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On a generalization of Lyapounov's theorem

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ABSTRACT

We provide a simpler proof of Gouweleeuw's theorem about the convexity of the range of an $\mathbb{R}^n$-valued vector measure $\vec{\mu}$ in terms of $\vec{\mu}$. We also discuss possible extensions of Gouweleeuw's results to vector measures with values in infinite-dimensional vector spaces and to unbounded vector measures.

1. INTRODUCTION

Let L be a (real) topological vector space, let $(\Omega, \mathcal{F})$ be a measurable space and let $\vec{\mu} : \mathcal{F} \to L$ be a countably additive measure[1]. The range $\mathcal{R}(\vec{\mu})$ of $\vec{\mu}$ is the set

$$\{\vec{\mu}(F) : F \in \mathcal{F}\}.$$

We say that $\vec{\mu}$ is *bounded* provided that the closure of $\mathcal{R}(\vec{\mu})$ in L is compact. If $A \subseteq \Omega$ is measurable then $\vec{\mu} \restriction A$ denotes the restriction of the measure $\vec{\mu}$ to A. Let $\mathbb{I}$ denote the closed unit interval $[0, 1]$.

By Halmos [6, Lemma 11], if L is finite dimensional (hence $L = \mathbb{R}^n$ for some n) then the boundedness of $\vec{\mu}$ implies that $\mathcal{R}(\vec{\mu})$ is in fact a compact subset of L. This is not trivial. This result cannot be generalized to infinite-dimensional vector spaces, as was shown by Lyapounov [8][2]. The counterexample of Lyapounov is based on the existence of an orthogonal basis for $L^2([-\pi, \pi])$

[1] If $L = \mathbb{R}$ then $\vec{\mu}$ is not necessarily positive.

[2] We are indebted to Henno Brandsma for translating Lyapounov's article for us.

consisting of functions that only assume the values $+1$ and -1. We will present a simpler description of it in §3.

A *$\vec{\mu}$-atom* is an element $F \in \mathcal{F}$ such that $\vec{\mu}(F) \neq 0$ while moreover for every measurable $E \subseteq F$ either $\vec{\mu}(E) = 0$ or $\vec{\mu}(E) = \vec{\mu}(F)$. The measure $\vec{\mu}$ is *nonatomic* if $\mathcal{F}$ contains no $\vec{\mu}$-atoms.

Lyapounov's famous convexity theorem in [7] asserts that if $L = \mathbb{R}^n$ then $\mathcal{R}(\vec{\mu})$ is a *convex* subset of L provided that all the coordinate measures of $\vec{\mu}$ (self-explanatory) are nonatomic (this is not trivial as well; see also Dubins and Spanier [3, Theorem 5], and Halmos [6, p. 421]). It was shown by Dubins and Spanier [3, Lemma 4.1] that $\vec{\mu}$ is nonatomic if and only if all of its coordinate measures are nonatomic. As a consequence Lyapounov's theorem can also be formulated as follows: if $\vec{\mu}$ is nonatomic then $\mathcal{R}(\vec{\mu})$ is convex.

It is not clear at all how this result should be generalized. One would like to obtain a characterization of the convexity of $\mathcal{R}(\vec{\mu})$ in terms of $\vec{\mu}$. A natural approach for solving this problem is the following one. Let A be the union of all the $\vec{\mu}$-atoms and let $B = \Omega \setminus A$. Then $\vec{\mu} \restriction A$ is purely atomic and $\vec{\mu} \restriction B$ is nonatomic. It is easy to see that $\mathcal{R}(\vec{\mu}) = \mathcal{R}(\vec{\mu} \restriction A) + \mathcal{R}(\vec{\mu} \restriction B)$. Since $\mathcal{R}(\vec{\mu} \restriction B)$ is convex by Lyapounov's theorem, a sufficient condition for the convexity of $\mathcal{R}(\vec{\mu})$ is that $\mathcal{R}(\vec{\mu} \restriction A)$ is convex. (It is easy to detect when $\mathcal{R}(\vec{\mu} \restriction A)$ is convex, see Rényi [9, p. 80 (exercise 48)].) But this sufficient condition is not necessary, as the following (trivial) example shows. Let $\Omega = \mathbb{I} \cup \{2\}$ and let $\mathcal{F}$ be the collection of Borel subsets of Ω. Let the measure σ on Ω be defined by $\sigma \restriction \mathbb{I} =$ Lebesgue measure and $\sigma(\{2\}) = \frac{1}{2}$. Then, adopting the above notation, $A = \{2\}$, $B = \mathbb{I}$, $\mathcal{R}(\sigma) = \mathbb{I}$ is convex but $\mathcal{R}(\sigma \restriction A) = \{0, \frac{1}{2}\}$ is not convex. So we see that this natural approach does not work. Interestingly, a slightly different approach does work under the additional assumption that $\vec{\mu}$ is $\mathbb{R}^n$-valued for some n and is nonnegative in the sense that all of its coordinate measures are nonnegative. Gouweleeuw [4] (see also [5]) decomposed Ω into measurable sets $A_0, A_1, \ldots, A_n, \ldots$ such that $\vec{\mu} \restriction A_0$ is nonatomic (hence $\mathcal{R}(\vec{\mu} \restriction A_0)$ is convex) and for every $i \geq 1$, $\vec{\mu} \restriction A_i$ is in essence a 1-dimensional measure. For those 1-dimensional pieces it is easy to detect when their ranges are convex, and, interestingly, $\mathcal{R}(\vec{\mu})$ is convex if and only if $\mathcal{R}(\vec{\mu} \restriction A_i)$ is convex for every $i \geq 1$ (Gouweleeuw [4] (see also [5])).

The aim of this paper is to present a simpler proof of Gouweleeuw's theorem. Our method, unlike Gouweleeuw's, also applies to measures that are not necessarily nonnegative. We will also discuss possible extensions of our results. For example, to vector measures with values in infinite-dimensional vector spaces and to unbounded vector measures. We will also discuss the convexity of matrix-k-ranges.

It should be mentioned that in case L is an infinite-dimensional Banach space an analytical condition is available describing the convexity and weak compactness of the range of $\vec{\mu}$ (see Diestel and Uhl [2, §IX.1]). This description runs as follows. If $\vec{\mu}$ is a countably additive bounded measure with values in a Banach space L then there is a bounded nonnegative measure ν such that $\nu(E) = 0$ if and only if $\vec{\mu}(E \cap F) = 0$ for all $F \in \mathcal{F}$. (For the case $L = \mathbb{R}^n$ and

$\vec{\mu} = (\mu_1, \ldots, \mu_n)$ one can take $\nu(E) = \sum_{i=1}^{n} |\mu_i|(E)$.) In terms of ν convexity and weak compactness of the range of $\vec{\mu}$ can now be described as follows. $\mathcal{R}(\vec{\mu} \restriction A)$ is weakly compact and convex if and only if for each $E \in \mathcal{F}$ with $\nu(E) > 0$ the operator $f \to \int_E f \, d\vec{\mu}$ on $L_\infty(\nu)$ is not one-to-one on the subspace of functions in $L_\infty(\nu)$ vanishing off E. This result is due to Knowles (see [2, Theorem IX.1.4]). In [2] it is shown how Lyapounov's theorem can be deduced easily from this result. In addition, it is shown in [2, Theorem IX.1.10] that if L is a Banach space with the Radon–Nikodým property, and $\vec{\mu} : \mathcal{F} \to L$ is a nonatomic countably additive measure of bounded variation, then the norm closure of $\mathcal{R}(\vec{\mu})$ is convex and norm compact.

