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EVERY KUNEN-LIKE L-SPACE HAS A NON-MONOLITHIC HYPERSPACE

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Abstract

We show that every space defined like Kunen's example of a compact L-space has a non-monolithic hyperspace. This answers a question of Bell's. This result is also relevant to a question of Arhangel'skiĭ's.

1. Introduction and Notations

The results in this paper were initiated by questions by Arhangel'skiĭ and Bell. The former asked the question when the hyperspace of a topological space is monolithic, and the latter gave a partial answer to this question. Let us first define our terms: by the hyperspace of a topological space X we will mean the space of all closed non-empty subsets of X (denoted by H(X)), endowed with the Vietoris topology. A base for this topology is given by the sets

$$\langle U_1, \ldots, U_n\rangle = \{F \in \mathrm{H(X)} : F \subset \bigcup_{i=1}^{n} U_i,\ \forall i \in \{1, \ldots, n\} : F \cap U_i \neq \emptyset\}$$

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where the U_i are open subsets of X and n is a natural number.

A space X is called *monolithic* if for every subset A of X we have that $nw(\mathrm{Cl}\, A) \leq |A|$. Here nw denotes the net weight, a cardinal invariant which coincides with weight for compact spaces. A space is called $\aleph_0$-monolithic if the above holds for countable subsets A.

In [1] Arhangel'skiĭ asked the question when H(X) is monolithic. In [2] Bell proved that in order for a T_1 space X to have a monolithic hyperspace it must be compact, monolithic and hereditarily Lindelöf (HL). He also proved that these conditions are sufficient if moreover X is a linearly ordered topological space, thereby showing that a compact and monolithic Suslin line is a non-metrisable example of a space with a monolithic hyperspace. Such examples can only exist if there are compact L-spaces (i.e. HL spaces that are not separable), so it was a natural question for him to ask whether Kunen's compact L-space, constructed under CH ([6]), has a monolithic hyperspace. The authors of this paper showed in [3] that one can modify the Kunen construction to obtain a compact monolithic L-space with a non-monolithic hyperspace. This showed that the necessary conditions obtained by Bell are not in general sufficient for X to have a monolithic hyperspace. However, the question whether a modification of this construction could yield a compact L-space with a monolithic hyperspace, or whether every such construction would have a non-monolithic hyperspace was left open. In this paper we will show that the latter holds: every "Kunen-like" compact space has a non-monolithic hyperspace. Of course all this still leaves open the original question by Arhangel'skiĭ to exactly characterise the X's that will have a monolithic hyperspace.

Also, the authors have recently shown ([5]) that under $\diamondsuit^+$ there are 2^{ω_1} non-homeomorphic spaces that are constructed like Kunen's compact L-space from [6]. Under CH there will be at least ω_1 many non-homeomorphic ones. This shows that the result of this paper is an improvement of that from [3]. One question from this last paper is still open, and we repeat it here:

Question. *If there is a compact L-space with a monolithic hyperspace, does it follow that there is a Souslin line?*

An affirmative answer to this question would imply that CH is not strong enough to produce a non-metrisable space with a monolithic hyperspace. For all undefined notions we refer the reader to [4] for definitions and further information.

Our arguments will work for inverse limits of compacta of a certain type, which we will call "Kunen systems" for short. $(X_\alpha, \pi^\alpha_\beta, \omega \leq \alpha < \omega_1)$ will be called a "Kunen system" if it has the following properties:

i: X_ω is an infinite compact zero-dimensional metric space.

ii: For all $\alpha < \omega_1$ there is a non-empty (closed) subset S_α of X_α such that $X_{\alpha+1}$ is homeomorphic to the topological sum of X_α and S_α, and the projection $\pi^{\alpha+1}_\alpha$ is the obvious one.

iii: If α is a limit ordinal less than ω_1, then X_α is homeomorhic to the inverse limit of the X_β's with β smaller then α, and the corresponding π^β_α's are the inverse limit projections from the limit to its components.

iv: For every countable set D of X_ω there is a (non-empty) closed nowhere dense set $B \subset X_\omega \setminus D$, and a $\beta < \omega_1$ such that $S_\beta \subset (\pi^\beta_\omega)^{-1}(B)$.

