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How many ω-bounded subgroups?

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Abstract

A topological space is said to be *ω-bounded* if each of its countable subsets has compact closure. It has been shown recently by Itzkowitz and Shakhmatov that for every compact Abelian group G of uncountable weight, and for every compact connected group G of uncountable weight, the set $\Omega(G)$ of dense ω-bounded subgroups of G satisfies $|\Omega(G)| \geqslant |G|$. These authors asked whether their estimate $|\Omega(G)| \geqslant |G|$ may be improved to $|\Omega(G)| = 2^{|G|}$ for some or all such G. In the present paper we answer this question affirmatively for all compact groups G which are either Abelian or connected and which satisfy in addition the condition $w(G) = (w(G))^{\omega}$. We show also that every compact group G with $w(G) \geqslant \log((2^{\mathfrak{c}})^+)$ satisfies $|\Omega(G)| > 2^{\mathfrak{c}}$. © 1997 Elsevier Science B.V.

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Respectfully dedicated to the memory of Maarten Maurice, 1935–1996

1. Introduction and notation

The least infinite cardinal is denoted ω, and $\mathfrak{c} = 2^{\omega}$. For a cardinal α and a set X, we write

$$[X]^{\alpha} = \{A \subseteq X\colon\ |A| = \alpha\} \quad \text{and} \quad [X]^{\leqslant\alpha} = \{A \subseteq X\colon\ |A| \leqslant \alpha\}.$$

By a *Tychonoff* space we mean a completely regular, Hausdorff space. We say that a (Hausdorff) space is *zero-dimensional* if its topology admits a base of open-and-closed

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subsets. We say that a space X is *ω-bounded* if the closure (in X) of each countably infinite subset of X is compact.

The topological groups we consider are assumed to satisfy the Hausdorff separation axiom; as is well known (cf. [8, 8.4]), this guarantees that they are Tychonoff spaces.

Given a group G and $A \subseteq G$, we denote by $\langle A\rangle$ the smallest subgroup of G containing A.

The set of dense, ω-bounded subgroups of a topological group G is denoted $\Omega(G)$.

The Abstract contains the statement of our principal result. In Section 2 we show for certain cardinals α the existence of weak-P-spaces X such that $w(X) = \alpha < 2^\alpha = |X|$, and we apply these considerations to prove in Section 3 our principal result; we give some related results in Section 4 and some unsolved problems in Section 5.

2. Weak-P-spaces of small weight

Definition 2.1. Let X be a Hausdorff space. Then

(a) X is a *P-space* if every union of countably many closed subsets of X is closed in X; and

(b) X is a *weak-P-space* if each of its countably infinite subsets is closed.

Lemma 2.2. *For infinite cardinals α and γ, the following conditions are equivalent:*

(a) *There is a Hausdorff, zero-dimensional, weak-P-space of cardinality γ and weight $\leqslant \alpha$;*

(b) *the compact space $\{0,1\}^\alpha$ contains a subspace X such that $|X| = \gamma$ and X is a weak-P-space (in the inherited topology); and*

(c) *the set $\{0,1\}^\alpha$ contains a set X with these properties: $|X| = \gamma$, and if $C \in [X]^\omega$ and $p \in X\backslash C$ then there is $\xi < \alpha$ such that $\pi_\xi(p) = p_\xi = 1$ and $\pi_\xi|C \equiv 0$.*

Proof. A set X as in (c) evidently has the property that each $C \in [X]^\omega$ is closed in the topology on X inherited from $\{0,1\}^\alpha$, so (c) $\Rightarrow$ (b). That (b) $\Rightarrow$ (a) is clear. To see that (a) $\Rightarrow$ (c) let $\mathcal{B}$ be a base of open-and-closed subsets of a weak-P-space X with $|X| = \gamma$ and $|\mathcal{B}| \leqslant \alpha$, and let e be the topological embedding $e : X \hookrightarrow \{0,1\}^{\mathcal{B}}$ defined by

$$e(x)_B = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \in X\backslash B. \end{cases}$$

Given C and p as in (c), since C is closed in X there is $B \in \mathcal{B}$ such that $p \in B \subseteq X\backslash C$ and it is then clear (identifying X with its homeomorph $e[X]$) that $p_B = 1$ and $\pi_B|C \equiv 0$, as required. $\square$

Notation 2.3. Let γ and α be infinite cardinals. Then

(a) *condition $P(\gamma,\alpha)$ holds* if there is a Tychonoff P-space of cardinality γ and weight $\leqslant \alpha$; and

(b) *condition w-$P(\gamma,\alpha)$ holds* if the conditions of Lemma 2.2 are satisfied.

