GENERAL TOPOLOGY, IN PARTICULAR DIMENSION THEORY, IN THE NETHERLANDS: THE DECISIVE INFLUENCE OF BROUWER'S INTUITIONISM

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Dedicated to the memory of our friend and colleague Maarten A. Maurice

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Introduction

Dutch work in dimension theory can be very naturally divided into two periods. The first period encompasses the contributions of LUITZEN EGBERTUS JAN BROUWER (1881–1966), whose work brought about a revolution: modern topology was born and with it dimension theory. The second period concerns the work of other Dutch mathematicians who worked in topology after BROUWER, when topology had become an established discipline.

In the first part of this paper we will discuss BROUWER's contributions to dimension theory. However, in order to understand the background of BROUWER's work in topology more fully it is necessary to concentrate also on the broader context of his entire scientific opus and in particular on the relation of his topological work with his intuitionistic foundational views. Usually the intuitionistic work and the topological work of L. E. J. BROUWER are considered as almost disjoint. In the first part of this paper we argue that this view is very one-sided. We briefly describe BROUWER's intellectual development and show, by considering BROUWER's contributions to dimension theory, not only that the topological work is closely related to BROUWER's pre-1917 intuitionism, but even that without his intuitionistic views BROUWER might never have turned to topology. The opposition between classical topology and intuitionism arises only after 1917 when BROUWER had become aware of the fact that a consequent intuitionism implied the loss of a considerable part of classical mathematics, not only with respect to method but also with respect to content. In this first part we will stay very close to BROUWER's precise but rather informal mathematical style. A translation of BROUWER's work into modern mathematical language would be misleading.

In the second part of the paper we briefly describe the development of the interest in The Netherlands in topology and in particular general topology after BROUWER. Although the events described all took place after Brouwer had turned away from classical topology, before World War II Brouwer's presence continued to influence the developments. As we will see FREUDENTHAL – whose influence on Dutch topology was considerable – came to The Netherlands because of BROUWER's intuitionism. We illustrate the Dutch work in dimension theory with a description of a particularly interesting development to which Dutch topologists contributed substantially: the history of DE GROOT's compactification problem.

Part I: L. E. J. Brouwer's Topological Work and its Relation with Intuitionism

1.1. DR. JEKYLL AND MR. HYDE?

With good reason BROUWER has been called one of the two greatest Dutch mathematicians, next to the seventeenth century genius CHRISTIAAN HUYGENS. Yet there seems to be something schizophrenic about him. His collected works consist of two volumes. The first volume contains BROUWER's philosophical and foundational work. The second volume contains mainly topological work. According to Heyting, the editor of the first volume, the two volumes are "almost disjoint" [36, p. xiii]. At first sight there are two BROUWERs, and both are famous, although in different circles. There is on the one hand the foundationalist BROUWER, who defended and developed a highly original branch of constructivistic mathematics: intuitionism. For an intuitionist a proof requires a mental construction. From the intuitionistic point of view actually infinite sets cannot be constructed mentally, which means, for example, that the principle of the excluded third, "p or non-p", is not generally valid so that the classical proof by contradiction is unreliable, because "non-non-p" does not necessarily imply p. At first sight there is, however, on the other hand, next to the foundationalist, another BROUWER, one of the major founders of modern topology. Like Stephenson's Mr. Hyde, who does everything that his alter-ego, Dr. Jekyll, resents, the topologist BROUWER without hesitation violates the intuitionistic rules and studies actually infinite sets and freely uses the principle of the excluded third. Rumour has it that some topologists tend to view BROUWER primarily as a topologist, whose foundational work was merely the result of a peculiar hobby, while, on the other hand, some foundationalists tend to consider the topological work as deviant.

An example of the latter tendency we find in VAN STIGT's publications. The historian VAN STIGT, an expert on BROUWER and the author of a major work on BROUWER's intuitionism [73], argues that the foundationalist BROUWER in fact realized that no one was going to listen to his foundational message unless he first proved to the world that he was a great mathematician. BROUWER's topological work was, in VAN STIGT's view, the result of a conscious attempt to show the world that he deserved to be taken seriously. As soon as the world took him seriously, after four or five very fertile years of topological work, and after Amsterdam University made him an ordinary professor in 1912, BROUWER returned to foundational studies. This thesis is the central theme of [71].[1] VAN STIGT's view represents one possibility to understand BROUWER's work as a whole. It can be supported by the fact that BROUWER's topological work was all done in a very short period, following the defense of his doctoral dissertation on the foundations of mathematics in 1907 and preceding his professorship in 1912. Yet, although VAN STIGT is probably right in the sense that BROUWER did

[1] In [73] the same argument returns, although less conspicuously. However, Professor VAN STIGT wrote us that in his view the apparent disagreement between the present paper and his publications is more a matter of emphasis and that it has never been his intention to maintain that BROUWER's foundational and topological work are wholly disjoint interests (private communication, June 30, 1994).

indeed view his topological work as a good means to acquire a reputation that would enable him to get attention for his foundational ideas, we will try to show that the opposition between foundational and the topological work should not be exaggerated and that the two parts of his work are very much coherent.

Below we will describe BROUWER's intellectual development chronologically and we will see that BROUWER the topologist and BROUWER the intuitionist are closely related: there is only one BROUWER, a great mathematician and a complex personality, but quite consistent. In Section 1.2 we show how in BROUWER's early views on the foundations of mathematics topological problems play a central role. Those views quite naturally led to his topological work. In Section 1.3 we argue that BROUWER's topological work is not opposed to his early intuitionism. Although BROUWER was aware of the fact that certain proofs would eventually have to be revised, his early intuitionism did not contradict his topological work. In Section 1.4 we describe BROUWER's contributions to dimension theory. Finally, in Section 1.5, we show that only in the course of World War I BROUWER became aware of the fact that his early intuitionism had been immature and that a further elaboration of his intuitionistic point of view required a rather drastic revision of classical mathematics.

1.2. BROUWER, THE PHILOSOPHER OF MATHEMATICS

1.2.1. *"Life, Art and Mysticism" or* BROUWER, *the mystic*

BROUWER, who enrolled at Amsterdam's municipal university in 1897, was a brilliant student, but he found mathematics as it was taught at the university extremely boring. Many years later, in 1946, he said that the classes he attended made him draw the conclusion that a mathematician was either "a servant of natural science or [...] a collector of truths. Truths, fascinating by their immovability but horrifying by their lifelessness, like stones from barren mountains of disconsolate infinity" [73, p. 25] (the translation is VAN STIGT's). BROUWER even very seriously considered to quit his university studies. Fortunately, in 1903, the remarkable GERRIT MANNOURY (1881–1966) was appointed as "privaat-docent" at the university. The erudite MANNOURY was a mathematician[2], but also a sceptical philosopher, whose favourite method of philosophical investigation was one of "playful doubt and questioning" [8, p. 5]. BROUWER attended MANNOURY's classes on the philosophy of mathematics and was fascinated by this "ever critical and easily switching relativist" (Ibidem). BROUWER and MANNOURY became and would remain friends, till MANNOURY's death more than sixty years later, which is remarkable because BROUWER was an emotional

[2] Actually MANNOURY was the first Dutch topologist. In [61] he proves a duality theorem about which HOPF later wrote ([43, p. 25]): "[the theorem] fully belongs in the circle of modern duality theorems and the fact that MANNOURY knew it in the year 1897 shows how far ahead of his time he was. It is on the whole unfortunate that he did not continue this work, after all he was close to Alexander's duality theorem."

individualist, a loner who offered those who associated with him usually only one choice: admiration or enmity. A friendship with him rarely lasted long [66, p. 46–47]. As a result of MANNOURY's classes mathematics became to BROUWER a challenge instead of a collection of lifeless truths and in 1904 he started to prepare his doctoral dissertation on the subject that would be his main concern for the rest of his life: the foundations of mathematics. As was first pointed out by VAN STIGT in his Ph. D. Thesis [70] and later by VAN STIGT [72] and VAN DALEN [28], BROUWER's views on the foundations of mathematics are closely related to the ideas that he expressed in a series of lectures that he gave at Delft University in 1904–1905. In those lectures, that were published in the form of a booklet, "Leven, Kunst en Mystiek" ("Life, Art and Mysticism") [13], BROUWER reacted to the views of BOLLAND, an arrogant but at that time quite popular Dutch Hegelian philosopher, who revered reason and considered himself to be its re-incarnation. BROUWER's "Life, Art and Mysticism" is such that the editor of BROUWER's philosophical and foundational work, HEYTING, published only some parts of it in Volume I of the Collected Works. VAN STIGT attempted to persuade him that the whole of it should be included, but HEYTING must have feared that publication of a text that is so strange and so different from a decent piece of mathematics, "that crazy booklet" as he denoted it to VAN STIGT, might hurt BROUWER's reputation. The booklet, which BROUWER always remained proud of, reveals to us a solipsist and a mystic, who views human beings as creatures that are tragically locked up in their own mind, unable to communicate with each other. Human reason and in particular mathematics and science are condemned because they enable mankind to exploit the earth and each other and in doing so man is alienated from his real self. For BROUWER salvation lies in introspection, in insight, in intuition. He preaches asceticism, the neglect of pleasure, possessions and honour. Striking, at least from a modern point of view, is the fact that BROUWER dedicates several pages to describe women as inferior beings, whose destiny it is to serve the superior male sex.

1.2.2. *"On the Foundations of Mathematics"*

In 1907, two years after the publication of "Life, Art and Mysticism", BROUWER defended his doctoral dissertation, "Over de Grondslagen der Wiskunde" ("On the Foundations of Mathematics") [14]. The dissertation is very much in line with the ideas from "Life, Art and Mysticism". In the dissertation the solipsist BROUWER describes true mathematics as a free creation that consists of mental constructions that are executed within the isolation of the human mind. Language is exterior to mathematics and merely an imperfect means to communicate one's mental constructions to others. Logicism and formalism are rejected as solutions to the crisis in the foundations of mathematics because they are based on an attempt to found mathematics in language. Logic is seen by BROUWER as an empirical science dealing with linguistic regularities. The agreement between

"Life, Art and Mysticism" and the dissertation would have been even greater if BROUWER's supervisor, KORTEWEG, had not refused to accept several passages which according to him had nothing to do whatsoever with mathematics and its foundations. For example, the following sentences do not occur in the final version of the dissertation: "And that is why science is only meaningful as a factor in the struggle of men against nature for their fellowmen by counting and measuring calculation, in other words, natural science has value as a *weapon*, but otherwise does not touch life, yes, it is there as disturbing as *everything* that is related to struggle. While mathematics, undertaken for its own sake, can acquire all the harmony [...] of music and architecture and can give all the illicit enjoyments that lie in the free unfolding of faculties without compulsion from outside".[3] N.B. BROUWER still condemns applied mathematics, like in "Life, Art and Mysticism", whereas the mathematics that he describes in the dissertation, pursued for its own sake, is something entirely different.

BROUWER's dissertation consists of three chapters. In the first chapter BROUWER describes how the science of mathematics can be constructed on the basis of a basic mathematical intuition. The second chapter deals with the relation of mathematics and experience, while the third chapter is devoted to the relation of mathematics and logic. We will concentrate on the first chapter.

1.2.3. *The basic intuition of mathematics and point sets on the continuum*

At heart the basic intuition of mathematics is the continuous flow of time in which we can distinguish different moments. In this respect BROUWER follows Kant in whose philosophy time is an "Anschauungsform"; also for Kant the a priori intuition of time precedes all experience. BROUWER describes this basic intuition as "a unity of continuity and discreteness". The intuition of continuity, of "fluidity", implies the discrete in the form of a before and an after. On the other hand, the discreteness corresponds to the apartness of moments in time and implies the continuous, i.e. the existence of a "between" which is never exhausted by the insertion of new moments [14, p. 17]. It was BROUWER's intention to show in the first chapter of his dissertation how mathematics can be constructed on the basis of this basic intuition. First BROUWER introduces the number system as a potentially infinite system of signs. The intuition of the discrete, of the before and after, yields the possibility to construct again and again a successor and thus we obtain the potentially infinite sequence of the natural numbers. The negative numbers are obtained in a similar way. The rational numbers are introduced as ratios of integers and the irrational numbers are also defined as "symbolic aggregates of

[3]"En daarom heeft de wetenschap ook alleen zin als factor in den strijd der mensen tegen de natuur om hun medemensen door tellende en metende berekening, m.a.w. de natuurwetenschap heeft waarde als *wapen*, maar raakt verder het leven niet, ja is er even storend, als *alles* wat aan strijd annex is. Terwijl wiskunde, om zichzelf bedreven, alle harmonie [...] van muziek en architectuur kan krijgen en al de ongeoorloofde genietingen kan geven, die liggen in de vrije ontplooiing van faculteiten, zonder dwang van buiten" [31, p. 30]

previously introduced numbers" [14, p. 16]. Soon, however, BROUWER relates the numbers to point sets, which for our purposes is important. The intuitively given continuum is the starting point. It is given in intuition and in a sense all its points are also given in intuition because we can cut the continuum wherever we want and in doing so we define a point on it. Yet the continuum is not given as a set of points that can all together be constructed individually. The continuum is like a "matrix", as BROUWER says, that can be filled with points, not arbitrarily, but in accordance with the nature of the continuum.

