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Projections of planar Cantor sets in potential theory

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1. INTRODUCTION

As far as projections of Cantor sets in the plane are concerned there are two types of interest: 'large' sets with 'small' projections and 'small' sets with 'large' projections. In Monterie [7] an example of each type is given related to logarithmic capacity:

Example 1. [7, §C.8] There exists a planar Cantor set K of positive capacity and a countable dense set of directions A such that the capacity of the projection of K is zero for every direction from A.

Example 2. [7, §C.6] There exists a Cantor set K in $\mathbf{R}$ with zero capacity such that every projection of $K \times K$ except those on the x and y-axes has positive capacity.

We will present a topological theorem about fairly arbitrary functions from $\mathcal{K}(\mathbf{R}^2)$ into $[0, \infty)$ that will include these two examples as special cases. Particularly interesting is that the construction of our examples uses only a few topological properties of capacity whereas Monterie's constructions require detailed knowledge of the capacity function in the form of a rather technical criterion for deciding which sets have capacity zero.

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Our results give us a lot of freedom in specifying which projections are 'large' and which are 'small'. For instance, Corollaries 5 and 7 imply:

Example 3. There exists, given any two countable disjoint sets of directions A and B, a planar Cantor set such that the capacity of the projection is zero for any direction from A and nonzero for any direction from B.

Example 4. There exists, given any countable compact set of directions A, a planar Cantor set such that the capacity of the projection in the direction θ is zero if and only if $\theta \in A$.

These results are not just valid for logarithmic capacity but for a wide range of capacities and measures. We employ the 'turning parallelogram' construction of Besicovitch [1] that has been used by several authors to obtain sets with projections of prescribed measure (see e.g. [8, 2, 5, and 6]). Our contribution is to show that such constructions can work even if no particular measure or capacity is specified. It suffices to know that the set function is upper semicontinuous and that it vanishes on finite sets.

2. PRELIMINARIES

We denote the space of compacta in the plane equipped with the usual Hausdorff metric by $\mathcal{K}(\mathbf{R}^2)$. The space of projection directions in the plane is the circle $S = \mathbf{R}/\pi\mathbf{Z}$. If $\theta \in S$ then p_θ is the projection of the plane onto the line through the origin that is perpendicular to θ. In other words,

$$p_\theta(x, y) = \frac{1}{2}(x - x\cos 2\theta - y\sin 2\theta,\ y + y\cos 2\theta - x\sin 2\theta).$$

Obviously, the function $p : S \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by $p(\theta, x, y) = p_\theta(x, y)$ is a continuous mapping and hence it generates a continuous map from $S \times \mathcal{K}(\mathbf{R}^2)$ to $\mathcal{K}(\mathbf{R}^2)$. By an interval in the plane we mean a nondegenerate line segment.

We now recall the definition of logarithmic capacity. If $A \in \mathcal{K}(\mathbf{R}^2)$ and $\mu \in \mathcal{P}(A)$ is a probability measure on A then the potential energy of μ is

$$I(\mu) = \int_A \int_A \log \frac{1}{|z - w|}\, d\mu(w)\, d\mu(z).$$

The capacity is a function from $\mathcal{K}(\mathbf{R}^2)$ into $[0, \infty)$ defined by

$$\operatorname{cap} A = \exp\left(- \inf_{\mu \in \mathcal{P}(A)} I(\mu) \right).$$

We will use only the following well-known properties of capacity (see Landkof [4] and Tsuji [9]):

1. Capacity is an upper semicontinuous function, i.e. the preimage of every interval $(-\infty, t)$ is open.
2. The capacity of finite sets is zero.
3. The capacity of a space that contains a nontrivial continuum is positive.

4. If $A \subset B$ then $\operatorname{cap} A \leq \operatorname{cap} B$.
5. If $\operatorname{cap} A = \operatorname{cap} B = 0$ then $\operatorname{cap} A \cup B = 0$.
6. $\operatorname{cap} p_\theta(A) \leq \operatorname{cap} A$.

3. CANTOR SETS

Before giving our main results we would like to present an example of a 'small' Cantor set which projects onto 'large' sets.

Proposition 1. *There exists a planar Cantor set of vanishing Lebesgue measure with the property that every projection is an interval.*

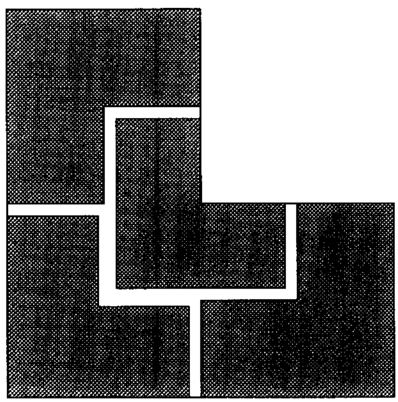

Figure 1.