Our main theorem (Theorem 3.3 below) gives a necessary condition for convexity of the range which is totally in terms of the measure $\vec{\mu}$ itself. This condition is in general not sufficient; however, in the case of a finite dimensional measure it is also sufficient.

2. ONE-DIMENSIONAL SETS

Our interest is in vector measures the ranges of which are bounded convex subsets of arbitrary locally convex metrizable vector spaces. It follows by Bessaga and Pełczyński [1, §§III.2 and III.3] that for every bounded convex subset C of such a vector space L there exists a linear function $f : L \to \mathbb{R}^\infty$ such that $f[\bar{C}] \subseteq \ell^2$ and $f \restriction \bar{C} : \bar{C} \to f[\bar{C}]$ is a homeomorphism (here $\bar{C}$ denotes the (compact) closure of C in L). This shows that in most interesting cases we can assume without loss of generality that the ranges of the vector measures under consideration are subsets of ℓ^2.

Let $(\Omega, \mathcal{F})$ be a measurable space, let L be a vector space and let $\vec{\mu} : \mathcal{F} \to L$ be a bounded countably additive measure. An arbitrary measurable set $A \subseteq \Omega$ is said to be a *$\vec{\mu}$-negligible* if for every measurable $B \subseteq A$ we have $\vec{\mu}(B) = 0$. (So if $L = \mathbb{R}$ then a $\vec{\mu}$-negligible set is precisely a nullset of the total variation of $\vec{\mu}$.) An arbitrary measurable set $A \subseteq \Omega$ is said to be *1-dimensional* provided that $\vec{\mu}(A) \neq 0$ and there exists a 1-dimensional linear subspace L_A of L such that

$$\mathcal{R}(\vec{\mu} \restriction A) \subseteq L_A.$$

In other words, for every measurable $B \subseteq A$ we have $\vec{\mu}(B) \in L_A$. Observe that if L_A exists then it is unique and is equal to the linear span of the vector $\vec{\mu}(A)$.

The concept of a 1-dimensional set is implicit in Gouweleeuw [5, Theorem 1.8].

A $\vec{\mu}$-atom E is 1-dimensional because $\mathcal{R}(\vec{\mu} \restriction E) = \{0, \vec{\mu}(E)\}$.

Let $A \subseteq \Omega$ be 1-dimensional. Then $\vec{\mu} \restriction A$ can be identified with a single $\mathbb{R}$-valued measure. In case $L = \ell^2$, the measure $\vec{\mu}$ can also be identified with one of its coordinate measures. (For every $i \geq 1$, the i-th coordinate measure μ_i of $\vec{\mu}$ is defined as follows: if $A \subseteq \Omega$ is measurable and $\vec{\mu}(A) = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots)$ then $\mu_i(A) = x_i$.) To see this, let $\vec{x} = (x_1, x_2, \ldots)$ be an element of L_A different from 0. Let $j \geq 1$ be the minimal index for which $x_j \neq 0$. It is easy to see that for every measurable $B \subseteq A$ and $i \in \mathbb{N}$ we have

$$\mu_i(B) = \frac{x_i}{x_j}\,\mu_j(B),$$

i.e., the measure μ_i is a multiple of μ_j. This remark explains the terminology and will also be used in the proof of Lemma 3.7.

A 1-dimensional set A is called *maximal* provided that for every measurable set $B \subseteq \Omega \setminus A$ with $\vec{\mu}(B) \neq 0$ the set $A \cup B$ is not 1-dimensional.

Lemma 2.1. *If A and B are 1-dimensional and $\vec{\mu}(A \cap B) \neq 0$ then $A \cup B$ is 1-dimensional and $L_A = L_B = L_{A \cup B}$.*

Proof. Since $\vec{\mu}(A \cap B) \neq 0$ and $\vec{\mu}(A \cap B) \in L_A \cap L_B$ it follows that $L_A = L_B$. It now easily follows that the range of the measure $\vec{\mu} \restriction (A \cup B)$ is contained in L_A, i.e. $A \cup B$ is 1-dimensional. □

Lemma 2.2. *Every 1-dimensional set is contained in a maximally 1-dimensional set.*

Proof. Let A be 1-dimensional. Let $\mathcal{B}$ be the collection of all 1-dimensional sets $B \subseteq \Omega \setminus A$ such that

$$(\exists \alpha_B > 0)(\vec{\mu}(B) = \alpha_B \vec{\mu}(A)).$$

By induction on $n \geq 0$ we will construct an increasing sequence of 1-dimensional sets $A_n \subseteq \Omega$ with $\mathcal{R}(\vec{\mu} \restriction A_n) \subseteq L_A$, as follows. Put $A_0 = A$. Suppose that A_n has been defined. Put $\mathcal{B}_n = \{B \in \mathcal{B} : B \subseteq \Omega \setminus A_n\}$. Let $s_n = 0$ if $\mathcal{B}_n = \emptyset$ and

$$(1) \qquad s_n = \sup\{\alpha_B : B \in \mathcal{B}_n\}$$

otherwise. Observe that $s_n < \infty$ because $\vec{\mu}$ is bounded. If $s_n = 0$ then put $A_{n+1} = A_n$. If $s_n \neq 0$ then pick an arbitrary element $B_n \in \mathcal{B}_n$ with $\alpha_{B_n} > s_n - (1/n)$ and put $A_{n+1} = A_n \cup B_n$. Observe that A_{n+1} is 1-dimensional and that $\mathcal{R}(\vec{\mu} \restriction A_{n+1}) \subseteq L_A$. This completes the inductive construction.

Let $A_{+\infty} = \bigcup_{n=0}^{\infty} A_n$. Then $A_{+\infty}$ is measurable and we claim that it is 1-dimensional. Indeed, let $S \subseteq A_{+\infty}$ be measurable and let $A_{-1} = \emptyset$. Then since L_A is closed in L (being 1-dimensional),

$$\vec{\mu}(S) = \lim_{m \to \infty} \sum_{n=0}^{m} \vec{\mu}(S \cap (A_n \setminus A_{n-1})) \in L_A.$$

This implies that $A_{+\infty}$ is 1-dimensional and also that $L_{A_{+\infty}} = L_A$.

We now claim that $\Omega \setminus A_{+\infty}$ does not contain any element of $\mathcal{B}$. Striving for a contradiction, assume that there exists an element $B \in \mathcal{B}$ with $B \subseteq \Omega \setminus A_{+\infty}$. Pick $n \geq 1$ such that $\alpha_B \geq (1/n)$. Observe that $B_n \cup B \in \mathcal{B}$, that $(B_n \cup B) \cap A_n = \emptyset$ and that

$$\alpha_{B_n \cup B} = \alpha_{B_n} + \alpha_B > s_n - \frac{1}{n} + \frac{1}{n} = s_n.$$

But this contradicts (1).