Remark 2.4.

(a) Condition (c) of Lemma 2.2 differs formally from the statement that some family of α-many open-and-closed subsets of X distinguishes points and countable subsets of X in the obvious sense. It is important for our application in Theorem 3.2 below that explicitly $p_\xi = 1$, while $\pi_\xi|C \equiv 0$. When condition w-$P(2^\alpha, \alpha)$ holds then according to (a) the set $\{0,1\}^\alpha$ admits a weak-P-space topology of weight α, and according to (c) the set $\{0,1\}^\alpha$ contains a set X with the indicated separation-by-projection property. It is nowhere asserted, of course, that in this latter case one may choose for X the set $\{0,1\}^\alpha$ itself.

(b) As is noted in [7, Problem 4J], a cozero subset of a P-space is open-and-closed. Thus every Tychonoff P-space is zero-dimensional, so condition $P(\gamma, \alpha)$ implies condition w-$P(\gamma, \alpha)$.

Theorem 2.5. *If* $1 < \alpha = \alpha^\omega$ *then condition* $P(2^\alpha, \alpha)$ *holds.*

Proof. Let $X = \{0,1\}^\alpha$ with its usual product topology and base $\mathcal{B}$ (of cardinality α), let $P\mathcal{B} = \{\bigcap \mathcal{A}\colon \mathcal{A} \in [\mathcal{B}]^\omega\}$, and let PX denote the set X with the topology generated by $P\mathcal{B}$. Then $w(PX) \leqslant |P\mathcal{B}| = \alpha^\omega = \alpha$. Since the base $P\mathcal{B}$ is closed under countable intersections, every G_δ-subset of PX is open in PX and hence PX is a P-space. □

Remark 2.6.

(a) We do not know if the condition $\alpha = \alpha^\omega$ is necessary for either or both of the conditions $P(2^\alpha, \alpha)$ and w-$P(2^\alpha, \alpha)$ to hold. It should be noted in any event that these conditions both fail for $\alpha = \omega$. Indeed a Tychonoff weak-P-space X of weight ω is a metrizable space without nontrivial convergent sequences, hence is discrete, hence satisfies $|X| = w(X) = \omega$.

(b) The Singular Cardinals Hypothesis (hereafter: SCH), a well-known consequence of GCH, asserts that if $\alpha \geqslant \mathfrak{c}$ and $\mathrm{cf}(\alpha) > \omega$, then $\alpha^\omega = \alpha$. Assuming SCH it is immediate from Theorem 2.5 that $\mathrm{cf}(\alpha) = \omega$ for every cardinal $\alpha \geqslant \mathfrak{c}$ for which condition w-$P(2^\alpha, \alpha)$ fails. Let us notice now a weaker statement, proved however in ZFC alone without assuming SCH.

Theorem 2.7. *Let* α *be the least cardinal such that* $\alpha > \mathfrak{c}$ *and condition* w-$P(2^\alpha, \alpha)$ *fails. Then* $\mathrm{cf}(\alpha) = \omega$ *(and hence* $\alpha < \alpha^\omega$*).*