Proof. Consider figure 1. Let us denote the large L-shaped region by A_0. We construct a decreasing sequence of compacta A_n as follows. The set A_1 consists of the union of the four shaded subregions in figure 1. Observe that every component of A_1 is similar in shape to A_0 with scale factor at most $\frac{1}{2}$. This means we can apply the same procedure that produced A_1 from A_0 to each of the components of A_1 to produce A_2. Continue this process indefinitely. It is obvious that $A = \bigcap_{n=0}^{\infty} A_n$ is a planar Cantor set. Since $\lambda(A_{n+1})/\lambda(A_n) = \lambda(A_1)/\lambda(A_0) < 1$ for every n we have $\lambda(A) = 0$, where λ stands for the 2-dimensional Lebesgue measure.

Observe that each straight line that intersects A_0 also intersects A_1. By induction we may conclude that each straight line that intersects A_0 also intersects every A_n and, hence, the intersection A (by compactness). Consequently, we have in each direction $p_\theta(A) = p_\theta(A_0)$, which are all intervals because A_0 is a continuum. $\square$

Let $\gamma : \mathcal{K}(\mathbf{R}^2) \to [0, \infty)$ be a function. We say that $A \in \mathcal{K}(\mathbf{R}^2)$ *has vanishing γ in almost all directions* if there is a $B \subset S$ of category I (i.e. a countable union of nowhere dense sets) such that $\gamma(p_\theta(A)) = 0$ for each $\theta \in S \backslash B$. Recall that the complement of a category I set in S contains a dense copy of the irrationals and hence every nonempty open subset has cardinality $\mathfrak{c}$.

In connection with Example 1 Monterie raises the question whether it is possible to obtain vanishing capacity in uncountably many directions for such an example. The following proposition shows that this is automatic.

Proposition 2. *If $\gamma : \mathcal{K}(\mathbf{R}^2) \to [0, \infty)$ is upper semicontinuous, $C \in \mathcal{K}(\mathbf{R}^2)$, and if A is a dense subset of S such that $\gamma(p_\theta(C)) = 0$ for every $\theta \in A$ then C has vanishing γ in almost all directions.*

Proof. Since γ is upper semicontinuous and p is continuous we have that $f(\theta) = \gamma(p_\theta(C))$ is upper semicontinuous as a function of θ. So $f^{-1}(\{0\})$ is a G_δ-set that is dense in S. $\square$

We now present our main result. Let us call an upper semicontinuous $\gamma : \mathcal{K}(\mathbf{R}^2) \to [0, \infty)$ that vanishes on finite sets a *pseudo-capacity*.

Theorem 3. *If γ is a pseudo-capacity and if A and B are disjoint σ-compacta in S such that A is countable then there is a planar Cantor set C such that $p_\theta(C)$ is a finite union of intervals for every $\theta \in B$ and $\gamma(p_\theta(C)) = 0$ for every $\theta \in A$. In addition, it can be arranged that $p_\theta(C)$ is connected for θ in any given compact subset of B.*

Proof. If A is empty then we apply Proposition 1.

Let $A \neq \emptyset$. Write B as the union of an increasing sequence of compacta $B_0, B_1, \ldots$. We may assume without loss of generality that $0 \in B_0$. We select a sequence $\theta_0, \theta_1, \ldots$ in S such that every element of A is listed infinitely many times, $\theta_n \neq \theta_{n+1}$ and $\theta_n \notin B_n$ for every n. If A consists of more than one point then we can use an enumeration of A. If A has only one element, say θ, then we put $\theta_n = \theta$ for n even and we select some other angle from the complement of B_n for n odd. Note that no θ_n equals 0.

We need some notation: if $X \in \mathcal{K}(\mathbf{R}^2)$ and $\varepsilon > 0$ then

$$D_\varepsilon(X) = X + ([-\varepsilon, \varepsilon] \times \{0\}) = \bigcup_{(x,y) \in X} [x - \varepsilon, x + \varepsilon] \times \{y\}.$$

We shall construct spaces whose components are of the form $D_\varepsilon(L)$ where L is a nonhorizontal interval, and the following definition will extract the intervals that generate such spaces.

$$\mathcal{L}(X) = \left\{ M : \begin{array}{l} M \text{ is a closed interval such that} \\ D_\varepsilon(M) \text{ is a component of } X \text{ for some } \varepsilon \end{array} \right\}.$$

We construct a decreasing sequence of compacta C_n and a sequence of positive numbers ε_n such that:

(1_n) C_n has finitely many components.