It follows similarly that there exists a 1-dimensional set $A_{-\infty} \supseteq A$ with the property that $\Omega \setminus A_{-\infty}$ contains no 1-dimensional set B having the following property:

$$(\exists \alpha_B < 0)(\vec{\mu}(B) = \alpha_B \vec{\mu}(A)).$$

We claim that $\hat{A} = A_{+\infty} \cup A_{-\infty}$ is maximally 1-dimensional. Clearly, $\hat{A}$ is 1-dimensional and $L_{\hat{A}} = L_A$ (Lemma 2.1). Now let $E \subseteq \Omega \setminus \hat{A}$ be measurable such that $\hat{A} \cup E$ is 1-dimensional. There exists $\alpha \in \mathbb{R}$ such that $\vec{\mu}(E) = \alpha \vec{\mu}(A)$. Now $\alpha \not> 0$ because $E \cap A_{+\infty} = \emptyset$ and $\alpha \not< 0$ because $E \cap A_{-\infty} = \emptyset$. We conclude that $\alpha = 0$, as required. $\square$

If A and B are maximally 1-dimensional sets and if $\vec{\mu}(A \cap B) \neq 0$ then by Lemma 2.1, $A \triangle B$ is $\vec{\mu}$-negligible. Such sets are considered to be the same maximally 1-dimensional sets. As a consequence, different maximally 1-dimensional sets intersect in a $\vec{\mu}$-negligible set. This implies that there are at most a countable number of maximally 1-dimensional sets, say $A_1, A_2, \ldots$. By removing $\vec{\mu}$-negligible sets if necessary, we may and will additionally assume that $A_i \cap A_j = \emptyset$ if $i \neq j$.

Put $A_0 = \Omega \setminus \bigcup_{n=1}^{\infty} A_n$. The collection $\{A_0, A_1, \ldots\}$ partitions Ω. Since every $\vec{\mu}$-atom is 1-dimensional and hence is contained in one of the A_n's, it follows that A_0 contains no $\vec{\mu}$-atoms and hence is nonatomic. As a consequence, $\mathcal{R}(\vec{\mu} \restriction A_0)$ is convex by Lyapounov's theorem.

The decomposition $\{A_0, A_1, \ldots\}$ of Ω constructed here is similar to a decomposition of Ω considered by Gouweleeuw [5]. There is an unimportant difference however. Our A_0 contains no 1-dimensional sets, in contrast to the nonatomic part of Gouweleeuw's partition that can contain 1-dimensional sets.

3. CONVEXITY OF RANGES OF MEASURES

We are interested in the question: When is $\mathcal{R}(\vec{\mu})$ convex? Let $A_0, A_1, \ldots$ be the partition of Ω constructed in §2.

We are interested in the following two statements:

($*$) for every $i \geq 0$, $\mathcal{R}(\vec{\mu} \restriction A_i)$ is convex; and

($**$) for every $i \geq 1$, $\mathcal{R}(\vec{\mu} \restriction A_i)$ is convex.

Clearly, $(*) \Rightarrow (**)$. A straightforward verification show that $(*)$ implies that $\mathcal{R}(\vec{\mu})$ is convex. So $(*)$ is a sufficient condition for the convexity of $\mathcal{R}(\vec{\mu})$. But it is in general not necessary. To show this, we first present a simpler description of Lyapounov's counterexample from [8].

Example 3.1. *Lyapounov's counterexample.* Let $(\Omega, \mathcal{F})$ be ($\mathbb{I}$, Borel sets) and let $L = \ell^2$. As usual, λ denotes Lebesgue measure on $\mathbb{I}$. Finally, let $\{R_n : n \in \mathbb{N}\}$ be an enumeration of all finite unions of closed subintervals of $\mathbb{I}$ with rational endpoints. Define $\vec{\mu} : \mathcal{F} \to \ell^2$ as follows:

$$\vec{\mu}(B) = (2^{-1} \cdot \lambda(B \cap R_1), \ldots, 2^{-n} \cdot \lambda(B \cap R_n), \ldots).$$

Then $\vec{\mu}$ is clearly a countably additive ℓ^2-valued measure on the Borel subsets of

$\mathbb{I}$. In addition, $\vec{\mu}$ is bounded since $\mathcal{R}(\vec{\mu})$ is contained in the compact Hilbert cube

$$Q = \{x \in \ell^2 \colon\ (\forall n)(|x_n| \le 2^{-n})\}.$$

We show that $\mathcal{R}(\vec{\mu})$ is not convex. To this end, let

$$\vec{x} = \vec{\mu}(\mathbb{I}) = (2^{-1} \cdot \lambda(R_1), \ldots, 2^{-n} \cdot \lambda(R_n), \ldots).$$

We claim that there is no Borel set $E \subseteq \mathbb{I}$ such that $\vec{\mu}(E) = \frac{1}{2}x$. Striving for a contradiction, assume that such E exists. Then

$$\begin{aligned}\vec{\mu}(E) &= (2^{-1} \cdot \lambda(R_1 \cap E), \ldots, 2^{-n} \cdot \lambda(R_n \cap E), \ldots) \\ &= (2^{-2} \cdot \lambda(R_1), \ldots, 2^{-(n+1)} \cdot \lambda(R_n), \ldots).\end{aligned}$$

We conclude that for every n,

$$\lambda(E \cap R_n) = \tfrac{1}{2} \cdot \lambda(R_n). \tag{2}$$

This is impossible, as the following argument shows. There exists $N \in \mathbb{N}$ such that $R_N = \mathbb{I}$. So by (2), $\lambda(E) = \frac{1}{2}$. Since λ is inner regular, there is a compact $K \subseteq E$ such that $\lambda(E \setminus K) < \frac{1}{8}$. Observe that $\lambda(K) > \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$. Since K is compact, there exists $n \in \mathbb{N}$ such that $K \subseteq R_n$ and $\lambda(R_n \setminus K) < \frac{1}{8}$. Observe that $\lambda(E \cap R_n) \ge \lambda(K) > \frac{3}{8}$. By (2) we therefore conclude that $\lambda(R_n) > 2 \cdot \frac{3}{8} = \frac{6}{8}$. On the other hand, $\lambda(R_n) < \frac{1}{8} + \lambda(K) \le \frac{1}{8} + \lambda(E) = \frac{5}{8}$. This is a contradiction.
We will now show that $\mathcal{R}(\vec{\mu})$ is not compact. For every $k \in \mathbb{N}$ let

$$B_k = \bigcup_{j=0}^{2^{k-1}-1} \left[\frac{2j}{2^k}, \frac{2j+1}{2^k}\right],$$

so $B_1 = [0, \frac{1}{2}]$, $B_2 = [0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}]$, $B_3 = [0, \frac{1}{8}] \cup [\frac{2}{8}, \frac{3}{8}] \cup [\frac{4}{8}, \frac{5}{8}] \cup [\frac{6}{8}, \frac{7}{8}]$, etc. We claim that $\vec{\mu}(B_n) \to \frac{1}{2}x$. It follows from this and from the above that $\mathcal{R}(\vec{\mu})$ is not compact. Choose R_i, and let $R_i = \bigcup_{j=1}^{n} A_j$ with $A_j = [\alpha_j, \beta_j]$, $A_k \cap A_j = \emptyset$ for $k \ne j$. For n large enough, in fact for $n > N_0$, where N_0 is the first integer such that $2^{-N_0} < \min_{1 \le j \le n}(\beta_j - \alpha_j)$, we have $\lambda(R_i \cap B_n) \approx \frac{1}{2}\lambda(R_i)$. More precisely $\lambda(R_i \cap B_n) \to \frac{1}{2}\lambda(R_i)$ as $n \to \infty$. But from this we have

$$\begin{aligned}\vec{\mu}(B_n) &= (2^{-1} \cdot \lambda(B_n \cap R_1), 2^{-2} \cdot \lambda(B_2 \cap R_2), \ldots) \\ &\to (2^{-2} \cdot \lambda(R_1), 2^{-3} \cdot \lambda(R_2), \ldots) = \tfrac{1}{2}x,\end{aligned}$$

as required.

