Is $\omega^* - \{u\}$ Absolutely Countably Compact?

JAN VAN MILL$^a$ AND JERRY E. VAUGHAN$^b$

$^a$Department of Mathematics  
Vrije Universiteit Amsterdam  
Amsterdam, The Netherlands  
and  
$^b$Department of Mathematical Sciences  
University of North Carolina at Greensboro  
Greensboro, North Carolina 27412

ABSTRACT: We construct an ultrafilter $u \in \omega^* = \beta \omega - \omega$ such that the subspace $\omega^* - \{u\}$ is not absolutely countably compact, and we show that under the continuum hypothesis, for every $u \in \beta \omega - \omega$, the subspace $\omega^* - \{u\}$ is not absolutely countably compact.

INTRODUCTION

We consider the following concept.

DEFINITION 1.1: (M.V. Matveev [3]) A space $X$ is called absolutely countably compact (acc) provided for every open cover $\mathcal{U}$ of $X$ and every dense $D \subseteq X$, there exists a finite set $F \subseteq D$ such that

$$\operatorname{St}(F, \mathcal{U}) = \bigcup \{ U \in \mathcal{U} : U \cap F = \emptyset \} = X.$$ 

Matveev proved (among other things) that

$$\text{compact} \implies \text{acc} \implies \text{countably compact}$$

and that neither arrow can be reversed. It is well-known that removing one point from $\omega^*$, the remainder of the Čech-Stone compactification of the integers, results in a countably compact subspace. Thus it is natural to ask the question: Are the spaces $\omega^* - \{u\}$ acc for all $u \in \omega^*$? In this paper we prove the following two results:

THEOREM 1.2: There exists $u \in \omega^*$ such that the subspace $\omega^* - \{u\}$ is not acc.

THEOREM 1.3: [CH] For every $u \in \omega^*$, the subspace $\omega^* - \{u\}$ is not acc.

Mathematics Subject Classification: 54D20, 54A35, 54G05.  
Keywords and phrases: countably compact, absolutely countably compact, ultrafilters, $\beta \omega - \omega$, $\omega$ by $\omega$ independent matrix of clopen sets, irreducible maps.
Still open is the question of whether the statement in Theorem 1.3 is a theorem of ZFC.

2. SOME LEMMAS

We begin with deriving a few general lemmas that will be important later. Let $X$ be a space with dense subset $D$. We say that a closed subset $T \subseteq X$ avoids $D$ if there is a family $\mathcal{U}$ of open subsets of $X$ such that the following conditions are satisfied:

1. $\bigcap \mathcal{U} = T$,
2. $|\mathcal{U}| = |D|$, and
3. for every $d \in D$ we have $|\{U \in \mathcal{U}: d \in U\}| < |\mathcal{U}|$.

Observe that if $T \subseteq X$ avoids $D$, then $T \cap D = \emptyset$.

Let $\kappa$ be an infinite cardinal. A subset $P$ of a space $X$ is called a $P_\kappa$-set if the intersection of fewer than $\kappa$ neighborhoods of $P$ is again a neighborhood of $P$. A $P$-set is a $P_{\omega_1}$-set and a $P$-point is a $P$-set singleton. We omit the simple proof of the following lemma.

**Lemma 2.1:** Suppose that $X$ is a compact space with weight $\kappa$. If $D \subseteq X$ is a dense subset of cardinality $\kappa$ and if $T \subseteq X - D$ is a closed $P_\kappa$-set then $T$ avoids $D$.

We now formulate and prove our main tool for recognizing spaces that are not acc. Recall that if $X$ is a space and if $p \in X$ then $\chi(p, X)$ is the character of $p$ in $X$, i.e., the smallest cardinality of a neighborhood base of $p$.

**Lemma 2.2:** Let $X$ be a compact $T_2$-space. If $D \subseteq X$ is dense, $T \subseteq X$ avoids $D$ and $p \in T$ is such that $\chi(p, T) = |D|$, then $X - \{p\}$ is not acc.

**Proof:** Let $U$ be a family of open neighborhoods of $T$ that witnesses the fact that $T$ avoids $D$. Let $\mathcal{V}$ be a neighborhood base for $p$ with $|\mathcal{V}| = |D|$. List $\mathcal{V}$ as $\mathcal{V} = \{V_U: U \in \mathcal{U}\}$ and put

$$\mathcal{W} = (X - T) \cup \{U - V_U: U \in \mathcal{U}\}.$$

Then $\mathcal{W}$ is clearly an open cover of $X - \{p\}$. Pick an arbitrary finite $F \subseteq D$. The family

$$\mathcal{U}_F = \{U \in \mathcal{U}: F \cap U \neq \emptyset\}$$

has cardinality less than $|D|$. Pick an arbitrary point

$$x \in \bigcap \{V_U \cap T: U \in \mathcal{U}_F\} - \{p\}.$$

Observe that such a point exists because $T$ is compact and character and pseudocharacter agree in compact spaces. We claim that $x \notin \operatorname{St}(F, \mathcal{W})$. To this end, pick an arbitrary element $W \in \mathcal{W}$ that intersects $F$. Since $x \in T$, we may clearly assume that $W$ is of the form $U - V_U$ for certain $U \in \mathcal{U}$. Then $U \in \mathcal{U}_F$ since $W$ meets $F$. But since $x \in V_U$, we have $x \notin W = U - V_U$. □

Let $X$ be compact, $D \subseteq X$ be dense and $p \in X - D$. Lemma 2.2 suggests the natural question of whether $X - \{p\}$ is not acc provided that $\{p\}$ avoids $D$. But this
is not true. Because countably compact spaces of countable tightness are acc, Matveev [3, Theorem 1.8], it follows that the ordinal space \( \omega_1 \) is acc. (Alternatively, use the Pressing Down Lemma.) Now simply observe that by Lemma 2.1, \( \{\omega_1\} \) avoids \( \omega_1 \) in the compact space \( \omega_1 + 1 \).