Proof. We assume otherwise, and for $\mathfrak{c} \leqslant \xi < \alpha$ we choose a base $\mathcal{S}_\xi$ for a Hausdorff weak-P-space topology on $\{0,1\}^\xi$ such that $|\mathcal{S}_\xi| \leqslant |\xi| < \alpha$; then for $C \in [\{0,1\}^\xi]^\omega$ and $p \in \{0,1\}^\xi \backslash C$ there is $S \in \mathcal{S}_\xi$ such that $p \in S \subseteq \{0,1\}^\xi \backslash C$. Departing temporarily from standard notation let π_ξ be the natural projection from $\{0,1\}^\alpha$ onto $\{0,1\}^\xi$, and set $\mathcal{S} = \{\pi_\xi^{-1}(S)\colon \mathfrak{c} \leqslant \xi < \alpha,\ S \in \mathcal{S}_\xi\}$. Then $|\mathcal{S}| \leqslant \alpha$, and we claim that the smallest topology on $\{0,1\}^\alpha$ in which the elements of $\mathcal{S}$ are declared open-and-closed is a weak-P-space topology. Indeed if $D \in [\{0,1\}^\alpha]^\omega$ and $q \in \{0,1\}^\alpha \backslash D$ then since $\mathrm{cf}(\alpha) > \omega$ there is $\xi < \alpha$ such that $\pi_\xi(q) \notin \pi_\xi[D]$; then with $p = \pi_\xi(q)$ and $C = \pi_\xi[D]$ and choosing $S \in \mathcal{S}_\xi$ such that $p \in S \subseteq \{0,1\}^\xi \backslash C$, we have $\pi_\xi^{-1}(S) \in \mathcal{S}$ and $q \in \pi_\xi^{-1}(S) \subseteq \{0,1\}^\alpha \backslash D$, as required. □

3. Counting the number of ω-bounded subgroups

Following Itzkowitz and Shakhmatov [14], we say that a compact group G is *product-like* if there is a continuous surjective homomorphism $\phi : G \twoheadrightarrow \prod_{\xi<\alpha} K_\xi$ with $\alpha = w(G)$ and with each K_ξ a nontrivial, metrizable (compact) group. While it is known (see the references cited below) that every compact group of uncountable weight which is either Abelian or connected is product-like, not every compact group of uncountable weight is product-like: one finds in [3, 4.10(d)] for each $\alpha \geqslant \omega$ examples of compact groups $G = G(\alpha)$ of weight α, which may be chosen to be totally disconnected or to satisfy $|G/C| = 2$ (with C the component of the identity in G), which admit no homomorphism onto any group $K_0 \times K_1$ with $|K_i| > 1$.

Lemma 3.1. *Let X be a compact space and $A \subseteq X$, and set*

$$\omega(A) = \bigcup \{\overline{D}^X \colon D \in [A]^\omega\}.$$

Then

(a) *$\omega(A)$ is the smallest ω-bounded subset of X containing A; and*

(b) *if X is a topological group and A a subgroup, then $\omega(A)$ is a subgroup of X.*

Proof. (a) The existence of a smallest ω-bounded subspace of X containing A is immediate from the fact that the intersection of any family of ω-bounded subsets of X is again ω-bounded. Evidently any ω-bounded superset of A contains $\omega(A)$, so it is enough to show that $\omega(A)$ is ω-bounded. This is clear: if $E \in [\omega(A)]^\omega$ and for each $x \in E$ we choose $D(x) \in [A]^\omega$ such that $x \in \overline{D(x)}^X$, then with $D = \bigcup_{x\in E} D(x)$ we have $D \in [A]^\omega$ and $\overline{E}^X \subseteq \overline{D}^X \subseteq \omega(A)$.

(b) Since clearly every $x \in \omega(A)$ satisfies $x^{-1} \in \omega(A)$, it is enough to show that $\omega(A)$ is closed under multiplication. For every topological group G and subsets E and F of G we have $\overline{E}^G \cdot \overline{F}^G \subseteq \overline{EF}^G$. Thus in the present case, taking $x_i \in \overline{D_i}^X \subseteq \omega(A)$ with $D_i \in [A]^\omega$ $(i = 0, 1)$ and setting $D = D_0 \cdot D_1$, we have $D \in [A]^\omega$ and $x_0x_1 \in \overline{D}^X \subseteq \omega(A)$, as required. □

Theorem 3.2. *Let α and γ satisfy condition w-$P(\gamma,\alpha)$ with $\alpha > \omega$ and let $K = \prod_{\xi<\alpha} K_\xi$ with each K_ξ a nondegenerate compact group. Then*

(a) *K admits 2^γ-many ω-bounded subgroups; and*

(b) *if there is a (fixed) group F such that each $K_\xi = F$ $(\xi < \alpha)$, then the subgroups of* (a) *may be chosen dense in K—that is, $|\Omega(K)| \geqslant 2^\gamma$.*

