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GENERAL TOPOLOGY

On Remainders Without Arcs

by

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Summary. Let X be a separable, completely metrizable space and let f be an autohomeomorphism of X. We prove that f is extendable to an autohomeomorphism of some metrizable compactification of X whose remainder contains no arcs.

All spaces under discussion are separable and metrizable. A well-known theorem of Engelking [2] states that every autohomeomorphism f of an n-dimensional space X can be extended to a homeomorphism $\overline{f} : C \to C$, where C is an n-dimensional compactification of X (and hence the remainder $C \setminus X$ is $\leqslant n$-dimensional). In [1] the question was investigated whether a similar result holds for infinite-dimensional spaces, i.e. whether for an infinite-dimensional space X and an autohomeomorphism f of X there exists a compactification C such that $C \setminus X$ is "small" from dimension theoretic perspective, while moreover, f can be extended to an autohomeomorphism of C. It is well known that the answer to this question is **no** for incomplete spaces. Let $\sigma = \{x \in \mathbb{R}^{\mathbb{N}} : x_i = 0 \text{ from some index on}\}$. Then σ is strongly countably dimensional while the remainder of every compactification of σ contains a copy of the Hilbert cube $Q = [0,1]^{\mathbb{N}}$ (see [7, Exercise 4.8.7] or [1]).

For complete spaces the situation looks more promising since Lelek [4] (see also [3]) proved that every complete space can be compactified through the addition of a strongly countably dimensional remainder. Let s denote the topological Hilbert space $\mathbb{R}^{\mathbb{Z}}$ and let α denote the "left shift" on s, i.e. $\alpha(x)_i = x_{i+1}$ for $i \in \mathbb{Z}$. In [1] the above question for complete spaces was answered negatively by showing that if C is a compactification of s such that α extends to a continuous $\overline{\alpha} : C \to C$ then the remainder $C \setminus s$ contains strongly infinite-dimensional continua. It was left as an open question whether $C \setminus s$ must, in fact, contain a copy of the Hilbert cube Q. The following theorem implies that every autohomeomorphism of Hilbert space is extendable to an autohomeomorphism of some compactification whose remainder contains no arcs (and hence no copies of Q).

THEOREM. *If X is a separable completely metrizable space and $\mathcal{F}$ is a countable collection of maps from X to X then there is a metrizable compactification C of X such that every element of $\mathcal{F}$ is continuously extendable over C and no arc in C meets $C \setminus X$.*

An arc is a homeomorphic image of the closed unit interval $I = [0, 1]$ and a map is a continuous function.

A result such as this theorem is uninteresting outside the class of separable metrizable spaces. Let X be a sufficiently nice noncompact space and let f be an autohomeomorphism of X. Then f can be extended to an autohomeomorphism of the Čech-Stone compactification βX of X. In addition, since X is sufficiently nice, $\beta X \setminus X$ contains no arcs (for simple reasons). Thus, our theorem is of interest only within the class of separable metrizable spaces and its proof is nontrivial precisely because of that restriction.

In our proof there is used the Wallman compactification whose definition we now recall. We call a closed basis $\mathfrak{W}$ for the topology of a space X a *Wallman basis* for X if $\mathfrak{W}$ is closed under finite intersections and if $\mathfrak{W}$ is normal (i.e. if A and B are disjoint members of $\mathfrak{W}$ then there are $V, W \in \mathfrak{W}$ such that $V \cup W = X$ and $V \cap B = A \cap W = \emptyset$). If $\mathfrak{W}$ is a Wallman basis for X then the underlying set for the *Wallman compactification* $\omega(\mathfrak{W})$ of X relative $\mathfrak{W}$ is the set of $\mathfrak{W}$-ultrafilters. If $W \in \mathfrak{W}$ then $\overline{W} = \{\mathcal{F} \in \omega(\mathfrak{W}) : W \in \mathcal{F}\}$. The collection $\{\overline{W} : W \in \mathfrak{W}\}$ functions as a closed basis for the topology on $\omega(\mathfrak{W})$. Since $\mathfrak{W}$ is normal, $\omega(\mathfrak{W})$ is Hausdorff and if $\mathfrak{W}$ is countable then $\omega(\mathfrak{W})$ is metrizable. We shall use the following well-known fact: if $f : X \to Y$ is a map and $\mathfrak{X}$ and $\mathfrak{Y}$ are Wallman bases on X, respectively Y, such that $f^{-1}[\mathfrak{Y}] \subset \mathfrak{X}$ then f extends to a map $\overline{f} : \omega(\mathfrak{X}) \to \omega(\mathfrak{Y})$. For more information about Wallman compactifications, see [8].