We next use this example to show that $(*)$ is not a necessary condition for the convexity of $\mathcal{R}(\vec{\mu})$.

Example 3.2. Let $\Omega = [0, \infty)$ and $\mathcal{F} =$ Borel sets. We adopt the notation in Example 3.1 and let λ denote Lebesgue measure on $\mathbb{R}$. Define $\vec{\sigma} : \mathcal{F} \to \ell^2$ by the following formula:

$$\begin{cases} \vec{\sigma} \restriction \mathbb{I} = \vec{\mu}, \\ \vec{\sigma} \restriction [n, n+1] = (\underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}}, 2^{-n} \cdot \lambda \restriction [n, n+1], 0, 0, \ldots) \quad (n \in \mathbb{N}). \end{cases}$$

It is clear that $\mathcal{R}(\vec{\sigma})$ is the subspace $\{x \in \ell^2 \colon\ (\forall n)(0 \le x_n \le 2^{-n})\}$ and hence is convex. We claim that for every n the interval $[n, n+1]$ is maximally 1-dimensional. Fix $n \in \mathbb{N}$. From the description of $\vec{\sigma}$ it is clear that all we need to show is that if $E \subseteq \mathbb{I}$ is Borel and $\lambda(E) > 0$ then $[n, n+1] \cup E$ is not 1-dimensional. To this end, fix an arbitrary Borel set $E \subseteq \mathbb{I}$ with positive Lebesgue measure. There clearly exist infinitely many $m \in \mathbb{N}$ for which $\lambda(E \cap R_m) > 0$. But this implies that $\vec{\mu}(E)$ is a vector with infinitely many nonzero coordinates, and therefore does not belong to the linear span of $\{\vec{\sigma}([n, n+1])\}$. So the sets $A_0 = \mathbb{I}$ and $A_n = (n, n+1]$ for $n \geq 1$ correspond to the partition of Ω considered in §2.

From this we see that the compactness and/or the convexity of $\mathcal{R}(\vec{\sigma})$ need not imply the corresponding properties for $\mathcal{R}(\vec{\sigma} \restriction A_0)$. Notice however that the sets $\mathcal{R}(\vec{\sigma} \restriction A_i)$ are convex for every $i \geq 1$. That this is no accident will be shown in Theorem 3.3.

Put $A = A_1$ and $B = \Omega \setminus A_1$. For later use, observe that $\mathcal{R}(\vec{\sigma})$ is compact, that $\mathcal{R}(\vec{\sigma} \restriction A)$ is compact but that $\mathcal{R}(\vec{\sigma} \restriction B)$ is *not* compact (see Example 3.1). That this can happen will complicate our life later on.

This example is nonatomic. It is easy however to modify the example so that it has infinitely many atoms. Simply give $\{n\}$ measure 2^{-n} for every $n \geq 2$.

Observe that from Diestel and Uhl [2, Theorem IX.1.10] it follows that the norm closure of $\mathcal{R}(\vec{\mu})$ is convex and norm compact.

We now formulate our main result.

Theorem 3.3. *Let $(\Omega, \mathcal{F})$ be a measurable space, let L be a metrizable locally convex vector space and let $\vec{\mu} : \mathcal{F} \to L$ a countably additive bounded measure. Finally, let $A_1, A_2, \ldots$ be the maximally 1-dimensional sets of Ω. If $\mathcal{R}(\vec{\mu})$ is convex then $\mathcal{R}(\vec{\mu} \restriction A_i)$ is compact and convex for every $i \geq 1$.*

Corollary 3.4. *Let $(\Omega, \mathcal{F})$ be a measurable space and let $\vec{\mu} : \mathcal{F} \to \mathbb{R}^n$ be a countably additive bounded measure and let $A_1, A_2, \ldots$ be the maximally 1-dimensional sets of Ω. Then the following statements are equivalent:*

(a) *$\mathcal{R}(\vec{\mu})$ is convex.*
(b) *$\mathcal{R}(\vec{\mu})$ is compact and convex.*
(c) *For every $i \geq 1$, $\mathcal{R}(\vec{\mu} \restriction A_i)$ is convex.*

Proof. Since $\mathcal{R}(\vec{\mu})$ is compact by Halmos [6, Lemma 11], the equivalence (a) $\Leftrightarrow$ (b) is trivial. Put $A_0 = \Omega \setminus \bigcup_{i=1}^{\infty} A_i$. Then $\vec{\mu} \restriction A_0$ is nonatomic. Hence $\mathcal{R}(\vec{\mu} \restriction A_0)$ is convex by Lyapounov's theorem. From this, the implication (c) $\Rightarrow$ (a) is trivial. Finally, that (a) $\Rightarrow$ (c) follows from Theorem 3.3. □

Remark 3.5. Corollary 3.4 was first proved by Gouweleeuw [5, 4] under the additional assumption that $\vec{\mu}$ is nonnegative (in the sense that its coordinate measures are all nonnegative).

Remark 3.6. From Corollary 3.4 we see that $(**)$ (and hence also $(*)$) is a necessary and sufficient condition for the convexity of $\mathcal{R}(\vec{\mu})$ in the finite dimensional case. It was shown earlier in this section that (*) is not a necessary condition for the convexity of $\mathcal{R}(\vec{\mu})$ in the infinite-dimensional case.

We now turn to the proof of Theorem 3.3. For symmetry reasons it suffices to show that $\mathcal{R}(\vec{\mu} \restriction A_1)$ is compact and convex. To simplify our notation, put $A = A_1$ and $B = \Omega \setminus A$, respectively. Then $\mathcal{R}(\vec{\mu}) = \mathcal{R}(\vec{\mu} \restriction A) + \mathcal{R}(\vec{\mu} \restriction B)$.

Lemma 3.7. *$\mathcal{R}(\vec{\mu} \restriction A)$ is compact.*

Proof. Since $\vec{\mu} \restriction A$ is in fact a bounded 1-dimensional measure, this follows from Rényi [9, p. 83 (exercise 50)]. □

We now turn to the more interesting part of the proof. To begin with, we first prove a special case of our main result.

We prove Theorem 3.3 *in the special case* $L = \mathbb{R}^n$.

Assume that for some n, $\mathcal{R}(\vec{\mu})$ is a bounded and convex subset of $\mathbb{R}^n$. We will prove that $S = \mathcal{R}(\vec{\mu} \restriction A)$ is convex by applying the following lemma which is Gouweleeuw [5, Lemma 1.31].

Lemma 3.8. *Let E be a vector space. Let $S, T, L \subseteq E$, where L is a linear subspace of E and $S \subseteq L$. Suppose that there is a point $p \in T$ such that the hyperplane $p + L$ intersects T in the point p only. Then if $S + T$ is convex, S is convex.*

Proof. Pick arbitrary $x_1, x_2 \in S$ and $\alpha \in (0, 1)$. Our aim is to show that the vector $\alpha x_1 + (1 - \alpha)x_2$ belongs to S.