In order to handle the continuum mathematically BROUWER applies a potentially infinite system of points (cuts) on the continuum that he calls the "dual scale". By means of this dual scale he turns the intuitively given continuum into a "measurable continuum". In order to construct this dual scale BROUWER proceeds as follows. In the intuitive continuum he constructs "a sequence of points having the order type of the positive and negative whole numbers". This sequence of points is potentially infinite and so is the sequence of intervals that it yields. Each of the intervals is now cut up by means of repeated dichotomies. BROUWER writes: "then if we add a point in every interval, then again in each of the intervals so obtained, and so on, we obtain the order type η on the continuum, which in this way comes to correspond with the system of the finite dual fractions" [14, pp. 17–18]. Those dual fractions are the elements of $\mathbb{Q}$ written in the binary system. At the time BROUWER believed that he could handle the continuum by means of this potentially infinite scale. He remarks that such a dual scale need not be dense everywhere, which means by definition that it does not necessarily penetrate into every segment of the continuum. However, he says, we can contract every segment not penetrated by the scale into one point. The result is, says BROUWER, that by definition the scale will be everywhere dense. In this way the scale turns the intuitive continuum into the "measurable continuum", which is in fact the standard-continuum and can be identified with the system of the real numbers. Below, in §1.5.1, we will return to this kind of reasoning, because in 1917 BROUWER would no longer be satisfied with it. However, we will first see how BROUWER in his dissertation further described the construction of mathematics.

1.2.4. *Lie groups in the dissertation*

In 1907 it was not at all BROUWER's intention to do away with the results of classical mathematics. On the contrary, he considered most of classical mathematics as sound. He still assumed, in the words of VAN STIGT, "that the classical treatment of 'real number' can be justified and form a sound basis for analysis and function theory" [73, pp. 320–321]. It is interesting to see how BROUWER introduces the arithmetical operations on the measurable continuum, whose construction we described above, and how he introduces various geometries. The answer is: by means of group theory and topological notions. Moreover, he felt this to be the only correct way. He wrote: "The arithmetical operations on the measurable continuum *ought to* be defined by means of group theory" [14, p. 98] (the emphasis is ours – T.K. & J.v.M.). Let us consider the introduction of addition. In order to give a group theoretic characterization of addition on the measurable continuum – with the group of translations in mind – BROUWER considers a group of bijections (or a uniform group of transformations in BROUWER's words) on the measurable continuum, that is

1. one-parameter continuous, "i.e. its transformations can be arranged in a linear continuum in such a way that, if a point moves continuously in this image-continuum, then the corresponding transformations bring about simultaneous continuous movements for the points of the continuum to which they are applied",
2. closed, i.e. if two denumerably infinite sequences of points $A_1, A_2, \ldots$, and $B_1, B_2, \ldots$ have on the measurable continuum respectively the limit points A and B, while there is a denumerable sequence of transformations that maps A_1B_1 on A_iB_i for $i = 1, 2, 3, \ldots$, then there is a transformation that maps A_1B_1 on AB.

BROUWER derives different properties of such a group of bijections. For example, the order of the points remains unchanged under all bijections. BROUWER's proof: Move continuously from the identity to transformation t (property 1). If t would change the order of the two different points A and B then the two points would have met each other during their continuous movement, which would contradict the uniformity of the group (i.e. that it consists of bijections).

If we choose some point as zero point we can denote the transformation that maps 0 on the point a by "$+a$". The image of a is denoted by $2a$, the image of $2a$ by $3a$, etc. There is also a point between 0 and a such that the transformation "$+b$" that carries 0 to b, carries b to a. BROUWER puts $b = a/2$. Etc. In this way each group of transformations of the measurable continuum generates another dual scale on the continuum. Because this scale is also dense it makes the continuum measurable in a new way and *with respect to this new measurable continuum* BROUWER can then identify the chosen group of transformations with the group of additions, or the group of translations. In a similar way he concentrates on the problem to find "the most general set of two one-parameter continuous uniform groups that can be combined to a two-parameter continuous group" [14, p. 23] and he finds the group of addition and multiplication combined on the scale: $x' = c_1x + c_2$ [14, p. 25].

BROUWER extends this approach by means of group theory in combination with topological notions to the different geometries. In this context he refers to HILBERT's famous speech at the 1900 International Congress of Mathematicians on important unsolved problems. HILBERT's fifth problem concerned Lie-groups.

LIE had considered groups of transformations on $\mathbb{R}^n$,

$$x'_i = f_i(x_1, x_2, \ldots, x_n; a_1, a_2, \ldots, a_n) \qquad (i = 1, \ldots, n).$$

Assuming differentiability of the functions LIE showed that if well-chosen extra axioms are added the transformation groups corresponding to different geometries can be derived. HILBERT considered the assumption of differentiability as unnatural and he proposed a more general approach by assuming merely continuity. In the dissertation BROUWER points out that his introduction of combined addition and multiplication on the measurable continuum constitutes a solution of the one-dimensional case of HILBERT's fifth problem. In 1902 HILBERT himself had solved the case of the Euclidean or non-Euclidean motions in the plane by considering – in BROUWER's words – "*the group of invertible uniform transformations leaving invariant the connections of incidence and forming a closed system* (i.e. if there are transformations in the group which make a definite set of points approximate another definite set of points, then there is also a transformation in the group transforming the first set into the second), and for which every "circle" (i.e. the set of points into which, after fixing one point, another point can still be transformed) *consists of an infinity of points*" [14, p. 52]. HILBERT's approach was also based on a combination of group theory and topological notions; he applied CANTOR's theory of point sets in combination with the Jordan-curve theorem [52]. It is not at all strange that BROUWER was fascinated by the possibility to look at groups of one to one continuous mappings and define the geometry that you want by means of the right topological restrictions. After the discovery of non-Euclidean geometries an introduction of geometry on the basis of some Euclidean intuition was for BROUWER out of the question. On the other hand, the axiomatic approach, which HILBERT applied in his famous "Grundlagen der Geometrie" – axioms are essentially arbitrary statements that implicitly define a geometry – was radically opposed to BROUWER's view of mathematics. HILBERT's 1902 paper showed BROUWER a different way to introduce the various geometries starting from continuous transformations, i.e. transformations that could be described directly in terms of the measurable continuum. Obviously, an important object of chapter one of the dissertation was to "show the construction of groups independent of differentiability to be *essential* in the construction of mathematics" as BROUWER wrote in 1906 (quoted and translated by VAN STIGT, [73, p. 39], – italic is ours, T.K. & J.v.M.). The result was that in the dissertation topological notions started to play a major role in founding mathematics.

1.3. THE RELATION BETWEEN TOPOLOGY AND INTUITIONISM

1.3.1. *The basic unity of* BROUWER*'s work*

Before we can turn to BROUWER's work in topology, we must return to the alleged opposition between BROUWER's foundational views and his topological practice. With respect to the relation of BROUWER's intuitionism and his topological work in the period 1908–1913 we will firstly argue that his intuitionism was still immature at that time. In the dissertation he did not suspect that a consequent application of his constructivistic ideas would imply that the classical continuum and the classical notion of function would have to be given up. He assumed that classical mathematics only needed a better foundation. And, on the other hand, he also assumed that classical logic could still be applied in mathematics provided the argument referred directly to mathematical constructions. There is undoubtedly some development in BROUWER's views on this point. In a 1908 paper he restricted the application of the principle of the excluded third and it is clear that in the period 1909–1913 he was aware of the fact that he was using non-constructivistic methods in his topological work and that this work would have to be revised eventually. Yet, only in 1917 BROUWER realized that intuitionistic mathematics would inevitably deviate considerably from classical mathematics. In §1.3.2 we will elaborate on this point.

VAN STIGT has argued that BROUWER consciously remained silent on intuitionistic foundational matters and turned to non-intuitionistic topology primarily to establish a reputation for himself. We would rather describe the situation that BROUWER faced in 1907, after the defense of his dissertation, as follows. In different ways the dissertation was unfinished. On the one hand the sketch that BROUWER gave of an intuitionistic construction of mathematics in the first chapter was very incomplete and, for example, for BROUWER, HILBERT's fifth problem remained a major unsolved problem. On the other hand the intuitionistic philosophy and the related general methodological considerations were also unfinished. Obviously, both aspects of BROUWER's "research programme" required further elaboration. However, BROUWER decided to work on the former aspect first. The fact that he felt the need to establish a reputation for himself may indeed have played a role. But it is clear that he was very naturally drawn into the problem of the Lie-groups and eventually into topology. We will discuss BROUWER's topological work and in particular his work in dimension theory in §1.4. In that section we will also show that BROUWER's work in dimension theory is constructivistic in the sense of the first part of the dissertation. BROUWER's style in topology is such that there is first of all a strong visual aspect, while the mental images corresponding to the topological transformations are obviously related to the intuitive continuum. BROUWER's topology is mental construction, although not always in the strict sense of BROUWER's later intuitionism. The proofs possess a great conceptual clarity based on clear images. Moreover, manifolds are constructed out of simplexes, and manifolds and continuous mappings are handled by means of potentially infinite systems of approximations similar to the way in which in the dissertation the continuum is handled by means of the dual scale. BROUWER's topology is not abstract, his notions always refer to mathematical systems that can be considered as mentally constructed. We will also show that the fact that there are instances in his topological work where BROUWER sins against his own intuitionistic views, does not run counter to the existence of a basic unity between the work in his dissertation and his topological work.

1.3.2. *Tertium non datur*

In the dissertation and for several years afterwards BROUWER considered the measurable continuum as a clear notion. Certain questions on infinite point sets, like the continuum hypothesis, he viewed as the result of turning mathematics into a logical system whereby the connection with mathematics is lost. He condemned the application of the principle of the excluded third in such situations. But in the dissertation he wrote "Further we emphasize that the syllogism and the other logical principles may be reckoned to hold for the language of logical reasonings on finite sets, on denumerably infinite sets, on domains in continua, but in any case exclusively on mathematically constructed systems; the conviction that we may rely on that applicability, is based on the certainty that mathematical systems are under discussion" [14, p. 75]. Elsewhere in the dissertation he expressed himself in the same vein saying that as long as classical logic corresponds to acts of mathematical construction classical logic can be applied. BROUWER wrote: "here[...] we safely apply the principles of *identity, syllogism, distribution, contradiction* and *tertium non datur*" [14, p. 88]. When BROUWER was handling the continuum in the dissertation he felt that by means of the dense dual scale on the intuitive continuum he had constructively sufficiently defined the classical continuum as it occurred in the mathematics of his time. From his later much more strictly constructivistic point of view, BROUWER's 1907 construction to turn the intuitive continuum into a measurable continuum is dubious. The dual scale in itself is a clear notion, but because of its potentially infinite character, we cannot know precisely which segments of the intuitive continuum it will not penetrate. When BROUWER in the dissertation contracts each segment of the intuitive continuum that is not penetrated by the dual scale into one point (see §1.2.3) he implicitly considers the dual scale as an actually infinite whole. In 1917 he realized that this was unacceptable. Another example of an argument that BROUWER in 1917 could no longer accept is his proof of the theorem that says that on the measurable continuum every bounded infinite set possesses a limit point. BROUWER writes in the dissertation: "From the measurability we conclude that every denumerably infinite set of points, lying in the segment determined by two points as its endpoints, has at least one limit point, i.e. at least one point such that on at least one of its sides in every segment contiguous to it there are other points of the set. (For otherwise there would be a shortest distance between points and this could be held only a finite number of times in the finite segement.)" [14, pp. 18–19]. The argument is based on a classical proof by contradiction.

At that time, however, because the proof refers directly to the measurable continuum, BROUWER saw no problem here.