Proof. (a) Let 0_ξ denote the identity of K_ξ, let $0_\xi \neq k_\xi \in K_\xi$, and (slightly abusing notation) find

$$X \subseteq \{0,1\}^\alpha = \prod_{\xi<\alpha} \{0_\xi, k_\xi\} \subseteq K$$

such that $|X| = \gamma$ and X has the property of Lemma 2.2(c): if $C \in [X]^\omega$ and $p \in X \backslash C$ then there is $\xi < \alpha$ such that $p_\xi = k_\xi$ and $x_\xi = 0_\xi$ for each $x \in C$. For $A \subseteq X$ let

$$K(A) = \omega(\langle A\rangle) = \bigcup\{\overline{D}^K\colon\ D \in [\langle A\rangle]^\omega\};$$

then by Lemma 3.1 each $K(A)$ is an ω-bounded subgroup of K, so to complete the proof of (a) it is enough to show for $A \subseteq X$ that $K(A) \cap X = A$. We will show that if $p \in X$ satisfies $p \in \overline{D}^K \subseteq K(A)$ with $D \in [\langle A\rangle]^\omega$, then $p \in A$. Since each $x \in D$ satisfies $x \in \langle C_x\rangle$ for some finite $C_x \subseteq A$, there is a countable subset $C = \bigcup_{x\in D} C_x$ of A such that $p \in \overline{\langle C\rangle}$. If $p \notin C$ then there is $\xi < \alpha$ such that $p_\xi = k_\xi$ and $x_\xi = 0_\xi$ for each $x \in C$, and for such ξ we have $x_\xi = 0_\xi$ for each $x \in \langle C\rangle$ and hence for each $x \in \overline{\langle C\rangle}$, so $0_\xi \neq k_\xi = p_\xi = 0_\xi$. This contradiction shows $p \in C \subseteq A$.

(b) The groups $K = F^\alpha$ and $K^\alpha = (F^\alpha)^\alpha$ are isomorphic and homeomorphic as topological groups. Let Δ denote the diagonal copy of $K = F^\alpha$ inside the group K^α. Since Δ and F^α are isomorphic and homeomorphic as topological groups there is by part (a) applied to Δ a faithfully indexed family $\{\Delta(\eta)\colon\ \eta < 2^\gamma\}$ of ω-bounded subgroups of Δ. Let Σ denote the Σ-product of K^α—that is, let

$$\Sigma = \{x \in K^\alpha\colon\ |\{\xi < \alpha\colon\ x_\xi \neq 0\}| \leqslant \omega\}$$

—note that Σ is a normal subgroup of K^α, and for $\eta < 2^\gamma$ let $D(\eta) = \langle \Sigma + \Delta(\eta)\rangle = \Sigma + \Delta(\eta)$. Since Σ and $\Delta(\eta)$ are ω-bounded the product space $\Sigma \times \Delta(\eta)$ is ω-bounded; hence its continuous image $D(\eta)$ is an ω-bounded subgroup of K.

It is clear from $\alpha > \omega$ that if p and q are distinct points of Δ then $(p+\Sigma) \cap (q+\Sigma) = \emptyset$. It follows that the family $\{D(\eta)\colon\ \eta < 2^\gamma\}$ is faithfully indexed: if η and ζ are different ordinals less than 2^γ and (say) that there is $p \in \Delta(\eta)\backslash\Delta(\zeta)$, then $p \in D(\eta)$ but $p \notin \Sigma + \Delta(\zeta) = D(\zeta)$. □

Remark 3.3. Every compact group G of weight $\alpha > \omega$ which is either Abelian or connected is product-like in the sense of Itzkowitz and Shakhmatov [14] given above; see for example [6, 5.5] or [14, 1.11] for the Abelian case, [6, Proof of 6.5] or [13] for the connected case. In the present instance, in order that we can apply Theorem 3.2 to find 2^{2^α}-many ω-bounded dense subgroups of G, we need not only a continuous epimorphism $\psi\colon G \twoheadrightarrow \prod_{i\in I} F_i$ with $|F_i| > 1$ and $|I| = \alpha$ but a stronger condition, namely $\psi\colon G \twoheadrightarrow F^\alpha$ with $|F| > 1$. The above-cited results from [6] give several instances in which this can be achieved, among them $\mathrm{cf}(\alpha) > \omega$. Since the condition $\alpha = \alpha^\omega$, a hypothesis of Theorem 2.5 above, yields $\mathrm{cf}(\alpha) > \omega$, we have from [6] (or from [13,14]) the following statement.