Since $p + x_1, p + x_2 \in S + T$ and $S + T$ is convex, the vector

$$u = \alpha(p + x_1) + (1 - \alpha)(p + x_2) = p + \alpha x_1 + (1 - \alpha)x_2 \in S + T.$$

Pick $s \in S$ and $t \in T$ such that $u = s + t$ and observe that

$$t = u - s = p + (\alpha x_1 + (1 - \alpha)x_2 - s) \in p + L.$$

Since $p + L$ intersects T in the point p only, we conclude that $t = p$ which implies that

$$s = \alpha x_1 + (1 - \alpha)x_2.$$

Since $s \in S$, we are done. □

Let $L_A^\perp$ denote the orthogonal complement of L_A in $\mathbb{R}^n$ and let $\pi : \mathbb{R}^n \to L_A^\perp$ denote the orthogonal projection. (Then L_A is the kernel of π.) For every $p \in \mathbb{R}^n$ write $\bar{p} = \pi(p)$. Since $T = \mathcal{R}(\vec{\mu} \restriction B)$ is compact by Halmos [6, Lemma 11] (here we use that $\vec{\mu}$ is $\mathbb{R}^n$-valued), there is a vector $m \in T$ with the property that for every $b \in T$ we have $\|\bar{m}\| \geq \|\bar{b}\|$.

In the proofs of the following lemmas we will make use of the following triviality: if $x, y \in \mathbb{R}^n$, $\|x-y\| \leq \|x\|$ and $\|x+y\| \leq \|x\|$ then $y=0$. Indeed, simply observe that

$$2\|x\|^2 \geq \|x-y\|^2 + \|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Lemma 3.9. *Let $E \subseteq B$ be measurable such that $\vec{\mu}(E) \in m + L_A$. If $F \subseteq B \setminus E$ is measurable and $\vec{\mu}(F) \in L_A$ then $\vec{\mu}(F) = 0$.*

Proof. Striving for a contradiction, assume that $\vec{\mu}(F) \neq 0$. Since A is maximally 1-dimensional, $A \cup F$ is not 1-dimensional. There consequently exists a subset $H \subseteq F$ such that $a = \vec{\mu}(H) \notin L_A$. Since $\vec{\mu}(F) \in L_A$ it follows that $b = \vec{\mu}(F \setminus H) \notin L_A$. Observe that $\bar{a} \neq 0$ and that $\bar{b} = -\bar{a}$ because $a + b \in L_A$. Since $\bar{m}$ is maximal, $\vec{\mu}(E \cup H) \in T$ and $\vec{\mu}(E \cup H) \in a + m + L_A$, we get $\|\bar{m}\| \geq \|\bar{m} + \bar{a}\|$. It follows similarly that $\|\bar{m}\| \geq \|\bar{m} + \bar{b}\| = \|\bar{m} - \bar{a}\|$. We conclude that $\bar{a} = 0$, which is a contradiction. □

Lemma 3.10. *Let $E \subseteq B$ be measurable such that $\vec{\mu}(E) = m$. If $F \subseteq B$ is measurable and $\vec{\mu}(F) \in m + L_A$ then $\vec{\mu}(E \triangle F) = 0$.*

Proof. Write $p = \vec{\mu}(E \setminus F)$, $q = \vec{\mu}(E \cap F)$ and $r = \vec{\mu}(F \setminus E)$, respectively. Observe that $\bar{p} + \bar{q} = \bar{q} + \bar{r}$ from which it follows that $\bar{p} = \bar{r}$. Since $\bar{m}$ is maximal and $\vec{\mu}(E \cup F) = p + q + r \in T$ it follows that

$$\|\bar{m}\| \geq \|\bar{m} + \bar{r}\|.$$

It follows similarly that

$$\|\bar{m}\| \geq \|\bar{q}\| = \|(\bar{p} + \bar{q}) - \bar{r}\| = \|\bar{m} - \bar{r}\|.$$

This implies that $\bar{r} = 0$, i.e., $r \in L_A$. As a consequence, $\bar{q} = \bar{m}$. So $r = 0$ by Lemma 3.9. Since $\bar{p} = \bar{r}$ we also have $p \in L_A$. So another application of Lemma 3.9 gives $p = 0$. □

If $m \in S$ then $\mathcal{R}(\vec{\mu} \restriction B) \subseteq L_A$ and so $\vec{\mu}(B) = 0$ since A is maximal. As a consequence, $\mathcal{R}(\vec{\mu} \restriction A) = \mathcal{R}(\vec{\mu})$ is convex. If $m \notin S$ then the hyperplane $m + L_A$ intersects T in the point m only (Lemma 3.10). So the convexity of $\mathcal{R}(\vec{\mu} \restriction A)$ then follows from Lemma 3.8.

Observe that the just proved special case of Theorem 3.3 is all we need for the proof of Corollary 3.4.

At this point of the proof, let us explain why the above arguments do not work in the infinite-dimensional case. The reason is that $\mathcal{R}(\vec{\mu} \restriction B)$ need not be compact in general (Example 3.2). So it is not clear how to construct a vector such as the vector m above. We overcome this difficulty by constructing suitable approximations of $\vec{\mu}$ by $\mathbb{R}^n$-valued measures. As we remarked at the beginning of §2, we may assume without loss of generality that $L = \ell^2$.

We now use the just proved special case of Theorem 3.3 to prove the general case.

Let $\{\vec{e}_n\colon n \in \mathbb{N}\}$ be the standard orthonormal basis for ℓ^2. By performing a rotation if necessary, we may assume without loss of generality that L_A corresponds to the first coordinate axis of ℓ^2, i.e., the set

$$\{x \in \ell^2\colon (\forall n \geq 2)(x_n = 0)\}.$$

For every n let R_n denote the linear span of

$$\{\vec{e}_1, \ldots, \vec{e}_n\}.$$

So $L_A = R_1$. Finally, for every n let $\rho_n : \ell^2 \to R_n$ denote the orthogonal projection; ρ_n is defined as follows:

$$\rho_n(x) = (x_1, \ldots, x_n, 0, 0, \ldots) \quad (x \in \ell^2).$$

Define the measures $\vec{\sigma}_n : \mathcal{F} \to \ell^2$ by

$$\vec{\sigma}_n(B) = \rho_n(\vec{\mu}(B)) \quad (n \in \mathbb{N}).$$

Observe that $\vec{\sigma}_n$ can be identified with an $\mathbb{R}^n$-valued measure.

Let $n \in \mathbb{N}$. Observe that A is a 1-dimensional $\vec{\sigma}_n$-set since $\mathcal{R}(\vec{\sigma}_n \restriction A)$ is contained in L_A. But it is presumably not maximally 1-dimensional. By Lemma 2.2, there is a maximally 1-dimensional $\vec{\sigma}_n$-set that contains A.