In 1908, a year after the publication of his dissertation, BROUWER returned to the question of the logical principles in "De onbetrouwbaarheid van de logische principes" ("The unreliability of the logical principles") [15]. In that paper BROUWER explicitly asks himself whether it is allowed in "purely mathematical constructions and transformations, to neglect for some time the idea of the mathematical system under construction and to operate in the corresponding linguistic structure, following the principles of *syllogism*, of *contradiction* and of *tertium exclusum*, and can we then have confidence that each part of the argument can be justified by recalling to the mind the corresponding mathematical construction?" [15, p. 109]. In opposition to what he had assumed in the dissertation BROUWER argues that with respect to finite systems all principles can be applied, but with respect to infinite systems the principle of the excluded third is not reliable. Yet, he writes, "we shall never, by an unjustified application of the principle, come up against a contradiction and *thereby* discover, that our reasonings were badly founded" ([15, p. 110]). The reason is that BROUWER accepted the principle of contradiction which says that p and non-p cannot be true at the same time. The conclusion p from a proof of non-non-p is unjustified intuitionistically but it can never lead to a contradiction because non-non-p excludes non-p as a possibility. Most of BROUWER's topological work done in the period 1909–1913, is perfectly in harmony with BROUWER's 1907 ideas. However, in that topological work he repeatedly applies the principle of the excluded third, which he rejected in 1908. At this point there seems to be some inconsistency. Yet, several years later, in 1919, BROUWER once more described the principle of the excluded third as unreliable, but at the same time acknowledged its heuristic value. He also wrote on that occasion: "In the writings quoted in note 2 (BROUWER refers here to all his intuitionistic work from before 1917 – T.K.& J.v.M.) I drew [...] only fragmentary consequences from the [...] intuitionistic conception of mathematics, I also repeatedly used the old methods in my simultaneous philosophy-free mathematical works, whereby, it is true, I aimed at deriving only such results of which I could hope that, after the execution of a systematic development of the intuitionistic set theory, they would find a place and claim a value in the new edifice of learning, possibly in a modified form".[4] In the same paper BROUWER also wrote that in his intuitionistic work written before 1917 he had not yet

[4] "Von den [...] intuitionistischen Auffassung der Mathematik habe ich übrigens in den in Anm. 2) zitierten Schriften bloss fragmentarische Konsequenzen gezogen, habe auch in meinen gleichzeitigen philosophiefreien mathematischen Arbeiten regelmassig die alten Methoden gebraucht, wobei ich allerdings bestrebt war, nur solche Resultate herzuleiten, von denen ich hoffen konnte, dass sie nach Ausfuhrung eines systematischen Aufbaues der intuitionistischen Mengenlehre, im neuen Lehrgebäude, eventuell in modifizierter Form, einen Platz finden und einen Wert behaupten würden" [36, p. 231]

realized the full consequences of intuitionism.[5] We see no good reason not to take these words seriously. There are no indications that BROUWER, while he was doing his topological work, suspected that in the end his intuitionistic point of view would force him tot give up considerable parts of classical mathematics. Of course he knew that all his topological proofs by contradiction would have to be reconsidered from a strict intuitionistic point of view, but the years 1909–1913 were so fertile and his classical approach meant already such an enormous improvement in rigour compared to the work of his predecessors in topology, that BROUWER had to postpone the intuitionistic revision of the proofs.

1.4. BROUWER, THE TOPOLOGIST

1.4.1. *The role of topology in mathematics and the invariance of dimension*

BROUWER very clearly expressed his opinion on the importance of topology in mathematics in his 1909 inaugural lecture, "Het wezen der meetkunde" ("The Nature of Geometry") [17], read when he became "privaat docent". In the lecture he argues that all the mathematical systems have been built up from the intuition of time, from the "measureless one-dimensional continuum conceived by one single subject" [17, p. 116]. In his view of mathematics, geometry plays a central role and within geometry, defined as that part of mathematics that "in particular, investigates and classifies sets, transformations and transformation groups in [...] spaces" of one or more dimensions [17, p. 116], the group of analysis situs is the most fundamental. He said also: "it is possible and desirable to give priority to the geometrical method also in parts of mathematics where this has not been realized" [17, p. 120] and "In the same way it will not be necessary to banish coordinates and formulas from other theories when they have been successfully based on analysis situs, but the treatment without formulas, the "geometrical" treatment, will be the point of departure, the analytical method will become a dispensable tool" [17, p. 120]. It is clear that for BROUWER topology had become the most fundamental part of mathematics and it was inevitable that he would work in it and that he would eventually also study the problem of dimensional invariance.

In 1874 GEORG CANTOR had proved that the set of all real algebraic numbers is countable while the set of all real numbers is not. A startling discovery: there appeared to be different types of infinity. Quite naturally CANTOR started a search for other types of infinite sets, for example by looking at higher-dimensional figures. To his own surprise in 1877 he succeeded in proving that 1- and n-dimensional figures can be put in a one-one correspondence. This was another striking result, which led him to the conclusion that "the difference between figures of different dimension numbers will have to be sought in entirely differ-

[5] He wrote: "In meinen in Anm.2) zitierten Schriften [...] in denen die Konsequenzen des Intuitionismus sich noch weniger deutlich für mich abgezeichnet hatten" [36, Footnote on p. 234].

ent aspects than in the number of independent coordinates, which is held to be characteristic" [55, p. 141]. DEDEKIND with whom CANTOR was corresponding in those years was not willing to give up the idea that the number of independent coordinates is characteristic for the dimension of a manifold and conjectured: A one-one correspondence between the points of two continuous manifolds of different dimension must be discontinuous. Several mathematicians submitted proofs of the dimensional invariance under continuous mappings and by the end of the 19th century most people thought that the matter had been settled. This conviction however, was based upon an underestimation of the complexity of the situation. In 1899 JURGENS and also LUROTH published on the matter and it became clear that the general problem was still widely open and only some special cases had been solved (See [55] for an excellent extensive treatment of the different proofs). This was still the situation when BROUWER held his inaugural lecture in 1909. Two pages are dedicated to analysis situs. He wrote: "An immediately related problem is, in how far spaces of different dimension are different for our group. Most probably this is always the case, but it seems extremely hard to prove, and probably will remain an unsolved problem for a long time to come." [17, p. 18]. It is clear that in 1909 the problem of the invariance of dimension occupied his mind.

1.4.2. *Criticism of Schoenflies*

In the first decade of this century ARTHUR SCHOENFLIES had attempted to give a thorough set-theoretic foundation of topology. We will restrict ourselves to a few remarks about that work. For a fuller treatment we refer to [55]. In SCHOENFLIES' work a central result is Jordan's theorem: a closed Jordan curve, i.e. the one-to-one continuous image of a circle, divides the plane into two domains with the image as their common boundary. A *domain* is an open connected set. At certain points SCHOENFLIES work is quite subtle. For example, he distinguishes by means of the notion of accessibility between simple closed curves and closed curves that are not simple. By definition a point P on the boundary of a domain is *accessible* if it can be reached from an arbitrary point in the domain by a finite polygonal path in the domain or an infinite polygonal path in the domain of which P is the only limit point. The notion of accessibility was, according to JOHNSON, motivated by the curve $y = \sin(1/x)$ for $-1 \leq x \leq +1$ together with the limit points on the y-axis $-1 \leq y \leq +1$. If we connect the points $(1, \sin 1)$ and $(-1, \sin -1)$ by means of an unproblematic curve segment that does not intersect that point set we obtain a closed curve[6]. Here a *closed curve* is by definition a bounded closed point set that divides the plane into two domains with the curve as their common boundary. [67, pp. 118–120]. The above-given closed curve divides the plane into two domains but it is peculiar in the sense that the points of it on

[6] Such a curve is sometimes called Warsaw circle by topologists. It would be interesting to know when and why this usage originated.

the y-axis are not accessible from the points of the two domains. Closed curves that are such that all their points are accessible from the two domains are called *simple* by SCHOENFLIES. An important result that SCHOENFLIES proved is the following: simple closed curves are *closed Jordan curves.*[7] When he studied Lie-groups, BROUWER relied on SCHOENFLIES' work. However, in the winter of 1908–1909 BROUWER suddenly discovered that SCHOENFLIES' results were not reliable. In "Zur Analysis Situs" [18], BROUWER gave a series of devastating counterexamples. BROUWER does not criticize SCHOENFLIES' theory of simple closed curves, but attacks his more general theory of closed curves. In the paper he gave the sensational example of a closed curve that splits the plane into three domains of which it is the common boundary. It is also the first example of an indecomposable continuum. SCHOENFLIES' general theory of closed curves and domains had to be rejected entirely.

1.4.3. *Dimension theory: "Beweis der Invarianz der Dimensionenzahl"*

It goes far beyond the purpose of the present paper to discuss BROUWER's entire work in topology. We will restrict ourselves to the two major papers on dimension theory. For a more extensive treatment of BROUWER's work in dimension theory we refer to [56] and for BROUWER's topological work as a whole to FREUDENTHAL's comments in [37][8]. BROUWER's "Beweis der Invarianz der Dimensionenzahl", submitted in June 1910 and published in 1911 [19] marks, according to FREUDENTHAL, the onset of a new period in topology. Although the paper is short and merely contains a simple proof of the invariance of dimension, "it is in fact much more than this – the paradigm of an entirely new and highly promising method, now known as algebraic topology. It exhibits the ideas of simplicial mapping, barycentric extension, simplicial approximation, small modification, and, implicitly, the mapping degree and its invariance under homotopic change, and the concept of homotopy class." [37, p. 436]. We will give a sketch of the proof. The main problem is to prove the central

Theorem In an m-dimensional manifold the one-to-one continuous image of an m-dimensional domain contains a domain in every neighbourhood of any of its points.

The theorem expresses a weak form of domain invariance and it implies the invariance of dimension. By the way, for BROUWER an m-dimensional manifold is not an abstract notion. In "Über Abbildung von Mannigfaltigkeiten" [20, p. 462] (which goes back at least to January 1910 according to FREUDENTHAL

[7] This is not trivial. Because it can be proved that a closed Jordan curve dividing the plane in two domains is accessible from both domains, we can say that closed Jordan curves are simple closed curves and in a sense SCHOENFLIES proved the converse of Jordan's theorem.

[8] Also DUDA's well-written paper [35] on the history of the notion of dimension is worth reading.

[37, p. 473]) BROUWER gives a constructive definition of an m-dimensional simplicial manifold. For BROUWER an m-dimensional simplicial manifold is – in modern terms – a connected m-dimensional topological manifold which possesses a locally finite simplicial subdivision. BROUWER's terminology is the following. He first defines an "m-dimensional element" as the one-to-one continuous image of a simplex of the m-dimensional number space $\mathbb{R}^m$. An m-dimensional manifold is constructed out of m-dimensional elements by joining them together in such a way that two elements are either disjoint or possess a common p-dimensional edge ($0 \leq p \leq m-1$), while in every vertex the elements meet in the same way in which in the m-dimensional number space the simplexes of a "simplex star" meet. Such a star is everywhere dense in some neighbourhood of some point O, and it is the union of a collection of simplexes that do not enter each others interiors and of which any two have at most a p-dimensional edge ($p < m$) in common [20, p. 454]. BROUWER's definition of an n-dimensional manifold clearly shows a constructivistic style.

Let us return to the central theorem in "Beweis der Invarianz der Dimensionenzahl". It is very natural to attempt to prove it by contradiction. That is what BROUWER did. He assumes that the mapping involved is such that there is a neighbourhood of a certain image point in which the image set is nowhere dense[9]. This means that in the m-dimensional manifold there exists an m-dimensional cube K which is mapped by a one-to-one continuous function ϑ onto a nowhere dense, connected, perfect set C. The problem of the invariance of dimension is in this way reduced to the question whether we can derive a contradiction by considering continuous mappings of an m-dimensional cube. In his proof BROUWER considers in a m-dimensional manifold the image of of a m-dimensional cube K under a single-valued continuous mapping α and proves the crucial

Lemma If the mapping α is such that the maximum displacement of the points of the cube is less than half an edge, there exists a concentric and homothetic cube that is contained entirely in the image set of the first cube.

BROUWER considers a simplicial subdivision of the cube K, he gives the simplexes an orientation, and defines a mapping β, which is equal to α on the vertices of the basic simplexes and elsewhere a linear extension of α defined by means of barycentric coordinates. Essentially BROUWER studies β (If β maps some simplexes on simplexes of a lower dimension BROUWER slightly modifies β). He defines a concentric and homothetic cube K' chosen small enough that the β-image of the boundary of K does not penetrate into it and he considers the set K_1 of points in K' that under β are not the image of points on the edges (these are the at most $(m-2)$-dimensional simplexes that occur in the boundary of the basic simplexes) of the simplexes. Any two points of K_1 can be connected by

[9] A lies nowhere dense in B means to BROUWER that a neighbourhood of a point of A always contains points not belonging to A.

a polygonal path lying entirely in the set. With respect to the points P of K_1 that are not the images of the points of the $(m-1)$-dimensional edges of the simplexes and which BROUWER calls the ordinary points of K_1, he determines in fact the degree of β. BROUWER considers for any ordinary point P, the number p of the positively oriented image-simplexes that cover P minus the number p' of negatively oriented simplexes. When we cross the edge of an image simplex this difference does not change, so $c = p - p'$ is a constant for all ordinary points of K_1. Moreover, c must be one, because it cannot change when β is changed continuously into the identity. Because we can make the simplicial subdivision as fine as we want, c is also equal to one for the original mapping α. This proves the lemma.

In order to get the contradiction needed to prove the domain invariance theorem, BROUWER now maps the nowhwere dense image set C under ϑ back into the cube. He does this also by means of simplicial approximation and barycentric extension. The set C is enclosed by an m-dimensional cube K and is subjected to a simplicial subdivision by dividing it up into nm little cubes which are divided into basic simplexes. The simplexes that contain points of C in their interior or onto their boundary form a set F. We now define a one-to-one continuous mapping η which maps C onto a nowhere dense subset of K as follows. η maps a vertex of a simplex belonging to F to a point whose image under ϑ lies in one of the simplexes that posses that vertex. The (other) points of C are given images by means of barycentric extension. Suppose now that $\eta \circ \vartheta(R) = P$, then we can choose n so large that for all R the distance RP remains below a given positive number. We can then apply the Lemma to the mapping $\eta \circ \vartheta$ and we have a contradiction.