It is easy to see that Kunen's L-space from [6] is an example of a limit of a Kunen system. There $X_\omega = 2^\omega$, with the usual product measure. Conditions ii and iii are automatic from the construction, while the B and the β from iv can be explicitly computed from the chosen enumeration of the closed subsets of X_ω and the chosen function g from ω_1 onto $\omega_1 \times \omega_1$: If D is a countable subset of X_ω, let B be any closed nowhere dense set of positive measure (the measure is the standard product measure on 2^ω) in the complement of D. This can be done by regularity of the measure, and by enlarging D to a countable dense subset of X_ω, if necessary. This B equals some F^ω_ξ of the chosen enumeration of the closed subsets of X_ω. Now β can be chosen to be $g^{-1}(\omega, \xi)$. It is obvious that condition (3) from [6] gives our condition iv. The condition that all S_α must be non-empty can be met by passing to a subsequence of the original inverse sequence. This gives a cofinal subsequence (having the same limit), because the resulting space is not metrisable.

Another way of seeing this is the following: for all x in X_ω we have that $(\pi_\omega)^{-1}(x) \subset X$ is a metrisable compact set in the limit (because it has measure 0), and hence:

$$\forall x \in X_\omega : \exists \alpha < \omega : \forall \beta > \alpha : (\pi^\beta_\omega)^{-1}(x) \cap S_\beta = \emptyset$$

If now $D \subset X_\omega$ is given, we can w.l.o.g. assume it is dense in X_ω. As D is a countable, by the above we find a level β such that all fibres from D miss S_β. Now we can take B to be $\pi^\beta_\omega(S_\beta)$.

It is easy to verify the following standard

Fact. *All spaces in a Kunen system are zero-dimensional, compact and metrisable. All the projections are closed, continuous and onto. (Only conditions i–iii are needed for this).*

Note that not all limits of Kunen systems are L-spaces, e.g. 2^{ω_1} is also such a space. As being an L-space is necessary for a non-metrisable space to have a monolithic hyperspace, our theorem is only interesting in the case that it is.

We will use the following notations for inverse systems $(X_\alpha, \pi^\alpha_\beta, \omega \le \alpha < \omega_1)$ satisfying the first three properties of a Kunen system: $\mathcal{B}_\omega$ will denote the (countable) set of all clopen subsets of X_ω. By X or X_{ω_1} we will mean the limit of the inverse system, and $\pi^{\omega_1}_\beta$ will denote the canonical projections from the limit onto its factors. For every subset A of X_ω, and for every $\beta \in [\omega, \omega_1]$, we will use the notation $\hat{A}_\beta$ for $(\pi_\omega^\beta)^{-1}(A) \subset X_\beta$. Finally, by $\mathcal{B}_\beta$ ($\beta \in [\omega, \omega_1]$) we will denote the set $\{\hat{B}_\beta : B \in \mathcal{B}_\omega\}$, considered as a subset of the hyperspace of X_β.

2. Preliminaries

In this section we will prove a few lemmas that will be needed for the proof of our main result.

Lemma 1. *Let X and Y be topological spaces, and let f be a closed and continuous map from X onto Y. Let B be a closed subset of Y. Then $f(\operatorname{Fr} f^{-1}(B)) = \operatorname{Fr} B$.*

Proof. We will first prove the inclusion from right to left by contradiction. Let $b \in \operatorname{Fr} B$ be such that $f^{-1}(b) \cap \operatorname{Fr} f^{-1}(B) = \emptyset$. So $f^{-1}(b) \subset \operatorname{Int} f^{-1}(B)$. By the closedness of f, there is an open subset U of Y, containing b, such that $f^{-1}(U) \subset \operatorname{Int} f^{-1}(B)$. Using the surjectivity of f, it is obvious that $U \subset B$. But this contradicts the fact that b is a boundary point of B. As for the other inclusion: Let x be a boundary point of $f^{-1}(B)$ and suppose that $f(x) \in \operatorname{Int} B$. Then $f^{-1}(\operatorname{Int} B)$ contains x and is contained in $\operatorname{Int} f^{-1}(B)$, which is impossible. □

In fact, we will only need the following simple corollary to this lemma:

Corollary 1. *Let X, Y and f be as in Lemma 2. Let $B \subset Y$ be closed and nowhere dense. Then $f(\operatorname{Fr} f^{-1}(B)) = B$. In particular: If B is non-empty, $\operatorname{Fr} f^{-1}(B) \neq \emptyset$.*