Lemma 3.11. *Let $n \in \mathbb{N}$ and let A_n be a maximally 1-dimensional $\vec{\sigma}_n$-set that contains A. Then there is a maximally 1-dimensional $\vec{\sigma}_{n+1}$-set A_{n+1} with $A \subseteq A_{n+1} \subseteq A_n$.*

Proof. By the above there exists a maximally 1-dimensional $\vec{\sigma}_{n+1}$-set C that contains A. Observe that $\mathcal{R}(\vec{\sigma}_{n+1} \restriction C) \cup \mathcal{R}(\vec{\sigma}_n \restriction A_n) \subseteq L_A$. Put $F = C \setminus A_n$. Assume first that F is not $\vec{\sigma}_n$-negligible. Then there exists a measurable subset $G \subseteq F$ with $\vec{\sigma}_n(G) \neq 0$. Since A_n is maximally 1-dimensional with respect to $\vec{\sigma}_n$, $A_n \cup G$ is not 1-dimensional and there consequently exists a measurable set $H \subseteq G$ such that $\vec{\sigma}_n(H) \notin L_A$. In other words, $\vec{\sigma}_n(H)$ is a vector having a nonzero coordinate in one of the dimensions 2 through n. But the vector consisting of the first n coordinates of $\vec{\sigma}_{n+1}(H)$ is $\vec{\sigma}_n(H)$. As a consequence, $\vec{\sigma}_{n+1}(H) \notin L_A$ as well. But this is impossible since $H \subseteq C$ and $\mathcal{R}(\vec{\sigma}_{n+1} \restriction C) \subseteq L_A$. So F is $\vec{\sigma}_n$-negligible. We claim that F is also $\vec{\sigma}_{n+1}$-negligible. To this end, let $G \subseteq F$ be an arbitrary measurable set. Since $\vec{\sigma}_{n+1}(G) \in L_A$ it is a vector of the form $(p, 0, \ldots, 0)$. But $\vec{\sigma}_n(G)$ is equal to the vector $\vec{\sigma}_{n+1}(G)$ with its $n+1$-th coordinate deleted. As a consequence, $\vec{\sigma}_n(G) = 0$ implies that $\vec{\sigma}_{n+1}(G) = 0$. So $A_{n+1} = C \setminus F = C \cap A_n$ is as required (recall that maximally 1-dimensional sets are determined up to a negligible set). $\square$

By Lemma 3.11 there exists for every n a maximally 1-dimensional $\vec{\sigma}_n$-set A_n such that

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \supseteq A;$$

let $B_n = \Omega \setminus A_n$.

Now put $\hat{A} = \bigcap_{n=1}^{\infty} A_n$. Then $\hat{A}$ is measurable and we claim that $E = \hat{A} \setminus A$ is a $\vec{\mu}$-nullset. Assume the contrary. Then, since A is maximally 1-dimensional, there is a measurable subset $F \subseteq E$ such that $p = \vec{\mu}(F) \notin L_A$. Since $\lim_{n\to\infty} \rho_n(p) = p$, there exists $n \in \mathbb{N}$ such that $\vec{\sigma}_n(F) = \rho_n(\vec{\mu}(F)) \notin L_A$. This contradicts the fact that A_n is a 1-dimensional $\vec{\sigma}_n$-set. We next claim that $\mathcal{R}(\vec{\mu} \restriction A) = \bigcap_{n=1}^{\infty} \mathcal{R}(\vec{\sigma}_n \restriction A_n)$, which is as required since for every n, $\mathcal{R}(\vec{\sigma}_n \restriction A_n)$ is convex (here we use the just proved special case of Theorem 3.3), and the intersection of an arbitrary family of convex sets is again convex. First observe that $\mathcal{R}(\vec{\mu} \restriction A) \subseteq \bigcap_{n=1}^{\infty} \mathcal{R}(\vec{\sigma}_n \restriction A_n)$. Next, fix an arbitrary vector $p \in \bigcap_{n=1}^{\infty} \mathcal{R}(\vec{\sigma}_n \restriction A_n)$. For every n, pick a measurable set $E_n \subseteq A_n$ with $\vec{\sigma}_n(E_n) = p$. For every n, put $F_n = E_n \cap A$ and $S_n = E_n \setminus A$, respectively. Since $\mathcal{R}(\vec{\mu} \restriction A)$ is compact (Lemma 3.7), we may assume without loss of generality that the sequence $(\vec{\mu}(F_n))_n$ converges, say to $\vec{\mu}(F)$, where $F \subseteq A$. Since

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} A_n = \hat{A} \supseteq A$$

and $\vec{\mu}(\hat{A} \setminus A) = 0$ it follows that

$$\lim_{n\to\infty} \vec{\mu}(A_n \setminus A) = 0.$$

Since $S_n \subseteq A_n \setminus A$ for every n, we conclude that $\lim_{n\to\infty} \vec{\mu}(S_n) = 0$. As a consequence,

$$p = \lim_{n\to\infty} \vec{\mu}(E_n) = \lim_{n\to\infty} \vec{\mu}(F_n) + \lim_{n\to\infty} \vec{\mu}(S_n) = \vec{\mu}(F),$$

i.e., $p \in \mathcal{R}(\vec{\mu} \restriction A)$, as desired.

We finish this section by showing that the assumption of boundedness of the measures under consideration is essential (in Theorem 3.3 as well as Corollary 3.4). This is shown by the following example.

Example 3.12. Consider on $\mathbb{Z}_- \cup \mathbb{R}^+$ with the σ-algebra of the Borel sets the following two measures:

μ_1 is counting measure on $\mathbb{Z}_-$, Lebesgue measure on $\mathbb{R}^+$;

μ_2 is counting measure on $\mathbb{Z}_-$, Cantor measure on $\mathbb{R}^+$ (see below).

Construct $\mu_2 \restriction \mathbb{R}^+$ as follows. Let K_0 be the standard Cantor middle third set on $\mathbb{I}$, and let $F_0 : \mathbb{I} \to \mathbb{I}$ be the standard Cantor function, defined by $F_0(x) = \frac{1}{2}$ if $\frac{1}{3} < x < \frac{2}{3}$, $F_0(x) = \frac{1}{4}$ if $\frac{1}{9} < x < \frac{2}{9}$ etcetera, and by continuous extension on the points in K_0. Now define $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(x) = i + F_0(x - i)$ if $x \in [i, i+1]$. Then F is continuous, and we define μ_2 as the measure on the Borel sets obtained by extension from $\mu_2((a, b]) = F(b) - F(a)$. Clearly $\vec{\mu}_1 \restriction \mathbb{R}^+ \perp \mu_2 \restriction \mathbb{R}^+$. Now put

$$\vec{\mu} = (\mu_1, \mu_2).$$

Then obviously, $\mathcal{R}(\vec{\mu}) = \mathbb{R}^+ \times \mathbb{R}^+$. We claim that $\mathbb{Z}_-$ is a maximally 1-dimensional set. Clearly

$$\mathcal{R}(\vec{\mu} \restriction \mathbb{Z}_-) = \{(n, n) \colon n \in \mathbb{N}\} \subset L := \{(x, x) \colon x \geq 0\}.$$

Arguing by contradiction, assume $A \supseteq \mathbb{Z}_-$ is a measurable set such that $\vec{\mu}(A \setminus \mathbb{Z}_-) \neq 0$ and $\mathcal{R}(\vec{\mu} \restriction B) \subseteq L$ for every measurable $B \subseteq A$. Let us denote the Cantor set on $\mathbb{R}^+$ by K. (I.e., K is the union of the standard Cantor middle third sets on each interval $[i, i+1]$, $i = 0, 1, 2, \ldots$.) The set $\mathbb{R}^+ \setminus K$ will be denoted by K^c. Now let $I = (0, x)$ and consider

$$\vec{\mu}(A \cap I) = \vec{\mu}(A \cap I \cap K) + \vec{\mu}(A \cap I \cap K^c).$$

As $A \cap I$, $A \cap I \cap K$ and $A \cap I \cap K^c$ are measurable subsets of A, we should have that

$$\vec{\mu}(A \cap I) \in L, \quad \vec{\mu}(A \cap I \cap K) \in L \quad \text{and} \quad \vec{\mu}(A \cap I \cap K^c) \in L.$$

However, $\vec{\mu}(A \cap I \cap K) = (0, \mu_2(A \cap I \cap K))$, $\vec{\mu}(A \cap I \cap K^c) = (\mu_1(A \cap I \cap K^c), 0)$, which are in L if and only if $\mu_2(A \cap I \cap K) = 0$ and $\mu_1(A \cap I \cap K^c) = 0$, and, moreover,

$$\vec{\mu}(A \cap I) = (\mu_1(A \cap I), \mu_2(A \cap I)) = (\mu_1(A \cap I \cap K^c), \mu_2(A \cap I \cap K)).$$

So $\vec{\mu}(A \cap I) = 0$. This holds for each $x > 0$. So $\vec{\mu}(A \cap \mathbb{R}^+) = 0$, which is a contradiction.