When one looks at this proof one notices the great conceptual clarity. Moreover, the approach by means of simplicial approximation is similar to the way in which in the dissertation the continuum was handled by means of the dual scale. In an interesting paper Dubucs has shown that in the first decade of this century two different trends can be distinguished in topology: a Cantorian set-theoretic current and a combinatorial current. Within the set-theoretic current topology is developed starting from very general abstract notions, while the combinatorial approach is clearly constructivistic in the sense that topology is developed starting from elementary objects like simplexes and unnecessary general abstract notions are avoided. BROUWER's approach by means of simplicial approximation is combinatorial and DUBUCS rightly argues that "the topological works of BROUWER are related to his constructivistic preoccupations, although they, obviously, were not written within a strict intuitionistic framework".[10] Illustrative with respect to the success and influence of BROUWER's work on this point is the following quotation from a well-known textbook on combinatorial or algebraic

[10] "les traveaux topologiques de BROUWER se relient à ses préoccupations constructivistes, bien qu'ils ne soient évidemment pas menés dans un cadre intuitioniste strict." [34, p. 134]

topology from 1934: "Although there are extended theories about arbitrary subsets of Euclidean space, we will not deal with such a general notion of figure. It would involve us in undesirable set theoretic difficulties. Narrow enough to avoid these difficulties and broad enough to encompass almost all interesting figures, is the notion of complex introduced by L. E. J. BROUWER".[11] SEIFERT and THRELLFALL exaggerate when they attribute the notion of simplicial complex entirely to BROUWER, but the quotation shows nicely how well BROUWER's work did fit within the combinatorial tradition, with its dislike of "undesirable set theoretic difficulties".

1.4.4. *Dimension theory: "Über den natürlichen Dimensionsbegriff"*

As we have seen, the notion of the degree of a mapping is implicitly present in "Beweis der Invarianz der Dimensionenzahl". It is explicitly given in "Über Abbildung von Mannigfaltigkeiten" [20, p. 462]. In that paper, after having introduced the degree of a mapping, BROUWER proves by means of it, among others, the theorem that reads: "A continuous mapping of an n-dimensional sphere into itself without a fixpoint possesses the degree -1 for even n and the degree $+1$ for odd n" [20, p. 471]. For an even-dimensional sphere the theorem implies that there exists a fixed point if the mapping can be transformed continuously into the identity. BROUWER had already proved the 2-dimensional case of this theorem in 1909 by means of methods due to CANTOR and SCHOENFLIES [16]. Let us, however, return to dimension theory. In his "Beweis der Invarianz der Dimensionenzahl" BROUWER does not use a clearly defined dimensional invariant. Yet, as ALEXANDROFF has pointed out, implicitly there is one: for a sufficiently small $\varepsilon > 0$ there does not exist a continuous mapping from the n-dimensional cube into an $(n-1)$-dimensional polyhedron which moves each point for at most ε [4, p. 618]. An exercise book and fragments from the beginning of 1910 show, according to FREUDENTHAL, that BROUWER was acquainted with a very natural definition that POINCARÉ had proposed: "A continuum is n-dimensional, if it can be dissected in separate pieces by one or more $(n-1)$-dimensional continua". [37, p. 548]. Probably BROUWER heard about the definition during the 1909/10 Christmas holydays that he spent in Paris. This was the leading idea of BROUWER's first approach to the problem of the dimensional invariance. That first approach did not succeed. However, after the publication of "Über den natürlichen Dimensionsbegriff" BROUWER returned to his first approach. The reason was the following. The copy of the "Mathematische Annalen" in which BROUWER's proof of the invariance of dimension appeared contained also an elegant and

[11]"Obwohl es ausgedehnte Theorien über beliebige Teilmengen des euklidischen Raumes gibt, werden wir es nicht mit einem so allgemeinen Begriff der Figur zu tun haben. Es würde uns in unerwünschte mengentheoretische Schwierigkeiten verwickeln. Eng genug, um diesen Schwierigkeiten zu entgehen, und weit genug, um fast alle interessante Figuren zu umfassen, ist der von L.E.J. BROUWER eingeführte Begriff des Komplexes [...]" [68, p. 4]

simple proof of the same theorem by LEBESGUE. BROUWER was highly irritated, because he did not feel like sharing the honours with somebody else. Moreover, BROUWER soon discovered that LEBESGUE's proof was wrong, but it was cold comfort, because the wrong proof had great intuitive appeal and the "paving principle" on which it was based was clearly a flash of genius. BROUWER was right and LEBESGUE was wrong but LEBESGUE's proof had marred BROUWER's pleasure. There followed correspondence between BROUWER and the editor of the Annalen, BLUMENTHAL and between BROUWER and LEBESGUE, who refused to admit his mistake; instead of that he ignored BROUWER's challenge to give a proof of the paving principle. He promised simple proofs, which he did not produce and he reacted on irrelevant points. The result was that BROUWER broke off his direct contact with LEBESGUE in the middle of 1911. However, for many years he continued to make critical references to LEBESGUE in print (For an extensive discussion of the BROUWER-LEBESGUE dispute we refer to [56]). The paving principle was not easy to prove. At the time BROUWER was maybe the only mathematician who could do it. LEBESGUE succeeded ten years later in 1921. According to his own testimony, BROUWER found a proof within a few days and with it another proof of the invariance of dimension. LEBESGUE phrased the paving principle as follows: "If each point of a domain D of n dimensions belongsto at least one of the closed sets $E_1, E_2, \dots, E_p$, finite in number, and if these sets are sufficiently small, then there are points common to at least $n+1$ of these sets" [56, p. 158], translation by Johnson). In the background must have been the insight that, for example, one cannot build a brick wall without points that belong to at least three bricks. The paving principle implies the invariance of dimension because we can choose the E_i's in such a way that every point of an n-dimensional Euclidean space is covered by at most $n+1$ sets. BROUWER's proof of the principle is based on his original approach to the invariance of dimension by means of POINCARÉ's definition. His new proof of the invariance of dimension and his proof of the paving principle appeared in 1913 in "Über den natürlichen Dimensionsbegriff" [21]. In that paper BROUWER first considers POINCARÉ's definition: "A continuum is n-dimensional, if it can be dissected in separate pieces by one or more $(n-1)$-dimensional continua" and he points out several objections to this definition. For example, a definition of a "continuum" is required. Such a definition should exclude, however, a double cone in Euclidean space, because otherwise it would be 1-dimensional. In order to overcome these objections, BROUWER gives a definition applicable to "normal sets" (in the sense of FRÉCHET), these are separable metric spaces with no isolated points. By definition such a normal set π is a *continuum* if for every two of its elements m_1 and m_2 there exists a closed connected set which is a subset of π and contains m_1 and m_2. BROUWER's definition of dimension is based on the notion of "separating set". In a normal set π two closed subsets ρ and ρ' of π are separated in π by a third closed subset π_1 of π, iff ρ, ρ' and

π_1 are mutually disjoint and every connected closed subset of π that has points in common with both ρ and ρ', also contains at least one point in π_1.[12] The definition then follows POINCARÉ's definition: The expression "normal set π possesses the general dimension degree n", where n is an arbitrary natural number, means that for all choices of ρ and ρ' there exists a separating set π_1 with dimension degree $n - 1$, while it is not the case that for each choice of ρ and ρ' there is also a separating set of lesser degree. π possesses dimension degree 0 if it contains no continuum as a part. The main goal of BROUWER's paper is then to justify this definition and to prove that n-dimensional manifolds, in his sense of the word (see §1.4.3), indeed possess dimension n. Because the concept of dimension degree is a topological invariant, the invariance of dimension immediately follows. BROUWER first gives his definition of dimension degree a non-inductive form and then he breaks up the proof in two parts. In the first part he proves that an n-dimensional manifold has a dimension degree which is at most n and in the second part he proves that the dimension degree of an n-dimensional manifold is at least n. BROUWER gives his definition a non-inductive form as follows. Two persons, A and B, subject a set π to the following "dimension operation". Individual A chooses inside π arbitrarily two closed subsets ρ and ρ' and subsequently B separates ρ and ρ' by means of a closed subset π_1. The whole process is then repeated with respect to π_1: A chooses inside π_1 two closed subsets ρ_1 and ρ_1' and B chooses inside π_1 a subset π_2 that separates ρ_1 and ρ_1'. In this way the process is repeated indefinitely until possibly a set π_h occurs that contains no continuum as a part. If independently of the choices of A, B can make such choices that we always wind up with a last π_h for which $h \leq n$ and, on the other hand, if independently of the choices of the π_v, A can choose the ρ's in such a way that for π_h we never have $h < n$, then we say that π possesses the *general dimension degree* n. In the case that there exists no n such that independently of the choices of A, B cannot push h below n, then we say that π possesses an *infinite general dimension degree*. If a point P of π possesses neigbourhoods with general dimension degree m and no neighbourhoods with a general dimension degree less than m, BROUWER says that π *possesses in* P *the dimension degree* m. If in every point of a set the dimension degree is the same the set possesses a *homogeneous dimension degree*. BROUWER then proceeds to prove the theorem: An n-dimensional manifold possesses the homogeneous dimension degree n. First BROUWER proves that in the course of the repeatedly executed dimension operation B can make such choices that $h \leq n$. The proof is based on the

[12]There is a small mistake in BROUWER's definition of separation. In the iff-clause the "closed" should be deleted and the "connected" should be understood in the modern sense. Freudenthal has shown that BROUWER knew about this mistake already in 1913. However, the fact that he did not in time correct the mistake in print caused him a lot of trouble later. In 1923 Urysohn came up with a counterexample, BROUWER wrote several notes and papers correcting the mistake and in his later quarrel with Menger this issue played an exaggerated role. See [56] for the details.

possibility that depending on the choices of A, B can construct well-chosen simplicial subdivisions of the π's. The second part is the most difficult part. BROUWER must prove that during the repeatedly executed dimension operation A can make such choices that h does not become less than n. This part of the theorem is reduced to proving LEBESGUE's paving principle for a simplex. For a more detailed description of the proof we refer to [56].

BROUWER's definition of the general dimension degree is very abstract and, as JOHNSON has rightly pointed out in [55, p. 173], contrary to his intuitionistic views of mathematics. Yet BROUWER does not argue abstractly with the definition; the paper primarily concerns a proof of the fact that the definition applies to n-dimensional manifolds. This proof is completely in line with the constructivistic style of his first paper on dimension theory. It is interesting to realize that BROUWER's constructivism undoubtedly prevented him from developing dimension theory any further. BROUWER's definition of dimension was the first one in a sequence. A decade later, in the first half of the 1920s, PAUL S. URYSOHN and KARL MENGER came up with definitions of dimension different from BROUWER's. MENGER, URYSOHN, ALEXANDROFF and others then started to develop a more abstract dimension theory, which, in the words of HUREWICZ and WALLMAN "justified the new concept by making it the cornerstone of an extremely beautiful and fruitful theory which brought unity and order to a large domain of geometry" [54, p. 4]. A discussion of those contributions goes far beyond the scope of this paper.

1.5. THE TURN OF 1917 AND AFTERWARDS

1.5.1. *Intuitionist set theory*

In 1912 BROUWER became professor at the University of Amsterdam. It meant the fulfilment of a great ambition. It also coincided with a shift of interest. After several years of intensive activity in topology BROUWER's interest shifted back to the foundations of mathematics. Undoubtedly his involvement with the reedition of SCHOENFLIES' texts on point set theory (published as [48]) also set him on this course. Then came the war, international communication was difficult and for some time BROUWER was not very active mathematically. Soon, however, things changed.

In 1917 BROUWER decided to publish some "Addenda and corrigenda" [22] to his (at that time ten years old) dissertation. Why? Let us return for a moment to the dissertation. In the dissertation point sets on the continuum are constructed by means of cuts. BROUWER wrote: "We can construct on the continuum discrete, individualized sets of points which are finite, of order type ω, of order type η or can be obtained from such sets of points by alternation or subordination. The number of these points is always denumerable, and likewise the number of intervals determined on the continuum by pairs of points from the set is denumerable.