Lemma 3.4. *Let G be an infinite compact group such that $w(G) = \alpha = \alpha^\omega$ and either G is Abelian or G is connected. Then there are a (metrizable) group F with $|F| > 1$ and a continuous surjective homomorphism $\psi\colon G \twoheadrightarrow F^\alpha$.*

Theorem 3.5. *Let $1 < \alpha = \alpha^\omega$ and let G be a compact group, either Abelian or connected, such that $w(G) = \alpha$. Then $|\Omega(G)| = 2^{2^\alpha}$.*

Proof. With ψ and F as in Lemma 3.4, from Theorems 2.5 and 3.2(b) we have that $|\Omega(F^\alpha)| \geqslant 2^{2^\alpha}$. Since ψ is an open map [8, 5.29] and each $D \in \Omega(F^\alpha)$ is dense in F^α, each group $\psi^{-1}(D)$ is dense in G; since G is compact and each $D \in \Omega(F^\alpha)$ is ω-bounded in F^α, each group $\psi^{-1}(D)$ is ω-bounded in G. This shows $|\Omega(G)| \geqslant 2^{2^\alpha}$. The inequality $|\Omega(G)| \leqslant 2^{2^\alpha}$ is clear, since $\Omega(G) \subseteq \mathcal{P}(G)$ and $|G| = 2^{w(G)} = 2^\alpha$ (cf. [9, 28.58(c)]). □

Remark 3.6. In earlier works [1,4] we have considered (in a fixed topological group G) the concept of a family $\{H_i\colon i \in I\}$ of dense subgroups which is *almost disjoint* in the sense that $H_i \cap H_j = \{0\}$ for distinct $i, j \in I$, or even *independent* in the sense that

$$H_i \cap \Bigl\langle \bigcup_{j\in I,\ j\neq i} H_j \Bigr\rangle = \{0\}.$$

This notion is pursued vigorously in [13] in the context of dense pseudocompact subgroups. It is worth noting that in the present context the dense ω-bounded subgroups defined in Corollary 3.3 cannot satisfy such disjointness conditions. Our next theorem, arising from [13, 1.13] and generalizing it, makes this statement specific; see also Hodel [10, 3.3] for an earlier, related argument.

Theorem 3.7. *Let $\alpha \geqslant \omega$, let X be a Tychonoff space with $w(X) \leqslant \alpha$, and let $\{D_\eta\colon \eta < \gamma\}$ be a faithfully indexed family of dense subsets of X with $\mathrm{cf}(\gamma) > 2^\alpha$. Then there is $A \in [\gamma]^\gamma$ such that $\bigcap_{\eta\in A} D_\eta$ is dense in X.*

Proof. Each D_η meets each nonempty (basic) open set so for $\eta < \gamma$ there is $E_\eta \subseteq D_\eta$ such that E_η is dense in X and $|E_\eta| \leqslant w(X) \leqslant \alpha$—that is, $E_\eta \in [X]^{\leqslant\alpha}$. Since $|[X]^{\leqslant\alpha}| \leqslant (2^\alpha)^\alpha = 2^\alpha$ and $\mathrm{cf}(\gamma) > 2^\alpha$, there are $E \in [X]^{\leqslant\alpha}$ and $A \in [\gamma]^\gamma$ such that $E_\eta = E$ for all $\eta \in A$. □

Since in any space X the intersection of (any family of) ω-bounded subsets is again ω-bounded, we can amalgamate Theorems 3.5 and 3.7 as follows.