Note that $\mathcal{R}(\vec{\mu})$ is convex, while $\mathcal{R}(\vec{\mu} \restriction \mathbb{Z}_-) = \{(n, n) \colon n \in \mathbb{N}\}$ is not convex. As $\mathbb{Z}_-$ is a maximally 1-dimensional set we see that our main results fail if μ_1 and μ_2 are not finite measures.

4. A CRITERION FOR CONVEXITY OF THE ONE-DIMENSIONAL PIECES

As we remarked in §2, the measure $\vec{\mu}$ on one of the pieces $\{A_1, A_2, \ldots\}$ is in essence $\mathbb{R}$-valued.

For nonnegative measures it is easy to detect when their ranges are convex. Let σ be an arbitrary nonnegative countably additive finite measure on $(\Omega, \mathcal{F})$. Then Ω contains at most a countable number of σ-atoms, say $F_1, F_2, \ldots$. We choose the ordering of the atoms in such a way that $\sigma(F_1) \geq \sigma(F_2) \geq \ldots$. Then $\mathcal{R}(\sigma)$ is convex if and only if for every $n \in \mathbb{N}$:

$$\sigma(F_n) \leq \frac{1}{2}\left(\sigma(\Omega) - \sum_{r=1}^{n-1} \sigma(F_r)\right).$$

This result is essentially Rényi [9, p. 80, exercise 48]. For a proof containing all details, see Gouweleeuw [5, Theorem 1.25].

The case of an $\mathbb{R}$-valued measure will be reduced to the case of a nonnegative measure, as follows.

Lemma 4.1. *If μ is an $\mathbb{R}$-valued measure then $\mathcal{R}(\mu)$ is convex if and only if $\mathcal{R}(|\mu|)$ is convex.*

Proof. Let $\mu = \mu_+ - \mu_-$ be the Jordan decomposition of the measure μ. Then its total variation $|\mu|$ satisfies $|\mu| = \mu_+ + \mu_-$. Let C_+ and C_- be measurable sets such that $C_+ \cap C_- = \emptyset$, $C_+ \cup C_- = \Omega$, C_+ is the support of μ_+, and C_- is the support of μ_-.

Suppose that $\mathcal{R}(|\mu|)$ is convex, and assume that $\mathcal{R}(\mu)$ is not convex. Then there are numbers $\alpha \notin \mathcal{R}(\mu)$ and $\beta \in \mathcal{R}(\mu)$ such that $0 < \alpha < \beta$ or $\beta < \alpha < 0$. Without loss of generality assume the former takes place. We can take for β the number $\mu_+(C_+)$, as $\mu(A) \leq \mu_+(C_+)$ for any measurable A. Consider $\gamma := \alpha + \mu_-(C_-)$. As $0 < \gamma < \mu_+(C_+) + \mu_-(C_-) = |\mu|(\Omega)$ and $\mathcal{R}(|\mu|)$ is convex we have $\gamma \in \mathcal{R}(|\mu|)$. So there are sets $C \subseteq C_+$ and $D \subseteq C_-$ such that

$$\gamma = \mu_+(C) + \mu_-(D) = \alpha + \mu_-(C_-).$$

Therefore,

$$\begin{aligned}\alpha &= \mu_+(C) + \mu_-(D) - \mu_-(C_-) = \mu_+(C) - \mu_-(C_- \setminus D)\\ &= \mu(C \cup (C_- \setminus D)) \in \mathcal{R}(\mu),\end{aligned}$$

which is a contradiction. So $\mathcal{R}(\mu)$ is convex.

Conversely, assume $\mathcal{R}(\mu)$ is convex and $\mathcal{R}(|\mu|)$ is not convex. Then there is a number $\alpha \notin \mathcal{R}(|\mu|)$ such that $0 < \alpha < |\mu|(\Omega) = \mu_+(C_+) + \mu_-(C_-)$. Consider $\gamma := \alpha - \mu_-(C_-)$. Obviously $-\mu_-(C_-) < \gamma < \mu_+(C_+)$. As $\mathcal{R}(\mu)$ is convex, we have $\mathcal{R}(\mu) = [-\mu_-(C_-), \mu_+(C_+)]$. So there is a measurable set A such that $\mu(A) = \gamma$. But then

$$\gamma = \mu_+(A \cap C_+) - \mu_-(A \cap C_-) = \alpha - \mu_-(C_-).$$

So

$$\alpha = \mu_+(A \cap C_+) + \mu_-(C_- \setminus A) = |\mu|((A \cap C_+) \cup (C_- \setminus A)).$$

This is a contradiction, so $\mathcal{R}(|\mu|)$ is convex. $\square$

Next, we consider the relation between the atoms of μ and $|\mu|$.

Lemma 4.2. *A measurable set $F \subseteq \Omega$ is an atom of μ if and only if it is an atom of $|\mu|$. Moreover, in that case $|\mu|(F) = |\mu(F)|$.*

Proof. Recall that for measurable sets A

$$|\mu|(A) = \max\left\{\sum_{i=1}^{\infty} |\mu(A)| : \{A_i\}_{i=1}^{\infty} \text{ is a measurable partition of } A\right\}.$$

Suppose F is an atom of μ. As F is an atom of μ, for any partition $\{A_i\}_{i=1}^{\infty}$ of F we have $\sum_{i=1}^{\infty} |\mu(A_i)| = |\mu(F)|$, as all but one of the numbers $\mu(A_i)$ are zero, and the non-zero number is $\mu(F)$. So, for an atom of μ we have $|\mu|(F) = |\mu(F)|$. Now the same argument shows that for any measurable set $E \subseteq F$ we have $|\mu|(E) = |\mu(E)|$. As $\mu(E)$ is either zero or $\mu(F)$ we obtain that $|\mu|(E)$ is either zero or $|\mu(F)|$. So F is indeed an atom of $|\mu|$.

Conversely, suppose F is an atom of $|\mu|$, and again take $E \subseteq F$ measurable. Then without loss of generality we may take $|\mu|(E) = |\mu|(F)$ and $|\mu|(F \setminus E) = 0$. As $|\mu(F \setminus E)| \leq |\mu|(F \setminus E) = 0$ we have $\mu(F \setminus E) = 0$. This gives $\mu(E) = \mu(F)$, so F is an atom of μ. $\square$

Combining the above results with the condition of Rényi mentioned earlier, we obtain the following.