P r o o f. Let S be the $\sin(1/x)$ compactification of the interval $J = (0, 1]$ and let $\mathfrak{S}$ be a countable Wallman basis for J such that $\omega(\mathfrak{S}) = S$. Let $p : S \to I$ be the extension of the identity on J. We shall construct inductively a sequence of countable Wallman bases $\mathfrak{W}_0 \subset \mathfrak{W}_1 \subset \cdots$ on X. Let $\mathfrak{W}_0$ be any countable Wallman basis. Assume that $\mathfrak{W}_n$ has been constructed and consider the metric compactification $C_n = \omega(\mathfrak{W}_n)$. Since X is complete the remainder is σ-compact and we can find a countable collection $\mathcal{A}_n$ of compacta in $C_n \setminus X$ such that for every $x \in C_n \setminus X$ and every neighbourhood U of x there is an $A \in \mathcal{A}_n$ with $x \in A \subset U$. For every $A \in \mathcal{A}_n$ we select a map $\psi_A : C_n \to I$ such that $\psi_A^{-1}(0) = A$. Let $\varphi_A : X \to J$ be the restriction of ψ_A. We extend the countable closed basis

$$\mathfrak{W}_n \cup \bigcup_{A \in \mathcal{A}_n} \varphi_A^{-1}[\mathfrak{S}] \cup \bigcup_{f \in \mathcal{F}} f^{-1}[\mathfrak{W}_n]$$

to a countable Wallman basis $\mathfrak{W}_{n+1}$ for X. Consider the countable Wallman basis $\mathfrak{W} = \bigcup_{n=0}^{\infty} \mathfrak{W}_n$, the metrizable compactification $C = \omega(\mathfrak{W})$, and the canonical maps $\pi_n : C \to C_n$. Since $f^{-1}[\mathfrak{W}]$ is obviously contained in $\mathfrak{W}$ for every $f \in \mathcal{F}$ these functions are extendable to maps from C to C.

Let $\alpha : I \to C$ be an imbedding whose image intersects $C \setminus X$ in a point x. We may assume that $x = \alpha(0)$. Put $y = \alpha(1)$. Since $x \neq y$ there are two disjoint elements V and W of $\mathfrak{W}$ such that $x \in \overline{V}$ and $y \in \overline{W}$. Select an $\mathfrak{W}_n$ that contains both V and W. Consequently, $\pi_n(x) \neq \pi_n(y)$ and we can find an $A \in \mathcal{A}_n$ that contains $\pi_n(x)$ but not $\pi_n(y)$. Since $\varphi_A^{-1}[\mathfrak{S}] \subset \mathfrak{W}_{n+1} \subset \mathfrak{W}$ we can extend φ_A to a continuous $\overline{\varphi}_A : C \to S$. Observe that

$$p \circ \overline{\varphi}_A | X = p \circ \varphi_A = \varphi_A = \psi_A | X = \psi_A \circ \pi_n | X$$

and hence $p \circ \overline{\varphi}_A = \psi_A \circ \pi_n$. Since $\pi_n(x) \in A$ and $\pi_n(y) \notin A$ we have $p(\overline{\varphi}_A(x)) = 0$ and $p(\overline{\varphi}_A(y)) > 0$. Thus the continuous function $\overline{\varphi}_A \circ \alpha : I \to S$ maps 0 into $S \setminus J$ and 1 into J, providing the desired contradiction.

QUESTION. *Let $f : s \to s$ be a homeomorphism. Does there exist a compactification C of s such that*

(1) *C is a Hilbert cube and $C \setminus s$ is a σZ-set without arcs (or even just a σZ-set),*

(2) *f can be extended to a homeomorphism $\overline{f} : C \to C$.*

It was shown in [5] that there exists a compactification of s that satisfies condition (1). Results from [6] show that for an arbitrary homeomorphism $f : s \to s$ there needs not exist a compactification C of s such that f is extendable to an autohomeomorphism of C, while moreover, the pair $(C, C \setminus s)$ is equivalent to the pair (Q, B), where B denotes the pseudo-boundary of Q.

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