Theorem 3.8. *Let $\alpha \geqslant \omega$ and let G be a compact group, either Abelian or connected, such that $w(G) = \alpha^\omega$. Then*

(a) $|\Omega(G)| = 2^{2^{(\alpha^\omega)}}$; *and*

(b) *if $\mathcal{D} \subseteq \Omega(G)$ satisfies $|\mathcal{D}| = \gamma$ with $\mathrm{cf}(\gamma) > 2^{(\alpha^\omega)}$ (for example, if $\gamma = (2^{(\alpha^\omega)})^+$ or $\gamma = 2^{2^{(\alpha^\omega)}}$), then there is $\mathcal{E} \subseteq \mathcal{D}$ such that $|\mathcal{E}| = \gamma$ and $\bigcap \mathcal{E} \in \Omega(G)$.* □

4. Concerning compact groups of large weight

It is known that

(a) every pseudocompact group of uncountable weight contains a proper dense subgroup [5];

(b) every Abelian, zero-dimensional pseudocompact group of uncountable weight contains a proper, dense, pseudocompact subgroup [3]; and

(c) every pseudocompact Abelian group G such that $|G| > \mathfrak{c}$ or $\omega_1 \leqslant w(G) \leqslant \mathfrak{c}$ has a proper, dense, pseudocompact subgroup [2].

Since the general question—does every pseudocompact group G of uncountable weight contain a proper, dense, pseudocompact subgroup?—remains unanswered, even when G is assumed compact and Abelian, the following theorems of Itzkowitz and Shakhmatov [14] assume particular interest. It will be noted that their hypothesized compact groups are not assumed to enjoy any product-like structure.

Theorem 4.1 [14].

(a) *Every compact group G with $|G| > \mathfrak{c}$ has a proper, dense, countably compact subgroup; and*

(b) *if $2^{\omega_1} > \mathfrak{c}$ then every compact group with uncountable weight has a proper, dense, countably compact subgroup.*

Informally, then, every sufficiently large compact group has a proper, dense countably compact subgroup. In what follows, using techniques suggested in part by an argument in [14], we give similar results concerning the existence of dense ω-bounded subgroups. For our stronger conclusions we need a stronger version of "sufficiently large"; thus our results do not imply Theorem 4.1. We note explicitly that in Theorem 4.2 and Corollary 4.3 below no algebraic or topological properties are imposed on the groups considered; in particular, a product-like structure is not assumed.

Theorem 4.2. *Let κ, λ and α satisfy $\log\kappa \leqslant \lambda$, $\kappa \leqslant \alpha$, and $\lambda^\omega \cdot 2^{\mathfrak{c}} < 2^\kappa$, and let G be a compact group with $w(G) \geqslant \alpha$. Then $\Omega(G)$ contains a set $\mathcal{W}$ which is isomorphic as an ordered set to the cardinal number $(\lambda^\omega \cdot 2^{\mathfrak{c}})^+$.*

Proof. It is enough to find a continuous homomorphism $\psi : G \twoheadrightarrow F$ onto a group F such that $\Omega(F)$ contains such a well-ordered set $\mathcal{W}$; for then, just as in the proof of Theorem 3.5, the family $\{\psi^{-1}(W)\colon W \in \mathcal{W}\}$ is as required in G.

According to [14] it is a "part of folklore", which first came to our attention explicitly in work of Shakhmatov [15], that for every compact group K and $\omega \leqslant \gamma \leqslant w(K)$ there is a closed, normal subgroup H of K such that $w(K/H) = \gamma$. Taking here $K = G$ and $\gamma = \kappa$ and choosing such H with $w(G/H) = \kappa$, and using $d(G/H) = \log(w(G/H)) = \log\kappa$ (cf. [11]) and $|G/H| = 2^{w(G/H)} = 2^\kappa$ (cf. [9, 28.58(c)]), we have from $\log\kappa \leqslant \lambda < 2^\kappa$ a dense subgroup A of G/H such that $|A| = \lambda$. Referring to Lemma 3.1 and writing $W_0 = \omega(A)$ we have $W_0 \in \Omega(G/H)$ and $|W_0| \leqslant \lambda^\omega \cdot 2^{\mathfrak{c}} < |G/H|$. We continue recursively, supposing that $\xi < (\lambda^\omega \cdot 2^{\mathfrak{c}})^+$ and that for $\eta < \xi$ groups $W_\eta \in \Omega(G/H)$ have been defined so that the map $\eta \to W_\eta$ is an order-isomorphism from ξ into $\Omega(G/H)$, and with each $|W_\eta| \leqslant \lambda^\omega \cdot 2^{\mathfrak{c}}$. If ξ is a limit ordinal, set $W_\xi = \bigcup_{\eta<\xi} W_\eta$; if $\xi = \zeta + 1$ choose $x \in (G/H)\backslash W_\zeta$ and set $W_\xi = \omega(\langle W_\zeta \cup \{x\}\rangle)$. This definition of $W_\xi \in \Omega(G/H)$ extends the order-isomorphism and satisfies $|W_\xi| \leqslant \lambda^\omega \cdot 2^{\mathfrak{c}}$. It is then clear, taking