In the proof of our theorem, we will be interested in $\mathcal{B}_\omega$, the countable set of all clopen subsets of X_ω in a Kunen system. This is a dense subset of $H(X_\omega)$. What happens if we consider the set $\mathcal{B}_\alpha (= \{(\pi^\alpha_\omega)^{-1}(B) : B \in \mathcal{B}\})$? We would like to have some elements of the closure of this set in the hyperspace of X_α. The next lemma will give a few sets in this closure that are relevant to our problem:

Lemma 2.[1] *Let X and Y be a compact Hausdorff spaces, and let Y be zero-dimensional. Denote by $\mathcal{B}$ the set of all clopen subsets of Y, and let f be a continuous mapping from X onto Y. Denote by $\mathcal{B}'$ the set $\{f^{-1}(B) : B \in \mathcal{B}\}$. Let $A \subset Y$ be closed and non-empty. Then $f^{-1}(A) \in \mathrm{Cl}_{H(X)} \mathcal{B}'$. If moreover $\mathrm{Fr}\, f^{-1}(A) \neq \emptyset$, we have $\mathrm{Fr}\, f^{-1}(A) \in \mathrm{Cl}_{H(X)} \mathcal{B}'$.*

Proof. As for the first part, let $\langle U_1, \dots, U_n\rangle$ be an arbitrary open neighbourhood of $f^{-1}(A)$. Take elements $x_i \in f^{-1}(A) \cap U_i$, for $i = 1, \dots, n$. We then have that $f^{-1}f(x_i) \subset U_1 \cup \cdots \cup U_n$, for all i. Because the mapping f is closed we can find clopen subsets B_i containing $f(x_i)$ and such that for all i we have $f^{-1}(B_i) \subset U_1 \cup \cdots \cup U_n$. So: $f^{-1}(B_1 \cup \cdots \cup B_n) \in \langle U_1, \dots, U_n\rangle \cap \mathcal{B}'$, as the x_i witness the necessary non-empty intersections.

Suppose now that $\mathrm{Fr}\, f^{-1}(A) \neq \emptyset$. Let's call this boundary C for short. We will now approximate C "from the outside": Let $\langle U_1, \dots, U_k\rangle$ be an arbitrary hyperspace neighbourhood of C. So $C \subset U = U_1 \cup \ldots \cup U_k$. This implies that $U \cup f^{-1}(A)$ is open, and hence the set

$$V = \{\, y \in Y : f^{-1}(y) \subset U \cup f^{-1}(A)\,\}$$

is an open subset of Y (by closedness of f), which contains A. We have that $U_i \cap f^{-1}(V) \cap C \neq \emptyset$ for every i, and we see that $U_i \cap f^{-1}(V) \not\subset f^{-1}(A)$. So pick, for every i, points x_i witnessing this. As a consequence $f(x_i) \in V \setminus A$, and by zero-dimensionality of Y we can pick clopen sets C_i such that $x_i \in C_i \subset V \setminus A$. So $f^{-1}(C_i) \subset U$, and $x_i \in f^{-1}(C_i) \cap U_i$. So $f^{-1}(C_1 \cup \ldots \cup C_k) \in \mathcal{B}' \cap \langle U_1, \dots, U_k\rangle$, as required. □

[1] We thank the referee for simplifying our original proof of this lemma

The last lemma of this section concerns an elementary fact about nowhere dense subsets in an inverse system:

Lemma 3. *Let the inverse system* $(X_\alpha, \pi^\beta_\alpha, \omega \leq \alpha < \omega_1)$*, satisfying conditions i–iii of a Kunen system, be given. Let* $A \subset X_\omega$ *be closed and nowhere dense. If* $\beta < \omega_1$*, and if* $(\pi^\alpha_\omega)^{-1}(A) \cap S_\alpha$ *is nowhere dense in* S_α *for all* $\omega \leq \alpha < \beta$*, then* $(\pi^\beta_\omega)^{-1}(A)$ *is nowhere dense in* X_β*.*