In each of these intervals, and also in its totality, the set may be dense or not (by dense we mean: of the order type η after every well-ordered or inversely well-ordered set has been contracted to a single point)" [14, p. 45]. In order to clarify the situation further BROUWER considers a dense dual scale constructed on a segment (which is considered as a unit segment). The complete dense dual scale corresponds to a (potentially infinite) tree that ramifies into two branches at all vertices. This tree can be used to handle the whole segment. An arbitrary point set on the segment, however, corresponds to a (potentially infinite) subtree. BROUWER draws the following figure

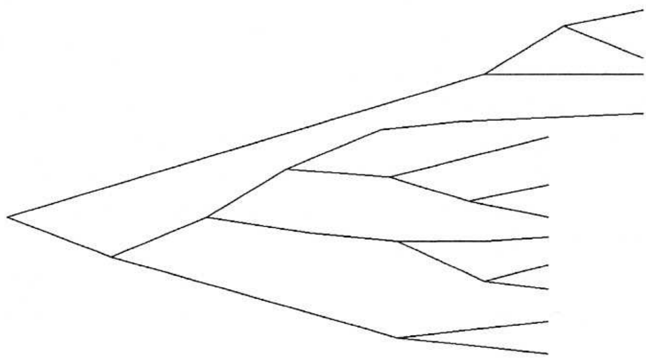

Figure 1

and writes: "If in this structure we cut off every branch which ceases to branch we are left either with nothing or with a continually multiplying dual branching. In the latter case the set is dense in the interval under consideration, in the former it is not." [14, p. 46]. Because the sets that can be constructed in this way are for an intuitionist the only well-constructed sets, the theorem that says that every closed point set can be divided into a perfect and a denumerable set (CANTOR's fundamental theorem) according to BROUWER needs no further proof for an intuitionist. In a review of the dissertation MANNOURY had already criticized BROUWER, saying that BROUWER's reasoning with respect to infinite sets now and then suffers from a certain vagueness [60, pp. 236–237]. BROUWER had not accepted that criticism. However, in 1917 he implicitly admitted that MANNOURY had been right. He wrote that the above described constructions of point sets are based upon "two essential suppositions, namely in the first place that the set can be constructed in such a way that it is individualized, i.e. so that the different infinitely proceeding branches of the tree produce different points and further that the individualized point set can be internally dissected, i.e that the process of breaking off branches which do not ramify any more, which must terminate after a denumerable number of steps, really can be effected. Now it is true that from the intuitionistic point of view the unrestricted comprehension axiom cannot be used [...]; therefore it is impossible to avoid special hypotheses about the way in which the point sets under consideration are constructed, and thereby about the limitation of the domain of set theory." [22, p. 146]. And he added the following revealing sentences: "This implies the right to consider, as contained in the construction principles, such hypotheses as are *desirable for the viability of the theory*. However it has lately become clear to me, as I hope to explain in a paper that will shortly appear, that the limits of set theory can be extended leaving out the two suppositions, mentioned above, on the construction principles, whilst preserving the viability of the theory" (the emphasis is ours – T.K. & J.v.M.). Obviously BROUWER had become aware of the fact that his 1907 point set theory was based on implicit assumptions that are not intuitively clear. Rejecting the mathematics based on those assumptions would endanger the viability of the theory and that is how he defended his early work. It is a weak defense and BROUWER must have known that. Assumptions added ad hoc in order to preserve the "viability of the theory" are at heart unacceptable in intuitionistic mathematics. Although it is not clear from his writings the problem must have bothered BROUWER considerably. Anyway, he solved it. Once more BROUWER brought about a revolution, this time in his own work. *Before 1917* BROUWER *had defended an intuitionistic philosophy of mathematics, but his mathematics had still been quite classical. About 1917* BROUWER *created intuitionistic mathematics*. BROUWER's first results are in a long 70 page paper "Begründung der Mengenlehre unabhangig vom logischen Satz vom ausgeschlossenen Dritten". The first part appeared in 1918 [23], the second part in 1919 [24]. [11] contains an introduction to those papers. It became clear that intuitionistic mathematics would be different from classical mathematics, not only with respect to methods but also with respect to its contents. After 1917 "CANTOR's fundamental theorem" was wrong [11, p. 234]. Before 1917 it had been evidently true (see [12, p. 140]). BROUWER explained this by saying that [12] was one of the early papers in which he was not fully aware of the consequences of intuitionism [11, p. 234, footnote]. There are no signs that BROUWER experienced the consequences as dramatic. Yet that is what they were, and not only in mathematics, also in BROUWER's personal relations with other mathematicians.

1.5.2. *After 1917*

We will not elaborate upon BROUWER's intuitionistic mathematics and restrict ourselves to a few remarks[13]. The intuitive continuum disappears from the scene. The generating processes of the order types of ω and η are explicitly extended to include free choices and a new notion of set, the "BROUWER species" is

[13] Also ARZARELLO [6] argues that BROUWER's topological work and his intuitionism are related. He attempts to show how the topological work influenced BROUWER's later post 1917 intuitionistic mathematics.

introduced. Real numbers are introduced by means of choice sequences and as a result classical real analysis has to be sacrificed to a considerable extent. Illustrative is the fact that in intuitionistic real analysis *every full function is uniformly continuous*. A full function is a function defined on a closed unit interval. This theorem is nowadays called the Fundamental Theorem. BROUWER found it difficult to find a satisfactory intuitionistic proof. But, also elsewhere, for example in dimension theory the consequences were tremendous. In 1928 BROUWER wrote that so far in fact only the contents of "Über den natürlichen Dimensionsbegriff" from 1913 had been intuitionistically rephrased in "Intuitionistische Einfuhrung des Dimensionsbegriffes" [25]. BROUWER said it as follows: "Because the acute investigations of URYSOHN, MENGER and ALEXANDROFF mainly possess a purely formal character, the intuitionistic (that is meaningful) part of dimension theory hardly extends beyond what had been reached in 1913"[14]. Nota bene that in this quotation most of dimension theory is discarded as meaningless. VAN STIGT describes the decade following the publication of BROUWER's intuitionistic foundation of set theory as the most productive stage of BROUWER's intuitionistic campaign. In terms of productivity that period was successful, but the success had a price. VAN STIGT writes: "It is a time when BROUWER took the centre stage as one of the chief contenders in the great Foundational Debate; a time of heated controversy which often turned into personal feuds and led to alienation from friends and fellow mathematicians." [73, p. 77]. Indeed. Very dramatic was BROUWER's conflict with HILBERT, the uncrowned king who ruled from Göttingen, at that time the centre of the mathematical world. At first BROUWER's intuitionistic mathematics did not attract much attention internationally. However, in the 1920s BROUWER's papers on intuitionistic mathematics started to appear in international journals. As a result the relationship between HILBERT and BROUWER, which had been cordial for many years – in 1919 HILBERT had even offered BROUWER a chair in Göttingen – rapidly deteriorated. HILBERT, who greatly respected BROUWER as a mathematician, realized that the intuitionistic attack on the principle of the excluded third was an attack on a substantial part of classical mathematics. He experienced BROUWER's work as a serious threat and he decided to act. HILBERT formulated his programme of metamathematics to prove the consistency of the classical mathematical theories. The confrontation of the two mathematicians reached a climax in 1928 when HILBERT was ill and worried that after his death his fellow-editor of the Mathematische Annalen, BROUWER, would gain too much influence. HILBERT dismissed BROUWER from the board of the Annalen. For an extensive treatment of the "Crisis of the Mathematische Annalen" we refer to [29]. BROUWER was a

[14]"Weil indessen die scharfsinnigen Untersuchungen von Urysohn, Menger und Alexandroff grossenteils einen bloss formalistischen Charakter besitzen, reicht der intuitionistische (d. h. sinnvolle) Bestand der Dimensionstheorie heute kaum ueber denjenigen von 1913 hinaus." [26, p.954]

sensitive man. The Annalen affair hurt him very much and it was followed by a period of despair and deep depression. Until his death in 1966 BROUWER's life had ups and downs. Although there were some successes and although he went on to publish about intuitionistic mathematics until 1955, the last part of his life was not very happy. Maybe one can say that through his personality and the course that he had set out for himself BROUWER was predetermined to live unhappily. With respect to his personality the following story is illustrative. According to VAN DALEN [30, p. 23], the topologist J. DE GROOT once told that after a meeting he drove BROUWER home. Underway BROUWER summed up all colleagues with whom he had at the time a bad relationship. Suddenly he said: But you and I also still have an unfinished quarrel! DE GROOT suggested to BROUWER not to pursue the matter any further.

1.6. CONCLUSIONS

It should be clear by now that it is quite possible to view BROUWER's foundational and mathematical work as a unity. A serious opposition between BROUWER's topological and foundational work can only be constructed if one abstracts from the chronology and if one opposes BROUWER's topological work from before 1917 with his foundational views from after 1917. This is, however, what one should not do. If one follows BROUWER's intellectual development chronologically one notices that before 1917 BROUWER's intuitionism was immature. It was a not yet fully developed view of the foundations of mathematics, which opposed other views of the nature of mathematics but which did not attack or doubt the bulk of classical mathematics. BROUWER's early intuitionism did not at all exclude his topological work. On the contrary, in the "construction of mathematics" as BROUWER viewed it in his 1907 dissertation, topological notions played a major role and BROUWER's topological work can be seen as a necessary consequence of this fact. To a large extent the topological papers are constructivistic in the style of the dissertation and there is a basic unity between that early intuitionism and the topological work. In this respect it is interesting that, although, on the one hand, BROUWER's foundational interest led him towards topology, on the other hand, in the case of dimension theory his constructivistic views, undoubtedly, prevented BROUWER from developing the theory any further. In 1917 the situation changed. BROUWER became aware of the full implications of intuitionism and he realized that mathematics had to be rebuilt completely and that is what he set out to do. At that moment his mathematics started to deviate seriously from the mainstream. Actually it is very fortunate that mature intuitionism was not born in 1907 from BROUWER's head like Pallas Athena from the head of Zeus. Because in that case his contribution to topology would probably have been much smaller.

Part II: Dutch Topology after L. E. J. Brouwer and de Groot's Compactification Problem

2.1. TOPOLOGY IN THE NETHERLANDS AFTER BROUWER

2.1.1. *Introduction*

In this part the emphasis is on Dutch contributions to dimension theory after BROUWER. However, we will first provide some background. We will make some introductory remarks on topological work in The Netherlands between the two world wars[15]. Those remarks will necessarily be incomplete, in particular with respect to applications of topology in other areas of mathematics. We will show that although BROUWER himself no longer actively contributed to topology, through his presence and the fact that great mathematicians often visited him, he nevertheless exerted considerable influence on Dutch topology. As for the developments after World War II we will restrict ourselves even further and we will merely briefly discuss the role of DE GROOT. Finally we will describe the history of DE GROOT's compactification problem.

2.1.2. *Between the world wars*

In the first part of this paper we saw that at some time before 1917 BROUWER stopped working in topology and turned his attention to an intuitionistic foundation of mathematics. Yet after World War II he ranked among the first mathematicians and he was frequently visited by foreign scholars. Moreover, in particular, Urysohn's dimension theory as it appeared in the first part of his "Mémoire sur les multiplicités cantoriennes" (Fundamenta Mathematicae, volumes 7 and 8) fascinated BROUWER [5, p. 116]. It seems to be precisely the kind of topology that he would have approved of if he had taken his intuitionism less seriously. The result was that in the period 1925–1926 P. ALEXANDROFF, MENGER and VIETORIS stayed with BROUWER. Their stay was influential. MENGER brought, for example, WITOLD HUREWICZ (1909–1956) from Vienna to Amsterdam. HUREWICZ, who became BROUWER's assistant, did his pioneering research in topology, for example on the theory of homotopy groups, in Amsterdam. In 1937 he left for the United States.

Remarkable is the winter 1925–26, when EMMY NOETHER spent the Christmas holidays in Blaricum, the village where BROUWER lived. NOETHER lectured there too and according to ALEXANDROFF the beginning of her group theory (her homomorphism and isomorphism theorems) reached its final stage in Blaricum [5, pp. 120–121]. Also the young BARTEL L. VAN DER WAERDEN participated in the discussions. It was there that NOETHER developed the view that group theory should be the foundation for combinatorial topology, and that numerical invariants like the Betti-numbers should be replaced by homology-groups, an

[15] For a more extensive treatment of topology in the Netherlands between the two world wars we refer to FREUDENTHAL [43].

idea which was finally accepted generally, although, for example, LEFSCHETZ was rather sceptical [5, p. 121].

The seminars organized on the occasion of the presence of the foreign guests also exerted influence on young Dutch mathematicians. For example, in 1925 in Amsterdam, DAVID VAN DANTZIG (1900–1959) attended a seminar led by BROUWER where ALEXANDROFF, MENGER, VIETORIS and HUREWICZ were present. It stimulated him to make a systematic study of topological groups, rings and fields from a unified point of view. He called this unified theory "Topological algebra". In 1931, at the University of Groningen, VAN DANTZIG defended a doctoral dissertation on topological algebra [32]. VAN DANTZIG discovered the solenoid. For a further discussion of VAN DANTZIG's work by VAN DER WAERDEN we refer to [9, pp. 231–233]. It is remarkable that VAN DANTZIG's work was totally independent of BROUWER's work. The later development of topological algebra is sketched in [74].

VAN DANTZIG's dissertation was written under the supervision of B. L. VAN DER WAERDEN (born in Amsterdam in 1903), who from 1929 to1931 was professor of mathematics in Groningen. From 1927 to 1929 VAN DER WAERDEN had been "Privatdozent" in Göttingen. D. J. STRUIK once recalled how EMMY NOETHER used to walk the streets of Göttingen accompanied by two young men, who were called in jest "die Unterdeterminanten". One of the "Unterdeterminanten" was VAN DER WAERDEN. He became internationally famous after he published his "Moderne Algebra I" and "Moderne Algebra II" in 1930 and 1931. VAN DER WAERDEN was very much aware of the importance of topology. He studied special problems in topological groups and also applications of topology to enumerative geometry. VAN DER WAERDEN left the Netherlands in 1931, but between the two world wars he considerably stimulated topology in the Netherlands. VAN DER WAERDEN also stimulated EGBERTUS R. VAN KAMPEN (1908–1942), who obtainded his doctoral degree in 1929 at Leiden University under the geometer W. VAN DER WOUDE with a dissertation on combinatorial topology [57]. To VAN KAMPEN, who left the Netherlands for the United States in 1931, we owe the (Seifert-)Van Kampen theorem for fundamental groups. The original proof is in [58]. For a discussion by SINGH VARMA we refer to [9, pp. 244–249] . VAN DER WAERDEN's influence is also clearly present in GERRETSEN's 1938 dissertation on the topological foundation of enumerative geometry [44].