(2_n) Every component of C_n has the form $D_{\varepsilon_n}(L)$ where L is an interval parallel to θ_n.

(3_n) Every closed subset of $p_{\theta_n}(C_n)$ has $\gamma < 1/n$.

(4_n) If $K = D_{\varepsilon_i}(L)$ is a component of C_i and $0 \leq i \leq n$ then $p_\theta(\bigcup \mathcal{L}(K \cap C_n))$ is an interval that contains $p_\theta(L)$ for each $\theta \in B_i$.

As basis step for the induction, we select an $\varepsilon_0 > 0$, a line segment L in the direction of θ_0, and we put $C_0 = D_{\varepsilon_0}(L)$.

We first verify that $C = \bigcap_{n=0}^{\infty} C_n$ meets the requirements. Let $\theta \in A$ and find an $n \geq 1$ such that $\theta = \theta_n$. Then we have $p_\theta(C) \subset p_{\theta_n}(C_n)$ and hence $\gamma(p_\theta(C)) < 1/n$. Since n can be chosen arbitrarily large we have $\gamma(p_\theta(C)) = 0$.

Now let θ be an element of some B_i and let $K = D_{\varepsilon_i}(L)$ be a component of C_i. According to (4_n) the projection $p_\theta(\bigcup \mathcal{L}(K \cap C_n))$ is an interval that contains $p_\theta(L)$ for each $n \geq i$. This statement implies that

$$p_\theta(K \cap C_n) = p_\theta(\bigcup \mathcal{L}(K \cap C_n) + ([-\varepsilon_n, \varepsilon_n] \times \{0\}))$$

is an interval for $n \geq i$. So $p_\theta(K \cap C) = \bigcap_{n=i}^{\infty} p_\theta(K \cap C_n)$ is an interval that is nondegenerate because it contains $p_\theta(L)$ and L is parallel to θ_i which is not equal to $\theta \in B_i$. Since C_i has only finitely many components K we have that $p_\theta(C)$ is a finite union of intervals. Observe that if $\theta \in B_0$ then $K = C_0$, and hence $p_\theta(C)$ consists of a single interval.

What remains is to perform the induction. Assume that C_{n-1} has been constructed. Since $\theta_n \notin B_n$ we can find an angle $\delta > 0$ such that $[\theta_n - \delta, \theta_n + \delta]$ is disjoint from B_n.

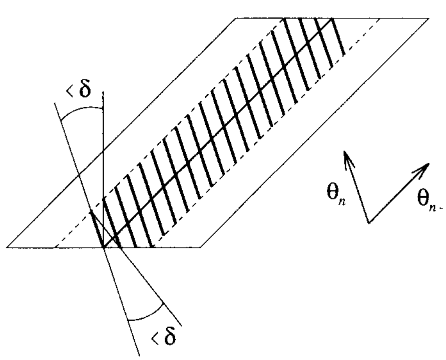

Figure 2.

Let $D_{\varepsilon_{n-1}}(L)$ be a component of C_{n-1} (represented by the large parallelogram in fig. 2). Find in $D_{\varepsilon_{n-1}/2}(L)$ a finite pairwise disjoint collection $\mathcal{M}_L$ of closed intervals parallel to θ_n as indicated in figure 2. We choose the line segments in $\mathcal{M}_L$ so close together that adjacent ones will overlap if we project along an angle outside the interval $[\theta_n - \delta, \theta_n + \delta]$. Consequently, for each $\theta \in B_n$,

$p_\theta(\bigcup \mathcal{M}_L)$ is an interval that obviously contains $p_\theta(L)$. It is easily seen that we can always arrange that $\mathcal{M}_L$ has more than one element each of which is at most half as long as L. This will guarantee that C will be a Cantor set. Since $\theta_{n-1} \neq \theta_n$ figure 2 is a faithful representation of the situation.