Proof. We will prove by induction that for all $\omega \leq \gamma \leq \beta$ we have that $(\pi^\gamma_\omega)^{-1}(A)$ is nowhere dense in X_γ. This is obvious if $\gamma = \omega$, so assume that $\gamma = \gamma'+1$ and that $(\pi^{\gamma'}_\omega)^{-1}(A)$ is nowhere dense in $X_{\gamma'}$. We have that $(\pi^\gamma_\omega)^{-1}(A) = (\pi^\gamma_{\gamma'})^{-1}(\pi^{\gamma'}_\omega)^{-1}(A)$, so this set is, by ii of the definition of a Kunen system, a disjoint sum of two nowhere dense sets: X_γ is homeomorphic to a disjoint sum of copies of $X_{\gamma'}$ and $S_{\gamma'}$; $(\pi^{\gamma'}_\omega)^{-1}(A)$ hits this last set in a nowhere dense set by assumption, and it is nowhere dense in $X_{\gamma'}$ as well (by the induction hypothesis). This implies that the disjoint sum is nowhere dense in X_γ.

Now assume that γ is a limit ordinal less than or equal to β, and that for all $\gamma' < \gamma$ we have that $(\pi^{\gamma'}_\omega)^{-1}(A)$ is nowhere dense in $X_{\gamma'}$. By iii of the definition of a Kunen system, we have that X_γ is the inverse limit of the $X_{\gamma'}$, so if there were a non-empty clopen set C contained in $(\pi^\gamma_\omega)^{-1}(A)$ there would be an index $\gamma_0 < \gamma$ such that $C = (\pi^\gamma_{\gamma_0})^{-1}(C')$ for some clopen $C' \subset X_{\gamma_0}$. So:

$$(\pi^\gamma_\omega)^{-1}(A) = (\pi^\gamma_{\gamma_0})^{-1}(\pi^{\gamma_0}_\omega)^{-1}(A) \supset (\pi^\gamma_{\gamma_0})^{-1}(C')$$

and hence, by surjectivity of $\pi^\gamma_{\gamma_0}$: $C' \subset (\pi^{\gamma_0}_\omega)^{-1}(A)$, contradicting this set being nowhere dense. This finishes the induction.
□

3. The Proof of the Main Result

Now we have everything needed to prove the following theorem:

Theorem. *Let $(X_\alpha, \pi^\beta_\alpha, \omega \leq \alpha < \omega_1)$ be a Kunen system. Let X denote its limit. Then H(X) is not $\aleph_0$-monolithic.*

Proof. Let us start the proof by noting that H(X) is the inverse limit of the system $(H(X_\alpha), \hat{\pi}^\beta_\alpha, \omega \leq \alpha < \omega_1)$, where $\hat{\pi}^\beta_\alpha$ is the map between $H(X_\beta)$ and $H(X_\alpha)$ that sends a subset A of X_β to $\pi^\beta_\alpha(A)$ (See [4, 3.12.27(f)]). It is clear that $\hat{\pi}^\beta_\alpha(\mathcal{B}_\beta) = \mathcal{B}_\alpha$ for all $\omega \leq \alpha < \beta \leq \omega_1$ (using the notation from the introduction), and by closedness of all mappings involved we will also have that: $\hat{\pi}^\beta_\alpha(\operatorname{Cl}\mathcal{B}_\beta) = \operatorname{Cl}\mathcal{B}_\alpha$ for all such α and β. By [4, 2.5.6] $\operatorname{Cl}\mathcal{B}_{\omega_1}$ is the inverse limit of the system $(\operatorname{Cl}\mathcal{B}_\alpha, \hat{\pi}^\beta_\alpha|\operatorname{Cl}\mathcal{B}_\beta, \omega \leq \alpha < \omega_1)$.

Let us call a pair (α, β), where $\omega \leq \alpha < \beta < \omega_1$, "bad" if the map $\hat{\pi}^\beta_\alpha|\operatorname{Cl}\mathcal{B}_\beta$ is not one-to-one. The following claim is essential to our proof:

Claim. *For every $\alpha \in [\omega, \omega_1)$ there is a $\delta < \omega_1$ such that (α, δ) is a bad pair.*