Another highly talented mathematician attracted by BROUWER's genius was HANS FREUDENTHAL (1905–1990), who came from Berlin to Amsterdam because of intuitionism. BROUWER had visited Berlin in the winter of 1926/27 and FREUDENTHAL was fascinated by BROUWER's foundational ideas. BROUWER invited FREUDENTHAL to come to Amsterdam. FREUDENTHAL became BROUWER's other assistant next to HUREWICZ, who had come to Amsterdam because of topology. FREUDENTHAL succeeded IRMGARD GAWEHN, an intriguing German lady. She had written an excellent Ph. D. thesis on topology with ROSEN-

THAL in Heidelberg: she found a purely topological characterization of all 2-manifolds without boundary. (Independently, similar results were obtained by several other mathematicians.)[16] FREUDENTHAL was a student of HOPF and he had written a dissertation on topology dealing with the so-called Freudenthal-ends of topological spaces [40]. In 1937 FREUDENTHAL became "conservator" at the University of Amsterdam. About to be naturalized when the second world war started, FREUDENTHAL became stateless and in 1940 he lost his position. Soon after the war FREUDENTHAL became professor of geometry at the University of Utrecht. He would stay in the Netherlands until his death in 1990. He left a lasting impression on Dutch mathematics through his mathematical, pedagogical and historical work. Although FREUDENTHAL came because of BROUWER's intuitionism, he for some time continued to work in topology. He proved, for example, the suspension theorems in HUREWICZ' theory of homotopy groups [41]. For a discussion by LEMMENS of FREUDENTHAL's contributions in this respect we refer to [9, pp. 336–340].

FREUDENTHAL's work stimulated A. VAN HEEMERT to write a doctoral dissertation on the R_n-adic generation of general-topological spaces with applications on the construction of non-separable continua [51]. Another pupil of FREUDENTHAL was J. DE GROOT who wrote a doctoral dissertation on compactifications [45]. On July 28, 1945, FREUDENTHAL wrote to HOPF: "I have a very industrious pupil, J. DE GROOT. He has done very beautiful topological things" (Freudenthal Archive, Box 60, Correspondence with Hopf 1928-1954). It is interesting that GERRETSEN, VAN HEEMERT and DE GROOT all defended their dissertations at Groningen University. The theses were supervised by G. SCHAAKE, a clever geometer, but not a topologist. In the case of GERRETSEN, this is not strange. GERRETSEN provided a rigid foundation for many results in enumerative geometry that SCHAAKE had obtained. In the case of VAN HEEMERT and DE GROOT it is remarkable. One is tempted to suppose that the reason will have been a combination of two things. BROUWER had turned to intuitionism. VAN DER WAERDEN had left the Netherlands for Leipzig and in 1939 FREUDENTHAL was only "conservator" at the University of Amsterdam without the "jus promovendi", while during the war he was even unemployed. A student interested in

[16] Later FREUDENTHAL said about GAWEHN: "She was a particularly beautiful woman, who had written a brilliant topological doctoral thesis with ROSENTHAL in Heidelberg. Later she showed up in philosophical circles in Berlin and also in that area she was considered to be a genius. However, in the Netherlands it soon became clear that she did not know anything about mathematics nor philosophy. [...] During the war she died in a mental institution. A possible explanation is that she possessed the capacity to identify to an exceptional degree with someone else with whom she was in love, a mathematician in Heidelberg and a philosopher in Berlin." [3, p. 118]. GAWEHN's topological dissertation was published in the Mathematische Annalen. FREUDENTHAL was rarely wrong in his judgement, but had a rather acute way of expressing himself. It seems probable that FREUDENTHAL's statement "she did not know anything about mathematics" is at least exaggerated. To our knowledge there is no proof whatsoever that GAWEHN did not write her dissertation herself.

a Ph. D. in topology was, obviously, forced to turn to a non-topologist for formal supervision. Moreover, it seems that VAN DER WAERDEN had left a positive climate in Groningen with respect to topology.

2.1.3. *After World War II: Johannes de Groot*

Because BROUWER had stopped working in topology, after 1930 FREUDENTHAL was in a sense topologist number one in the Netherlands. In spite of the positive influence that he exerted on the development of the subject in the Netherlands, for VAN DER WAERDEN topology was not a main concern. Moreover, he soon left the country. As for FREUDENTHAL, although he continued to work in topology for some time, his interests were not exclusively topological. He also worked in the foundations of mathematics, in geometry and in other areas. That is why after World War II, the author of the above-mentioned thesis on compactification, JOHANNES DE GROOT (1914–1972), took over the role of leading Dutch topologist. DE GROOT almost exclusively concentrated on topology, with great success. After World War II, DE GROOT dominated Dutch topology for more than two decades. In 1948 he was appointed full professor at the Technological University of Delft. From 1952 until his death in 1972 he was professor at the University of Amsterdam. For an extensive survey of his work, a list of publications and a list of doctoral dissertations written under him, we refer to BAAYEN and MAURICE [7]. He was an impressive personality, creative, hardworking, fair and reliable, with in certain respects an aristocratic life style. He liked expensive hotels, expensive port wine and he drove a Mercedes two-seater. Many a young topologist remembers being transported by DE GROOT after having undergone a continuous transformation to fit in the back-space of the Mercedes. Internationally he became quite well known. DE GROOT was always full of ideas which he would share with everyone. It is characteristic of DE GROOT that when he felt that his death was near, he dictated to his wife his last mathematical instructions for his students. The fact that ten of DE GROOT's Ph. D. students afterwards pursued academic careers (not necessarily in topology) illustrates on the institutional level his positive influence on Dutch mathematics. They are: P. C. Baayen, T. J. Dekker, J. Ch. Boland, H. de Vries, T. van der Walt, M. A. Maurice, A. B. Paalman-de Miranda, J. M. Aarts, E. Wattel and A. Verbeek.

2.2. DE GROOT'S COMPACTIFICATION PROBLEM

2.2.1. *Dimension functions*

A complete survey of DE GROOT's work is given in [7]. We will restrict ourselves to one example: the compactification problem. In order to be able to discuss that example it is necessary that we briefly mention some results from modern dimension theory. Throughout the remaining part of this paper, all topological spaces are assumed to be separable and metrizable.

Dimension functions in some sense measure the topological complexity of a topological space. A space with dimension 2 is more complex than a space with dimension 1, etc. Different dimension functions measure different things and some dimension functions are more interesting than others. To begin with, we briefly review some important facts from dimension theory.

As we saw above, the first definition of a dimension function was given by BROUWER [21]. His concept refers to the notion of a *partition* of a topological space. If A and B are closed subsets of a topological space X then a closed subset C of X is called a partition between A and B if there exist open subsets $U, V \subseteq X$ such that

$$A \subseteq U, B \subseteq V, U \cap V = \emptyset \text{ and } X \setminus C = U \cup V.$$

We also say that C *separates* A and B. Motivated by the simple observation of POINCARÉ [64] that solids can be partitioned by surfaces, surfaces by lines, and lines by points, BROUWER's definition of the *Dimensionsgrad*, $\operatorname{Dimgrad} X$, of a space X in fact is the following:

$\operatorname{Dimgrad} X = 0$	$\Leftrightarrow$	every subcontinuum of X is a point,
$\operatorname{Dimgrad} X \leq n \ (\geq 1)$	$\Leftrightarrow$	every pair A, B of disjoint closed subsets of X can be separated by a set S with $\operatorname{Dimgrad} S \leq n-1$,
$\operatorname{Dimgrad} X = n$	$\Leftrightarrow$	$\operatorname{Dimgrad} X \leq n$ and $\operatorname{Dimgrad} X \not\leq n-1$,
$\operatorname{Dimgrad} X = \infty$	$\Leftrightarrow$	$\operatorname{Dimgrad} X > n$ for $n = -1, 0, 1, \ldots$.

It is easy to see that if X and Y are homeomorphic spaces then $\operatorname{Dimgrad} X = \operatorname{Dimgrad} Y$, i.e., Dimensionsgrad is a topological invariant. BROUWER showed that $\operatorname{Dimgrad} \mathbb{I}^n = n$ from which it follows that $\mathbb{I}^n$ is not homeomorphic to $\mathbb{I}^m$ if $n \neq m$.

Examples of spaces X with $\operatorname{Dimgrad} X = 0$ are the rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{P}$ and the Cantor middle third set $\mathbf{C}$. It is easily seen that every product of spaces with Dimensionsgrad 0 has Dimensionsgrad 0. So, in particular, $\mathbb{Q}^\infty$, the product of infinitely many copies of $\mathbb{Q}$, has Dimensionsgrad 0.

Let ℓ^2 denote separable HILBERT space and put

$$E = \{x \in \ell^2 : x_i \text{ is rational for every } i\}.$$

This is called *Erdös' space* (see ERDÖS [39]). We will show that $\operatorname{Dimgrad} E = 0$. To this end, let $K \subseteq E$ be a continuum. The inclusion $i \colon \ell^2 \hookrightarrow \mathbb{R}^\infty$ is injective and continuous and so $i[K]$ is a subcontinuum of $\mathbb{Q}^\infty$; in addition, $i[K]$ has the same cardinality as K. Since the Dimensionsgrad of $\mathbb{Q}^\infty$ is 0, it follows that $|i[K]| = 1$ and so $|K| = 1$. We conclude that $\operatorname{Dimgrad} E = 0$.

There is something pathological about E. It can be shown that if $U \subseteq E$ is open and *bounded* (in the sense that the set $\{\|x\| : x \in U\}$ is bounded) then

U is not closed (ERDÖS [39]; see also ENGELKING [38, 1.2.15]). So the open neighbourhood $\{x \in E : \|x\| < 1\}$ of 0 in E does not contain an open and closed neighbourhood of 0 in E. This distinguishes E from spaces such as $\mathbb{Q}$, $\mathbb{P}$ and $\mathbf{C}$. In modern terminology, E is *not* of dimension 0, but $\mathbb{Q}$, $\mathbb{P}$ and $\mathbf{C}$ are. To make this precise, for a space X define its *large inductive dimension* (or BROUWER–Čech dimension), $\operatorname{Ind} X$, as follows:

$$
\begin{array}{lcl}
\operatorname{Ind} X = -1 & \Leftrightarrow & X = \emptyset, \\
\operatorname{Ind} X \leq n \ (\geq 0) & \Leftrightarrow & \text{every pair } A, B \text{ of disjoint closed subsets of } X \\
 & & \text{can be separated by a set } S \text{ with } \operatorname{Ind} S \leq n-1, \\
\operatorname{Ind} X = n & \Leftrightarrow & \operatorname{Ind} X \leq n \text{ and } \operatorname{Ind} X \not\leq n-1, \\
\operatorname{Ind} X = \infty & \Leftrightarrow & \operatorname{Ind} X > n \text{ for } n = -1, 0, 1, \ldots.
\end{array}
$$

This dimension function is obviously similar to the Dimensionsgrad but is better since it distinguishes between spaces such as $\mathbb{P}$ and E. It is easy to show that $\operatorname{Ind} \mathbb{Q} = \operatorname{Ind} \mathbb{P} = \operatorname{Ind} \mathbf{C} = 0$ but, interestingly. $\operatorname{Ind} E \neq 0$ (in fact, $\operatorname{Ind} E = 1$, see [38, 1.2.15]).

We saw (cf. footnote 9) that BROUWER claimed that he meant his definition of Dimensionsgrad to read exactly as the definition of the large inductive dimension; that it read otherwise was caused by a clerical error. Going into the heated discussions that were the result of this claim, would triple the length of this article and we will therefore not do it. It turns out that for locally compact, locally connected spaces (including the Euclidean spaces $\mathbb{R}^n$) the Dimensionsgrad and the large inductive dimension take the same value.

There are two other strongly related dimension functions that are of interest, namely, the *small inductive dimension* (or Menger–Urysohn dimension), and the *covering dimension* (or LEBESGUE covering dimension). The small inductive dimension $\operatorname{ind} X$ of a space X is defined as follows:

$$
\begin{array}{lcl}
\operatorname{ind} X = -1 & \Leftrightarrow & X = \emptyset, \\
\operatorname{ind} X \leq n \ (\geq 0) & \Leftrightarrow & \text{every singleton subset } \{x\} \text{ of } X \text{ and every closed} \\
 & & \text{subset } A \subseteq X \text{ with } x \notin A \text{ can be separated by a} \\
 & & \text{set } S \text{ with } \operatorname{ind} S \leq n-1, \\
\operatorname{ind} X = n & \Leftrightarrow & \operatorname{ind} X \leq n \text{ and } \operatorname{ind} X \not\leq n-1, \\
\operatorname{ind} X = \infty & \Leftrightarrow & \operatorname{ind} X > n \text{ for } n = -1, 0, 1, \ldots.
\end{array}
$$

This dimension function is obviously very strongly related to the large inductive dimension.