Proposition 4.3. *Let μ be an $\mathbb{R}$-valued measure on $(\Omega, \mathcal{F})$. Then Ω contains at most a countable number of atoms $F_1, F_2, \ldots$. We choose the ordering of the atoms in such a way that $|\mu(F_1)| \geq |\mu(F_2)| \geq \cdots$. Then $\mathcal{R}(\mu)$ is convex if and only if for every $n \in \mathbb{N}$:*

$$|\mu(F_n)| \leq \frac{1}{2}\left(|\mu|(\Omega) - \sum_{r=1}^{n-1} |\mu(F_r)|\right).$$

So it is rather trivial to verify in concrete situations whether condition $(*)$ is met. One has to check whether countably many $\mathbb{R}$-valued measures have convex ranges and that can be done by looking at their atoms.

5. CONVEXITY OF MATRIX-k-RANGES

In this section we only consider $\mathbb{R}^n$-valued measures. Let $V \subseteq \Omega$ be measurable and let $k \geq 2$. An *ordered measurable k-partition* of V is an ordered collection $V_1, \ldots, V_k$ of measurable subsets of V with $V_i \cap V_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{k} V_i = V$. Let $\Pi_k(V)$ denote the collection of all ordered measurable k-partitions of V and let $\Pi_k = \Pi_k(\Omega)$. In this section we are interested in the question when the *matrix-k-range*

$$\mathcal{MR}_k(\vec{\mu}) = \{(\vec{\mu}(P_1), \ldots, \vec{\mu}(P_k)) \in (\mathbb{R}^n)^k \colon (P_1, \ldots, P_k) \in \Pi_k\}$$

is convex. ($\mathcal{MR}(\vec{\mu})$ is called the matrix-k-range of $\vec{\mu}$ because each element of $(\mathbb{R}^n)^k$ can be identified with an $n \times k$-matrix.) Our results provide simpler proofs of results due to Gouweleeuw [5]. It is clear that without loss of generality we may assume that $k \geq 2$.

The results presented here generalize those in §3 (in the special case that $L = \mathbb{R}^n$) because $\mathcal{MR}_2(\vec{\mu})$ is convex if and only if $\mathcal{R}(\vec{\mu})$ is convex.

Our pattern of reasoning is similar to the one in §3. Let $A_0, A_1, \ldots$ be the partition of Ω constructed in §2. Dubins and Spanier [3] proved that the compactness and convexity of $\mathcal{MR}_k(\vec{\mu})$ follow from Lyapounov's theorem, provided that the measures $\mu_1, \ldots, \mu_n$ are nonatomic. So we have no problems with $\mathcal{MR}_k(\vec{\mu} \restriction A_0)$. That set is always compact and convex.
Assume that

$(**)$ for every $i \geq 1$, $\mathcal{MR}_k(\vec{\mu} \restriction A_i)$ is convex.

A straightforward verification shows that $\mathcal{MR}_k(\vec{\mu})$ is convex. So $(**)$ is a sufficient condition for the convexity of $\mathcal{MR}_k(\vec{\mu})$. Interestingly, this condition is also necessary.

As we remarked in §2, the measure $\vec{\mu}$ on one of the pieces $\{A_1, A_2, \ldots\}$ is in essence $\mathbb{R}$-valued. For $\mathbb{R}$-valued measures it is easy to detect when their matrix-k-ranges are convex.

Let σ be an arbitrary countably additive finite measure on $(\Omega, \mathcal{F})$. Then Ω

contains at most a countable number of σ-atoms, say $F_1, F_2, \ldots$. Again we choose the ordering of the atoms in such a way that $|\sigma(F_1)| \geq |\sigma(F_2)| \geq \cdots$. Then $\mathcal{MR}_k(\sigma)$ is convex if and only if for every $n \in \mathbb{N}$:

$$|\sigma(F_n)| \leq \frac{1}{k}\left(|\sigma|(\Omega) - \sum_{r=1}^{n-1} |\sigma(F_r)|\right).$$

See Rényi [9, p. 80, exercise 48] and Gouweleeuw [5, Theorem 1.25] for the case that σ is nonnegative. The general case is treated as in §4.

We now come to the interesting part of our considerations. Condition $(**)$ is also necessary for the convexity of $\mathcal{MR}_k(\vec{\mu})$. So assume that $\mathcal{MR}_k(\vec{\mu})$ is convex. For symmetry reasons it suffices to show that $\mathcal{MR}_k(\vec{\mu} \restriction A_1)$ is convex. We again put $A = A_1$ and $B = \Omega \setminus A$, respectively. Then $\mathcal{MR}_k(\vec{\mu}) = \mathcal{MR}_k(\vec{\mu} \restriction A) + \mathcal{MR}_k(\vec{\mu} \restriction B)$. We argue as in §3.

Observe that $S_k = \mathcal{MR}_k(\vec{\mu} \restriction A)$ is contained in the linear subspace

$$\hat{L}_A = \{(x_1, \ldots, x_k) \in (\mathbb{R}^n)^k \colon (\forall i \leq k)(x_i \in L_A)\}$$

of $(\mathbb{R}^n)^k$. Put $T_k = \mathcal{MR}_k(\vec{\mu} \restriction B)$. Then $S_k + T_k$ is convex and, as in §3, we prove S_k convex by applying Lemma 3.8.

We adopt the notation of §3. Let $b = \vec{\mu}(B)$ and put

$$p = (m, b - m, \underbrace{0, 0, \ldots, 0}_{k-2 \text{ times}}) \in (\mathbb{R}^n)^k.$$

We claim that the hyperplane $p + \hat{L}_A$ intersects T_k in the point p only. Observe that $p \in \mathcal{MR}_k(\vec{\mu})$ because if $E \subseteq B$ is such that $\vec{\mu}(E) = m$ then $E, B \setminus E, \emptyset, \ldots, \emptyset$ is in Π_k and

$$(\vec{\mu}(E), \vec{\mu}(B \setminus E), \vec{\mu}(\emptyset), \ldots, \vec{\mu}(\emptyset)) = p.$$

Assume that $F_1, \ldots, F_k$ is in $\Pi_k(B)$ and that

$$(\vec{\mu}(F_1), \ldots, \vec{\mu}(F_k)) \in p + \hat{L}_A.$$

Pick an element $(\xi_1, \ldots, \xi_k) \in \hat{L}_A$ such that

$$\begin{aligned}(\vec{\mu}(F_1), \ldots, \vec{\mu}(F_k)) &= (m, b - m, 0, \ldots, 0) + (\xi_1, \ldots, \xi_k) \\ &= (m + \xi_1, b - m + \xi_2, \xi_3, \ldots, \xi_k).\end{aligned}$$

Then $\vec{\mu}(F_1) = m + \xi_1$ and hence $\vec{\mu}(E \triangle F_1) = 0$ by Lemma 3.10. So without loss of generality we may assume that $E = F_1$. Observe that the sets $E, F_3, \ldots, F_k$ are pairwise disjoint and that $\vec{\mu}(F_i) \in L_A$ for every $i \geq 3$. So Lemma 3.9 implies that $\vec{\mu}(F_i) = 0$ for every $i \geq 3$. We conclude that $\vec{\mu}(F_2) = \vec{\mu}(B \setminus E) = b - m$, because $F_1, \ldots, F_k$ is in Π_k. We conclude that the hyperplane $p + \hat{L}_A$ intersects T_k in the point p only. So we are done by another application of Lemma 3.8.

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