To prove this, let α be an arbitrary ordinal in $[\omega, \omega_1)$. For every $\gamma \in [\omega, \alpha)$ let D_γ be a countable dense subset of S_γ. Consider the set $D = \bigcup_{\gamma \in [\omega,\alpha)} \pi^\gamma_\omega(D_\gamma)$. Then D is a countable subset of X_ω ($D = \emptyset$ if γ happens to be ω), so we can find a closed nowhere dense set $B \subset X_\omega \setminus D$ and a $\beta < \omega_1$ such that $S_\beta \subset \hat{B}_\beta$. Note that, by the definition of $X_{\beta+1}$, the interior of $\hat{B}_{\beta+1}$ will be non-empty, as it will contain a clopen copy of S_β. Now we have for all $\gamma \in [\omega, \alpha)$:

$$S_\gamma \cap \hat{B}_\gamma \subset (\pi^\gamma_\omega)^{-1}(X \setminus D) \cap S_\gamma \subset (\pi^\gamma_\omega)^{-1}(X \setminus \pi^\gamma_\omega(D_\gamma)) \cap S_\gamma = \\ = X_\gamma \setminus D_\gamma \cap S_\gamma = S_\gamma \setminus D_\gamma$$

From this we may conclude that $S_\gamma \cap \hat{B}_\gamma$ is nowhere dense in S_γ, for all $\gamma \in [\omega, \alpha)$. Hence by Lemma 2 we may conclude that $\hat{B}_\alpha$ is nowhere dense in X_α. This implies, by the previous remark, that $\alpha < \beta + 1$. We will now show that $(\alpha, \beta + 1)$ is bad: First, applying Lemma 2 to X_ω, $X_{\beta+1}$, B and $\pi^{\beta+1}_\omega$, we see that $\hat{B}_{\beta+1} \in \mathrm{Cl}\,\mathcal{B}_{\beta+1}$ and also $\mathrm{Fr}\,\hat{B}_{\beta+1} \in \mathrm{Cl}\,\mathcal{B}_{\beta+1}$. Also note that these are different sets, as $\mathrm{Int}\,\hat{B}_{\beta+1} \neq \emptyset$. Second, we apply Corollary 2 to X_α, $X_{\beta+1}$, $\hat{B}_\alpha$ and $\pi^{\beta+1}_\alpha$. From this we obtain that

$$\pi^{\beta+1}_\alpha(\mathrm{Fr}\,\hat{B}_{\beta+1}) = \hat{B}_\alpha = \pi^{\beta+1}_\alpha(\hat{B}_{\beta+1})$$

the second equality being obvious. So this shows that $\hat{\pi}^{\beta+1}_\alpha$ is not one-to-one on $\mathrm{Cl}\,\mathcal{B}_{\beta+1}$, as required.

To conclude the proof we show that the closure in H(X) of the countable set $\mathcal{B}_{\omega_1}$ is not metrisable: to the contrary, suppose that $\mathrm{Cl}\,\mathcal{B}_{\omega_1}$ were metrisable. Then by [4, 3.2.H(e)], and the above inverse limit decomposition of $\mathcal{B}_{\omega_1}$, there exists an index $\alpha \in [\omega, \omega_1)$ and a continuous mapping f_α from $\mathrm{Cl}\,\mathcal{B}_\alpha$ to $\mathrm{Cl}\,\mathcal{B}_{\omega_1}$ such that $f_\alpha \hat{\pi}^{\omega_1}_\alpha$ equals the identity on $\mathrm{Cl}\,\mathcal{B}_{\omega_1}$. So in particular we will have that $\pi^{\omega_1}_\alpha$ will be injective. But by the claim we can find a δ such that (α, δ) is bad. But we have that $\hat{\pi}^{\omega_1}_\alpha = \hat{\pi}^\delta_\alpha \hat{\pi}^{\omega_1}_\delta$ and this is not one-to-one as a composition of a surjective and a non-injective mapping. This contradiction concludes the proof of the theorem. □

We'd like to conclude with a few remarks. It is well-known that Kunen's space is Corson compact. In fact it must be, as follows from [7, Thm. 6]: If the limit of a Kunen system has countable tightness, then it is Corson compact. It is also well-known that Suslin lines are *not* Corson compact (recall that a compact and monolithic Suslin line has a monolithic hyperspace). So one might be tempted to think that this is the "real" reason for the hyperspace of a Kunen space to be non-monolithic. But in fact there is a consistent counterexample to this: Let L be a monolithic compact Suslin line. By [7] L can be mapped irreducibly onto a Corson compact space X. By irreducibility, X is also an L-space, and it is easy to see that having a monolithic hyperspace is preserved by continuous maps between compacta. So under ¬SH there is a Corson compact space *with* a monolithic hyperspace. (Of course MA and ¬CH will imply that there are no such spaces.)

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