In the first part of this paper we saw that quite a different dimension function was introduced by LEBESGUE. Before we will present its formal definition, we give some background. Let us try to cover the square $\mathbb{I}^2$ with finitely many "small" rectangles in such a way that as few of the retangles as possible have points in common. At first one might come up with something as shown in the left part of Figure 2, where no more than four of the rectangles intersect. A moments reflection shows however that we can do better since the covering in the right part of Figure 1 shows that it can be done in such a way that no more than three of the rectangles intersect. Interestingly, this is how far one can go, even if one replaces "rectangle" by "arbitrary closed set", or by "arbitrary open set". This observation, the paving principle, is the basis for LEBESGUE's dimension function.

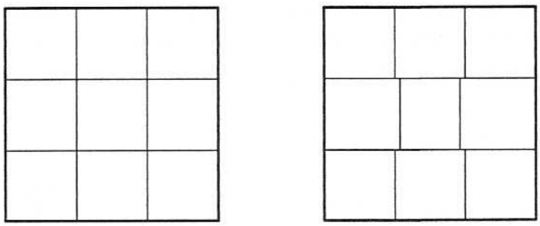

Figure 2

Let $\mathcal{U}$ be a cover of a space X and let $n \geq -1$. We say that the *order* of $\mathcal{U}$ is at most n, $\operatorname{ord}(\mathcal{U}) \leq n$, if for every $x \in X$,

$$|\{U \in \mathcal{U} : x \in U\}| \leq n+1.$$

If $\mathcal{U}$ is a cover of a space then a cover $\mathcal{V}$ of X is called a *refinement* of $\mathcal{U}$ provided that for every element $V \in \mathcal{V}$ there exists an element $U \in \mathcal{U}$ such that $V \subseteq U$. Now for a space X we define its *covering dimension* (or Čech–LEBESGUE dimension) $\dim X$ as follows:

$\dim X \leq n \; (\geq -1)$	$\Leftrightarrow$	for every finite open cover $\mathcal{U}$ of X there exists a finite open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\operatorname{ord}(\mathcal{V}) \leq n$.
$\dim X = n$	$\Leftrightarrow$	$\dim X \leq n$ and $\dim X \not\leq n-1$,
$\dim X = \infty$	$\Leftrightarrow$	$\dim X \not\leq n$ for every $n \geq -1$.

This dimension function differs from the previous three because it is not inductively defined. But interestingly, it turns out that for a given space X the dimension functions Ind, ind and $\dim$ take the same value. This result is known as the *coincidence theorem.*

Theorem 2.2.1 *For every separable metrizable space X we have*

$$\operatorname{ind} X = \operatorname{Ind} X = \dim X.$$

For a proof of this theorem, see ENGELKING [38, 1.7.7].

The following result due to BROUWER, LEBESGUE, MENGER and URYSOHN, is known as the *fundamental theorem of dimension theory.*

Theorem 2.2.2 *For every natural number n we have*

$$\operatorname{ind} \mathbb{R}^n = \operatorname{Ind} \mathbb{R}^n = \dim \mathbb{R}^n = n.$$

We will finish this section by presenting a particularly simple and transparant proof of the fact due to BROUWER that if $n \neq m$ then $\mathbb{I}^n \not\approx \mathbb{I}^m$.

A family of pairs of disjoint closed sets $\tau = \{(A_i, B_i) : i \in G\}$ of a space X is called *essential* if for every family $\{L_i : i \in G\}$, where L_i is a partition between A_i and B_i for every i, we have $\bigcap_{i \in G} L_i \neq \emptyset$; if τ is not essential then it is called *inessential.*

Consider $\mathbb{I}^n$ and for $i \leq n$ its faces

$$A_i = \{x \in \mathbb{I}^n : x_i = 1\} \text{ and } B_i = \{x \in \mathbb{I}^n : x_i = 0\}.$$

We will show that the collection of opposite faces $\{(A_i, B_i) : i \leq n\}$ of $\mathbb{I}^n$ is essential.

Theorem 2.2.3 *If C_i is a partition between A_i and B_i for every i then $\bigcap_{i=1}^n C_i \neq \emptyset$.*

PROOF: Assume that C_i is a partition between A_i and B_i for $i \leq n$ such that $\bigcap_{i=1}^n C_i = \emptyset$. For each $i \leq n$ we can find a continuous function $\xi_i \colon \mathbb{I}^n \to \mathbb{I}$ such that

$$\xi_i(A_i) = \{0\}, \xi_i(B_i) = \{1\} \text{ and } \xi_i^{-1}(1/2) = C_i.$$

Define $f \colon \mathbb{I}^n \to \mathbb{I}^n$ by $f(x) = (\xi_1(x), \ldots, \xi_n(x))$. Then f is continuous and does not take on the value $(1/2, \ldots, 1/2)$. For every $x \in \mathbb{I}^n \setminus \{(1/2, \ldots, 1/2)\}$ the ray from $(1/2, \ldots, 1/2)$ through x intersects the "boundary" $B = \bigcup_{i=1}^n A_i \cup \bigcup_{i=1}^n B_i$ of $\mathbb{I}^n$ in precisely one point, say $r(x)$. The function $r \colon \mathbb{I}^n \setminus \{(1/2, \ldots, 1/2)\} \to B$ is easily seen to be continuous. The function $g = r \circ f \colon \mathbb{I}^n \to B$ has the following properties:

$$g((0,1)^n) \cap (0,1)^n = \emptyset, \text{ and for every } i \leq n, g[A_i] \subseteq B_i \text{ and } g[B_i] \subseteq A_i.$$

Therefore, g has no fixed-point, which contradicts the BROUWER fixed point theorem.

So Theorem 2.2.3 shows that $\mathbb{I}^n$ has an essential family of pairs of disjoint closed sets of cardinality n. The question naturally arises whether the number n is best possible.

Theorem 2.2.4 *Let $n \in \mathbb{N}$. Then every family consisting of at least $n+1$ pairs of disjoint closed sets of $\mathbb{I}^n$ is inessential.*

PROOF: Let A and B be disjoint closed subsets of $\mathbb{I}^n$ and let $E \subseteq \mathbb{R}$ be dense.

We claim that there is a partition D in $\mathbb{R}^n$ between A and B such that

$$D \subseteq \{x \in \mathbb{R}^n : (\exists i \leq n)(x_i \in E)\}.$$

This is easy. Every point $x \in A$ has a neighbourhood of the form $\prod_{i=1}^n (a_i, b_i)$ with $a_i, b_i \in E$ for every $i \leq n$ such that $\prod_{i=1}^n [a_i, b_i] \cap B = \emptyset$. There is a finite family $\mathcal{F}$ of these neighbourhoods whose union covers A, and the boundary D of this union is contained in the union of the boundaries of the elements of $\mathcal{F}$. We conclude that D is the required partition between A and B.

Now let $\tau = \{(A_i, B_i) : i \leq n+1\}$ be a family consisting of $n+1$ pairs of disjoint closed subsets of $\mathbb{I}^n$. There exist $n+1$ pairwise disjoint dense subsets of $\mathbb{R}$. By the above there exist partitions D_i in $\mathbb{R}^n$ between A_i and B_i such that

$$D_i \subseteq \{x \in \mathbb{R}^n : (\exists i \leq n)(x_i \in E_i)\} \qquad (i \leq n+1).$$

Since the E_i are pairwise disjoint, a straightforward verification yields $\bigcap_{i=1}^n D_i = \emptyset$.

Observe that in Theorems 2.2.3 and 2.2.4 we formulated a topological property of $\mathbb{I}^n$ shared by no $\mathbb{I}^m$ for $m \neq n$. In particular we obtain:

Corollary 2.2.1 (Brouwer [21]) *Let $n, m \in \mathbb{N}$. If $n \neq m$ then $\mathbb{I}^n$ is not homeomorphic to $\mathbb{I}^m$.*

These remarks suggest the following definition: for a separable metrizable space X define its *partition degree*, $\operatorname{part} X \in \{-1, 0, 1, \ldots\} \cup \{\infty\}$, by

$\operatorname{part} X \leq n\ (\geq -1)$	$\Leftrightarrow$	every family of $n+1$ pairs of disjoint closed subsets of X is inessential,
$\operatorname{part} X = n$	$\Leftrightarrow$	$\operatorname{part} X \leq n$ and $\operatorname{part} X \not\leq n-1$,
$\operatorname{part} X = \infty$	$\Leftrightarrow$	$\operatorname{part} X \not\leq n$ for every $n \geq -1$.

In view of Theorems 2.2.3 and 2.2.4 it follows that $\operatorname{part} \mathbb{I}^n = n$ for every n. The following result (ENGELKING [38, Theorem 1.7.9]) therefore comes as no surprise.

Theorem 2.2.5 *For every separable metrizable space X we have*

$$\dim X = \operatorname{part} X.$$

This concludes our survey of dimension theory. For more information and historical comments, see HUREWICZ and WALLMAN [54] and ENGELKING [38]. We can now turn to the compactification problem.

2.2.2. *The compactification problem*

In his thesis, DE GROOT [45] posed an interesting problem in dimension theory. For a very accurate treatment of the problem and for many historical comments see the recent book by AARTS and NISHIURA [2].

In order to phrase DE GROOT's problem, let us again look at the definition of the small inductive dimension ind:

$$
\begin{array}{lll}
\operatorname{ind} X = -1 & \Leftrightarrow & X = \emptyset, \\
\operatorname{ind} X \le n\ (\ge 0) & \Leftrightarrow & \text{every singleton subset } \{x\} \text{ of } X \text{ and every closed} \\
 & & \text{subset } A \subseteq X \text{ with } x \notin A \text{ can be separated by a} \\
 & & \text{set } S \text{ with } \operatorname{ind} S \le n-1, \\
\operatorname{ind} X = n & \Leftrightarrow & \operatorname{ind} X \le n \text{ and } \operatorname{ind} X \not\le n-1, \\
\operatorname{ind} X = \infty & \Leftrightarrow & \operatorname{ind} X > n \text{ for } n = -1, 0, 1, \ldots.
\end{array}
$$

The definition of ind starts with assigning the value -1 to the empty space and then proceeds inductively. In a sense, the empty set is negligible or uninteresting and therefore serves as the basis for the inductive definition. What happens if one replaces the empty set by another space that for some reason is considered to be negligible or uninteresting and then defines a similar dimension function with that space as the starting point? One should be cautious here because we certainly want the new dimension function to be *topological*, i.e., to have the property that homeomorhic spaces have the same dimension. (Since there is only one empty set, for ind we have no problem.) Once a space is uninteresting, so is every space topologically homeomorphic to it. What we really want is to think of a whole *topological* class[17] of spaces to be negligible and to assign to every element of that class the dimension -1. Let us play with this idea a little bit. Let us take for our class of uninteresting spaces those spaces that contain at most one point[18]. The dimension function that we get from this class does not differ very much from the small inductive dimension. The real line $\mathbb{R}$ has dimension 1, $\mathbb{R}^2$ has dimension 2, etc. There is only one difference: a space containing precisely one point has dimension -1 instead of dimension 0 in the case of ind. So this new dimension function should not be considered because there is no real difference with ind. DE GROOT [45] took the class $\mathcal{C}$ consisting of all *compact* spaces as the starting point of a new dimension function[19]. He wanted to measure the "degree of compactness" of an arbitrary topological space. So in this setting, compact spaces are "uninteresting". The dimension function one obtains in this way is called *the compactness degree* $\operatorname{cmp} X$ of a topological space X: it is the dimension of a space *modulo* the class of all compact spaces, and is formally defined as follows:

[17] A class of spaces $\mathcal{C}$ is called *topological* if whenever $X \in \mathcal{C}$ and $X \approx Y$ then $Y \in \mathcal{C}$.
[18] Such spaces are quite uninteresting.
[19] This class is topological of course.

$\operatorname{cmp} X = -1$	$\Leftrightarrow$	X is compact,
$\operatorname{cmp} X \leq n\ (\geq 0)$	$\Leftrightarrow$	every singleton subset $\{x\}$ of X and every closed subset $A \subseteq X$ with $x \notin A$ can be separated by a set S with $\operatorname{cmp} S \leq n-1$,
$\operatorname{cmp} X = n$	$\Leftrightarrow$	$\operatorname{cmp} X \leq n$ and $\operatorname{cmp} X \not\leq n-1$,
$\operatorname{cmp} X = \infty$	$\Leftrightarrow$	$\operatorname{cmp} X > n$ for $n = -1, 0, 1, \ldots$.

So the first class of "uninteresting" separable metrizable spaces are those spaces X for which $\operatorname{cmp} X = 0$, and they are precisely the noncompact spaces that have a base $\mathcal{B}$ for the open sets such that the boundary of every element $B \in \mathcal{B}$ is compact. Those spaces are called *rimcompact* in the literature. So every locally compact space has $\operatorname{cmp} \leq 0$; in particular, $\operatorname{cmp} \mathbb{R}^n = 0$ for every n. So cmp is very much different from ind, although it is defined in a very similar way. One easily proves by induction that for every separable metrizable space X, $\operatorname{cmp} X \leq \operatorname{ind} X$. So the equality $\operatorname{cmp} \mathbb{R}^n = 0$ shows that the gap between cmp and ind can be arbitrarily large.