$F = G/H$ and $\psi: G \twoheadrightarrow G/H = F$ the canonical homomorphism, that the family $\mathcal{W} := \{W_\xi\colon \xi < (\lambda^\omega \cdot 2^{\mathfrak{c}})^+\}$ is as required in $\Omega(F)$. □

Corollary 4.3. *Let $\alpha \geqslant \log((2^{\mathfrak{c}})^+)$ and let G be a compact group with $w(G) = \alpha$. Then*
(a) $|\Omega(G)| > 2^{\mathfrak{c}}$; *and*
(b) *if every $\lambda < 2^\alpha$ satisfies $\lambda^\omega < 2^\alpha$ then $|\Omega(G)| \geqslant |G|$.*

Proof. (a) Take $\kappa = \lambda = \log((2^{\mathfrak{c}})^+)$ in Theorem 4.2.
(b) It is enough to show that every λ such that $2^{\mathfrak{c}} \leqslant \lambda < 2^\alpha$ and $\alpha \leqslant \lambda < 2^\alpha$ satisfies $|\Omega(G)| \geqslant \lambda^+$, for then $|\Omega(G)| \geqslant 2^\alpha = |G|$. Given such λ this follows from Theorem 4.2 upon taking $\kappa = \alpha$. □

5. Some remaining questions

We have shown that conditions $P(2^\alpha, \alpha)$ and w-$P(2^\alpha, \alpha)$ hold when $1 < \alpha = \alpha^\omega$, that they fail for $\alpha = \omega$, and that the least cardinal $\alpha > \mathfrak{c}$ where w-$P(2^\alpha, \alpha)$ fails satisfies $\mathrm{cf}(\alpha) > \omega$. This exhausts our knowledge about these properties. In particular, we did not settle the following questions.

Question 5.1. Do conditions $P(2^\alpha, \alpha)$ and w-$P(2^\alpha, \alpha)$ hold for all cardinals $\alpha > \omega$? For all cardinals $\alpha \geqslant \mathfrak{c}$? For all cardinals α such that $\mathrm{cf}(\alpha) > \omega$?

Question 5.2. Are conditions $P(2^\alpha, \alpha)$ and w-$P(2^\alpha, \alpha)$ equivalent?

We showed $|\Omega(G)| = 2^{2^\alpha}$ for certain compact groups G with $w(G) = \alpha$ for which w-$P(2^\alpha, \alpha)$ holds, but we are not convinced of the necessity of our hypotheses. Accordingly, we ask these questions.

Question 5.3. Does $|\Omega(G)| = 2^{2^\alpha}$ hold for every compact, product-like group G with $w(G) = \alpha > \omega$? What if $\mathrm{cf}(\alpha) > \omega$?

Question 5.4. Does $|\Omega(G)| = 2^{2^\alpha}$ hold for every compact (not necessarily product-like) group G with $w(G) = \alpha > \omega$?

Remark 5.5 (Added 1 November 1996). After reading a preliminary version of this manuscript, Alan Dow of York University (Canada) informed us in April 1996, that he can show: there are models of ZFC in which $2^{\aleph_1} \geqslant \aleph_3$ and condition w-$P(\aleph_3, \aleph_1)$ fails. This result gives a consistent negative answer to Question 5.1. He has also noted that (in ZFC) every cardinal α such that $\mathrm{cf}(\alpha) > \omega$ satisfies condition w-$P(\alpha^+, \alpha)$. (Indeed, there is a set $X = \{f_\xi\colon \xi < \alpha^+\} \subseteq \alpha^\alpha$ such that for $\xi < \eta < \alpha$ there is $\zeta = \zeta(\xi, \eta)$ satisfying $f_\xi(\zeta') < f_\eta(\zeta')$ whenever $\zeta < \zeta' < \alpha$; not only is X a weak-P-space of cardinality α^+ and weight α, but also every countable subset of X is discrete and C-embedded in X.) We are grateful to Alan Dow for permission to cite these statements here.

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