ordinal space ω_1 is acc. (Alternatively, use the Pressing Down Lemma.) Now simply observe that by Lemma 2.1, $\{\omega_1\}$ avoids ω_1 in the compact space $\omega_1 + 1$.

DEFINITION 2.3: A continuous function $f: X \to Y$ of X onto Y is called *irreducible* provided $f(A) \neq Y$ for every proper closed subset $A \subseteq X$.

The next lemma is well known (see [1, Exercise 3.1C(a)]).

LEMMA 2.4: If $f: X \to Y$ of X onto Y is continuous and X is compact, then there exists a closed set $X_0 \subseteq X$ such that $f(X_0) = Y$ and $f \restriction X_0 : X_0 \to Y$ is irreducible.

Recall that the *π-character* of a point x in a space X (denoted $\pi\chi(x, X)$) is the smallest cardinality of a family $\mathcal{U}$ of open subset of X such that every neighborhood of x contains a member of $\mathcal{U}$.

The next lemma is also well known; see, e.g., Juhász [2, p.64].

LEMMA 2.5: If $f: X \to Y$ is irreducible, and X is compact, then for all $x \in X$, $\pi\chi(x, X) \geq \pi\chi(f(x), Y)$.

DEFINITION 2.6: An indexed family $\{A^i_j : i \in I, j \in J\}$ of clopen subsets of ω^* is called a *J by I independent matrix* if:

(1) the rows of the matrix are pairwise disjoint, i.e., for all distinct $j_0, j_1 \in J$ and $i \in I$ we have that $A^i_{j_0} \cap A^i_{j_1} = \emptyset$,

(2) if F is a finite subset of I and $f \in J^F$ then

$$\bigcap\{A^i_{f(i)} : i \in F\} \neq \emptyset.$$

K. Kunen proved that there exists a $\mathfrak{c}$ by $\mathfrak{c}$ independent matrix of clopen subsets of ω^* (see [4, Lemma 3.3.2]).

PROOF OF THEOREM 1.2

Let D be a dense subset of ω^* having cardinality $\mathfrak{c}$, and put $D = \{d_\alpha : \alpha < \mathfrak{c}\}$. Let $\{A^\alpha_\beta : \alpha, \beta \in \mathfrak{c}\}$ be a $\mathfrak{c}$ by $\mathfrak{c}$ independent matrix of clopen subsets of ω^*. For each row $\alpha < \mathfrak{c}$, pick two sets $A^\alpha_{\beta_0}, A^\alpha_{\beta_1}$, so that

$$(A^\alpha_{\beta_0} \cup A^\alpha_{\beta_1}) \cap \{d_\beta : \beta < \alpha\} = \emptyset,$$

and define $B^\alpha_0 = A^\alpha_{\beta_0}$ and $B^\alpha_1 = A^\alpha_{\beta_1}$. Thus $\{B^\alpha_i : \alpha \in \mathfrak{c}, i \in 2\}$ is a $\mathfrak{c}$ by 2 independent matrix. Let

$$T = \bigcap\{(B^\alpha_0 \cup B^\alpha_1) : \alpha < \mathfrak{c}\}.$$

Then T avoids D. By compactness, for every $x \in 2^\mathfrak{c}$, we have $\cap\{B^\alpha_{x(\alpha)} : \alpha < \mathfrak{c}\} \neq \emptyset$, hence there is a natural mapping $f: T \to 2^\mathfrak{c}$ which is easily seen to be continuous and onto. By Lemma 2.4, there exists $S \subseteq T$ such that $f \restriction S : T \to 2^\mathfrak{c}$ is onto and irreducible. By Lemma 2.5 every point in S has character $\mathfrak{c}$ in S and hence in

T. An application of Lemma 2.2 now shows that for $u \in S$ we have $\omega^* - \{u\}$ is not acc.

PROOF OF THEOREM 1.3

Assume CH, and let $u \in \omega^*$.

We first prove the theorem in the special case that u is a P-point. Since all P-points in ω^* are topologically equivalent [5, p.171], and since the density of ω^* is ω_1, by Lemma 2.2 it suffices to construct a P-point $p \in \omega^*$ which is a nonisolated point in some nowhere dense closed P-set $P \subseteq \omega^*$. By [4, Lemma 1.4.3] there is a nowhere dense closed P-set P in ω^* which is homeomorphic to ω^*. We can therefore let p be any P-point of P.

We now use the special case to prove the general case. By [5, p.79], it follows that we can write $\omega^* - \{u\}$ as the disjoint union of two nonempty open sets U and V each having u in their closure. Since ω^* has no (ω, ω)-gaps, we may without loss of generality assume that u is a P-point in $U \cup \{u\}$. By Parovičenko's characterization of ω^* [4, Corollary 1.2.4], it easily follows that $U \cup \{u\}$ is homeomorphic to ω^*. By the previous case it now follows that U is not acc. But then clearly $\omega^* - \{u\} = U \cup V$ is not acc as well.

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