Not every space is rimcompact as the space

$$\mathbb{I}^2 \setminus (\{1\} \times (0,1))$$

shows. The question therefore naturally arises whether there are spaces with arbitrarily large compactness degree.

Lemma 2.2.1 *For every n let X_n be the space $\mathbb{Q} \times \mathbb{I}^n$. Then $\operatorname{cmp} X_n = n$.*

See AARTS and NISHIURA [2, Example 5.10(d)] for details.

Let X be a space. It is sometimes useful to study X as a subspace of a larger compact space Y. If X is dense in Y then we call Y a *compactification* of X. The set $Y \setminus X$ is called the *remainder* of the compactification Y. If X is not compact then there are many ways to compactify it. The largest compactification of X, its Čech–Stone compactification βX, is an interesting object to study but it is very large (in fact, it is not metrizable unless X is compact and metrizable). For a locally compact space X, its Alexandrov one-point compactification αX is the "smallest" compactification and it is metrizable iff X is separable and metrizable. For our purposes, αX is a "good" compactification. Since the class of spaces that can be compactified with one point coincides with the rather special class of all locally compact spaces, the question naturally arises what spaces can be compactified by adding a "small" object. An interesting partial result was obtained by ZIPPIN [75].

Theorem 2.2.6 *Let X be a separable metrizable space. Then X is rimcompact and topologically complete if and only if X has a compactification γX such that $\gamma X \setminus X$ is countable.*

So every topologically complete rimcompact space X has a compactification γX for which $\gamma X \backslash X$ is "small" because it has countably many points only. There are other interesting concepts of smallness besides having small cardinality. In dimension theory, a space is "small" if it has "small" topological complexity, i.e., has small dimension. So the "smallest" spaces in dimension theory are the zero-dimensional spaces.

It can be shown that a countable space is zero-dimensional. This follows e.g. from the Countable Closed Sum Theorem (ENGELKING [38, Theorem 1.5.3]), but it can also be seen directly. Indeed, let X be countable and let A and B be arbitrary disjoint closed subsets of X. There is a Urysohn function $f\colon X \to \mathbb{I}$ such that $f[A] = 0$ and $f[B] = 1$. Since $f[X]$ is countable, there exists $r \in \mathbb{I} \setminus f[X]$. Then $f^{-1}([0,r])$ is an open and closed subset of X that contains A but misses B.

So ZIPPIN's Theorem 2.2.6 in dimension theoretic terms says in particular that a rimcompact topologically complete separable metrizable space can be compactified by a adding a zero-dimensional set. In the framework of dimension theory it is therefore very natural to ask for a characterization of those spaces X which admit a compactification γX for which $\gamma X \setminus X$ has zero-dimensional remainder. This question was answered by DE GROOT [45] in his thesis (see also AARTS and NISHIURA [2, §VI.3]).

Theorem 2.2.7 *Let X be a separable metrizable space. Then X has a compactification Y such that $Y \setminus X$ is zero-dimensional if and only if X is rimcompact.*

(This result, in a slightly different setting, is also true for non-metrizable spaces. See FREUDENTHAL [42] and AARTS and NISHIURA [2, §VI.3] for more details.) This result is very satisfying since the "external" property of having a compactification with zero-dimensional remainder can be characterized by an "internal' property, namely, the property of being rimcompact.

As we said before, DE GROOT wanted to measure the "degree of compactness" of an arbitrary topological space. The rimcompact spaces have compactness degree ≤ 0 and are therefore close to being compact. This can now also be expressed in another way: the rimcompact spaces can be compactified by adding a set with low topological complexity, namely, a zero-dimensional set.

Since not every space is rimcompact, the question question naturally arises whether the spaces X with a compactification γX such that $\dim(\gamma X \setminus X) \leq 1$ can be characterized internally. This motivates the following definition.

Definition 2.2.1 *Let X be a space. The* compactness deficiency $\operatorname{def} X$ *of X is defined by*

$$\operatorname{def} X = \min\{\dim(\gamma X \setminus X) : \gamma X \textit{ is a compactification of } X\}.$$

It is clear that $\operatorname{def} X$ too measures a "degree of compactness" of an arbitrary topological space X. If $\operatorname{def} X$ is large then it is impossible to compactify X by adding a set of small topological complexity, i.e., X is far from being compact. Notice that in order to compute $\operatorname{def} X$ for a given space X one needs to consider the outside world ($\operatorname{def} X$ is external), while to compute $\operatorname{cmp} X$ one needs to perform computations in the space X itself ($\operatorname{cmp} X$ is internal). DE GROOT [45] conjectured that both invariants agree for all spaces.

Conjecture 2.2.1 (de Groot) *For every separable metrizable space X we have* $\operatorname{cmp} X = \operatorname{def} X$.

This conjecture has puzzled several generations of topologists, both in and outside the Netherlands. Observe that a compact space has only one compactification (itself) so that Theorem 2.2.7 can be reformulated as follows:

Theorem 2.2.8 *If X is a separable metrizable space then* $\operatorname{cmp} X = -1, 0$ *if and only if* $\operatorname{def} X = -1, 0$.

So the conjecture looks very promising. The following inequality (which can be proved rather directly) supports it even more (DE GROOT [45], [2, Theorem 5.8]):

Theorem 2.2.9 *For every separable metrizable space X,*

$$\operatorname{cmp} X \leq \operatorname{def} X.$$

There were several attempts to resolve DE GROOT's Conjecture. One was to split the compactification problem. That is, new invariants were introduced that take their values between cmp and def and natural questions that came up were tried to be answered. We will not list all the different attempts, that would lead us too far. We will focus instead on the following invariant that turned out to be fundamental later. If A is a subset of a topological space then $\partial(A)$ denotes its *topological* boundary, i.e., $\partial(A)$ is the closure of A minus its interior. The following concept is due to SKLYARENKO [69] who proposed it in 1960 as a candidate for the problem of finding an internal characterization of the compactness deficiency.

Definition 2.2.2 *Let $n \in \{-1, 0, 1, \ldots\}$. A separable metrizable space X has* $\operatorname{Skl} \leq n$ *if X has an indexed base $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ for the open sets such that for every subset $F \subseteq \mathbb{N}$ with $n+1$ elements the intersection*

$$\bigcap_{i \in F} \partial(B_i)$$

is compact.

By elementary arguments, one can prove that Skl takes its values between cmp and def (AARTS and NISHIUARA [2, Theorem 6.9]):

Theorem 2.2.10 *For every separable metrizable space X,*

$$\operatorname{cmp} X \leq \operatorname{Skl} X \leq \operatorname{def} X.$$

So the state of the art in 1960 was that there were (at least) two related concepts that were candidates for finding an internal characterization of the compactness deficiency. The first real breakthrough came in 1982 when POL [65] gave an example that answered the at that time fourty years old compactification problem of DE GROOT in the negative.

Example 2.2.1 (Pol's example) *There is a separable metrizable space X with* $\operatorname{cmp} X = 1$ *and* $\operatorname{def} X = 2$.

In an unpublished paper, HART [49] modified POL's construction and for every n obtained a space X_n with $\operatorname{cmp} X_n = 1$ and $\operatorname{def} X_n \geq n$. The second breakthrough came in 1988, when KIMURA [59] proved that SKLYARENKO's invariant for the problem of finding an internal characterization of the compactness deficiency is the right one.

Theorem 2.2.11 (Kimura's theorem) *For every separable metrizable space X,*

$$\operatorname{Skl} X = \operatorname{def} X.$$

This is pretty much the complete story of an interesting compactification problem posed by DE GROOT in 1942 and finally solved by POL in 1982 and KIMURA in 1988.

Of the interesting generalizations of DE GROOT's problem, we will mention the following one only. As we said before, cmp is dimension modulo the class consisting of all compact spaces. It is possible to define various other interesting dimension functions by varying the class of spaces that one considers uninteresting. If we take the class of all *topologically complete* spaces then we arrive at a particularly interesting situation. Define the *completeness degree* $\operatorname{icd} X$ of a space X as follows:

$\operatorname{icd} X = -1$	$\Leftrightarrow$	X is topologically complete,
$\operatorname{icd} X \leq n \ (\geq 0)$	$\Leftrightarrow$	every singleton subset $\{x\}$ of X and every closed subset $A \subseteq X$ with $x \notin A$ can be separated by a set S with $\operatorname{icd} S \leq n-1$,
$\operatorname{icd} X = n$	$\Leftrightarrow$	$\operatorname{icd} X \leq n$ and $\operatorname{icd} X \not\leq n-1$,
$\operatorname{icd} X = \infty$	$\Leftrightarrow$	$\operatorname{icd} X > n$ for $n = -1, 0, 1, \ldots$.

This dimension function measures the "degree of completeness" of a topological space. In a similar way one can define the *completeness deficiency* $\operatorname{cdef} X$ of X by

$$\operatorname{cdef} X = \min\{\dim(\delta X \setminus X) : \delta X \text{ is a completion of } X\};$$

DE GROOT's problem can be revived by asking for the relation between icd and cdef. This does not look very promising since in the compact case equality does not hold. Interesting, in the complete case, equality *does* hold, as was shown by AARTS.

Theorem 2.2.12 (Aarts [1]) *For every separable metrizable space X,*

$$\operatorname{icd} X = \operatorname{cdef} X.$$

For more detailed information and historical comments on DE GROOT's conjecture and its ramifications, see AARTS and NISHIURA [2].

2.2.3. *Metric characterizations of dimension*
In the Netherlands some work was also done on characterizations of dimension in terms of special metrics. This work was initiated by DE GROOT. A metric ρ on a set X is called *non–Archimedean* if

$$\rho(x,z) \leq \max\{\rho(x,z), \rho(y,z)\}$$

for all $x, y, z \in X$. The following result was proved by DE GROOT [46] in 1956.

Theorem 2.2.13 *Let X be a separable metrizable space. Then the following statements are equivalent:*

1. $\dim X \leq 0$,
2. *X has an admissible non-Archimedean metric.*

The same result was obtained earlier by HAUSDORFF [50]. Far reaching generalizations of it were found by NAGATA [62],[63].

Another interesting result in the same spirit was published by DE GROOT [47] in 1957.

Theorem 2.2.14 *Let X be a separable metrizable space. Then the following statements are equivalent:*

1. $\dim X \leq n$,
2. *X has an admissible totally bounded metric ρ such that for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ in X there are indices i, j, k such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x, y_k)$.*

2.3. PH. D. THESES ON DIMENSION THEORY IN THE NETHERLANDS

In the Netherlands, at least six Ph. D. theses were written on dimension theory:

	Title	Year	Supervisor
J. de Groot	Topologische Studiën	1942	G. Schaake
H. de Vries	Compact spaces and compactifications	1962	J. de Groot
J. M. Aarts	Dimension and deficiency in general topology	1966	J. de Groot
J. Bruijning	Some characterizations of topological dimension	1980	J. I. Nagata
J. J. Dijkstra	Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility	1983	J. van Mill
P. Borst	Transfinite classifications of weakly infinite-dimensional spaces	1986	J. van Mill

We already discussed the problem that was posed in DE GROOT's thesis in detail in §2.2.2. The thesis of DE VRIES dealt among other things with weight and dimension preserving compactifications. In AARTS's thesis the completeness degree was introduced (among other things); the first proof of Theorem 2.2.12 was published there. After DE GROOT's sudden and untimely death in 1972, he was succeeded at the University of Amsterdam by Professor J. I. NAGATA, an expert in dimension theory. BRUIJNING's thesis, written under supervision of NAGATA, dealt with several combinatorial characterizations of dimension (see also BRUIJNING and NAGATA [27]). DIJKSTRA did his undergraduate work under NAGATA at the University of Amsterdam; he wrote his thesis on dimension theory and infinite-dimensional topology under VAN MILL when NAGATA left Amsterdam to return to Japan (see also DIJKSTRA and VAN MILL [33]). For BORST similar remarks apply. He also did his undergraduate work under NAGATA at the University of Amsterdam, and continued his research on infinite-dimension theory under VAN MILL (see also BORST and DIJKSTRA [10]).

From the relatively small number of Ph. D. theses written on dimension theory in the Netherlands, we see that it was never very popular there. But it is fair to say that dimension theory in the Netherlands was alive ever since the pioneering work of BROUWER and that the contributions of Dutch topologists to it were substantial.

2.4. CONCLUSION

We did not tell the complete story of Dutch topology. We concentrated on general topology and in particular dimension theory and even there we had to restrict ourselves. The complete history of the Dutch contributions to topology and its applications remains to be written. Such a history would, of course, also have to encompass work outside of general topology, but related to it. Our story shows that, although BROUWER stopped working in topology after World War I, without his presence The Netherlands very probably would have contributed considerably less to the development of topology. We also saw how important FREUDENTHAL and DE GROOT were. As a matter of fact without BROUWER's intuitionism FREUDENTHAL would not have come to The Netherlands and without FREUDENTHAL DE GROOT's work would have gone into another direction. It is remarkable that the first part of our paper shows that without his intuitionistic views BROUWER possibly would not have turned to topology at all, while the second part shows that without BROUWER's intuitionism FREUDENTHAL would not have exerted his great influence on Dutch topology. In other words we reach the surprising conclusion that without intuitionism the history of topology in The Netherlands would have been totally different.

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