**Definition 2.3:** A continuous function \( f : X \to Y \) of \( X \) onto \( Y \) is called **irreducible** provided \( f(A) \neq Y \) for every proper closed subset \( A \subseteq X \).

The next lemma is well known (see [1, Exercise 3.1C(a)]).

**Lemma 2.4:** If \( f : X \to Y \) of \( X \) onto \( Y \) is continuous and \( X \) is compact, then there exists a closed set \( X_0 \subseteq X \) such that \( f(X_0) = Y \) and \( f|X_0 : X_0 \to Y \) is irreducible.

Recall that the \( \pi \)-character of a point \( x \) in a space \( X \) (denoted \( \pi x(x, X) \)) is the smallest cardinality of a family \( \mathcal{U} \) of open subset of \( X \) such that every neighborhood of \( x \) contains a member of \( \mathcal{U} \).

The next lemma is also well known; see, e.g., Juhász [2, p.64].

**Lemma 2.5:** If \( f : X \to Y \) is irreducible, and \( X \) is compact, then for all \( x \in X \), \( \pi x(x, X) = \pi x(f(x), Y) \).

**Definition 2.6:** An indexed family \( \{A^i_j : i \in I, j \in J\} \) of clopen subsets of \( \omega^* \) is called a \( J \) by \( I \) independent matrix if:

1. The rows of the matrix are pairwise disjoint, i.e., for all distinct \( j_0, j_1 \in J \) and \( i \in I \) we have that \( A^i_{j_0} \cap A^i_{j_1} = \emptyset \).
2. If \( F \) is a finite subset of \( I \) and \( f \in J^F \) then
   \[ \bigcap \{A^i_{f(i)} : i \in F\} \neq \emptyset. \]

K. Kunen proved that there exists a \( c \) by \( c \) independent matrix of clopen subsets of \( \omega^* \) (see [4, Lemma 3.3.2]).

**Proof of Theorem 1.2**

Let \( D \) be a dense subset of \( \omega^* \) having cardinality \( c \), and put \( D = \{d_\alpha : \alpha < c\} \). Let \( \{A^\alpha_\beta : \alpha , \beta \in c\} \) be a \( c \) by \( c \) independent matrix of clopen subsets of \( \omega^* \). For each row \( \alpha < c \), pick two sets \( A^\alpha_{\beta_0} , A^\alpha_{\beta_1} \), so that

\[ (A^\alpha_{\beta_0} \cup A^\alpha_{\beta_1}) \cap \{d_\beta : \beta < \alpha\} = \emptyset, \]

and define \( B^\alpha_0 = A^\alpha_{\beta_0} \) and \( B^\alpha_1 = A^\alpha_{\beta_1} \). Thus \( \{B^\alpha_i : \alpha \in c, i \in 2\} \) is a \( c \) by 2 independent matrix. Let

\[ S = \bigcap \{B^\alpha_0 \cup B^\alpha_1 : \alpha < c\}. \]

Then \( S \) avoids \( D \). By compactness, for every \( x \in 2^c \), we have \( \bigcap \{B^\alpha_x(0) : \alpha < c\} = \emptyset \), hence there is a natural mapping \( f : T \to 2^c \) which is easily seen to be continuous and onto. By Lemma 2.4, there exists \( S \subseteq T \) such that \( f|S : S \to 2^c \) is onto and irreducible. By Lemma 2.5 every point in \( S \) has character \( c \) in \( S \) and hence in
T. An application of Lemma 2.2 now shows that for \( u \in S \) we have \( \omega^* - \{u\} \) is not acc.

**PROOF OF THEOREM 1.3**

Assume CH, and let \( u \in \omega^* \).

We first prove the theorem in the special case that \( u \) is a \( P \)-point. Since all \( P \)-points in \( \omega^* \) are topologically equivalent [5, p.171], and since the density of \( \omega^* \) is \( \omega_1 \), by Lemma 2.2 it suffices to construct a \( P \)-point \( p \in \omega^* \) which is a nonisolated point in some nowhere dense closed \( P \)-set \( P \subseteq \omega^* \). By [4, Lemma 1.4.3] there is a nowhere dense closed \( P \)-set \( P \) in \( \omega^* \) which is homeomorphic to \( \omega^* \). We can therefore let \( p \) be any \( P \)-point of \( P \).

We now use the special case to prove the general case. By [5, p.79], it follows that we can write \( \omega^* - \{u\} \) as the disjoint union of two nonempty open sets \( U \) and \( V \) each having \( u \) in their closure. Since \( \omega^* \) has no \( (\omega, \omega) \)-gaps, we may without loss of generality assume that \( u \) is a \( P \)-point in \( U \cup \{u\} \). By Parovičenko’s characterization of \( \omega^* \) [4, Corollary 1.2.4], it easily follows that \( U \cup \{u\} \) is homeomorphic to \( \omega^* \). By the previous case it now follows that \( U \) is not acc. But then clearly \( \omega^* - \{u\} = U \cup V \) is not acc as well.

**REFERENCES**