Let $\mathcal{M}$ be the union of the $\mathcal{M}_L$'s. Then $F = p_{\theta_n}(\bigcup \mathcal{M})$ is a finite set and hence $\gamma = 0$ for all subsets of F. Using the fact that γ is upper semicontinuous we select for each subset G of F an $\varepsilon_G > 0$ such that every element of $\mathcal{K}(\mathbf{R}^2)$ that is ε_G-close in the Hausdorff metric to G has $\gamma < 1/n$. If we let ε_n be less than the minimum of the ε_G's then we have that every element of $\mathcal{K}(\mathbf{R}^2)$ that is contained in the closed ε_n-ball around F has $\gamma < 1/n$. In addition, we may assume that $\varepsilon_n < \frac{1}{2}\varepsilon_{n-1}$ and that ε_n is less than half the distance between the closest points of F. We put $C_n = D_{\varepsilon_n}(\bigcup \mathcal{M})$. Figure 3 shows the components of C_n inside a component of C_{n-1}.

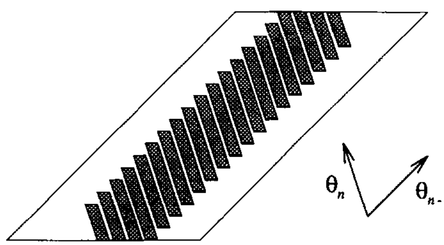

Figure 3.

Since $\mathcal{M}$ is finite, condition (1_n) is satisfied. Since $\varepsilon_n < \frac{1}{2}\varepsilon_{n-1}$ we have $C_n \subset C_{n-1}$. Obviously, $p_{\theta_n}(C_n)$ is a subset of the ε_n-ball around F and hence condition (3_n) is satisfied. Since the ε_n-balls around the points of F are pairwise disjoint, condition (2_n) is satisfied. If we combine condition (4_{n-1}) with the fact that $p_\theta(\bigcup \mathcal{M}_L)$ is an interval that contains $p_\theta(L)$ for each $\theta \in B_n$ then we find (4_n). The proof is now complete. □

Note that the only property of γ we really need for the proof of the theorem and its corollaries is that finite collinear sets are interior points of sets of the form $\gamma^{-1}([0, t))$.

In Example 1, it is not clear what the capacity is in the other directions. One might conjecture that if the capacity vanishes in almost all directions then it will be zero in all directions. The following corollaries give counterexamples.

Corollary 4. *If γ is a pseudo-capacity and if A is a countable dense subset of $(0, \pi)$ then there is a planar Cantor set C such that $p_0(C)$ is an interval and $\gamma(p_\theta(C)) = 0$ for every $\theta \in A$. This implies that C has vanishing γ in almost all directions.*

Corollary 5. *If γ is a pseudo-capacity and if A and B are disjoint coutable dense subsets of S then there is a planar Cantor set C such that $p_\theta(C)$ is a finite union of intervals for every $\theta \in B$ and $\gamma(p_\theta(C)) = 0$ for every $\theta \in A$.*

If we substitute $\gamma = \operatorname{cap}$ in these corollaries then we find $\operatorname{cap} C > 0$ because projection cannot increase capacity and intervals have positive capacity – reproducing Example 1. Our results show in addition that the function $\operatorname{cap} p_\theta(A)$ can be severely discontinuous in θ.

A natural question (raised also in [7, §C.1]) is whether a set with positive capacity can have projections of capacity zero in all directions. This question was answered by Kaufman [3] for capacities associated with potentials of the form $1/|x|^\alpha$. We include the straightforward adaptation of Kaufman's proof to the logarithmic case.

Proposition 6. *If* $\operatorname{cap} C > 0$ *then the Lebesgue measure of* $\{\theta \in S : \operatorname{cap} p_\theta(C) = 0\}$ *is zero.*

Proof. Let $\operatorname{cap} C > 0$. This statement is equivalent with the existence of a $\mu \in \mathcal{P}(C)$ with finite energy, i.e.

$$I(\mu) = \int_C \int_C \log \frac{1}{|x-y|} \, d\mu(x) \, d\mu(y) < \infty.$$

Define $C_\theta = p_\theta(C)$ and $\mu_\theta \in \mathcal{P}(C_\theta)$ by $\mu_\theta(B) = \mu(p_\theta^{-1}(B) \cap C)$ for B a Borel set in C_θ and $\theta \in S$. Consider the energy of μ_θ:

$$\begin{aligned} I(\mu_\theta) &= \int_{C_\theta} \int_{C_\theta} \log \frac{1}{|u-v|} \, d\mu_\theta(u) \, d\mu_\theta(v) \\ &= \int_C \int_C \log \frac{1}{|p_\theta(x-y)|} \, d\mu(x) \, d\mu(y). \end{aligned}$$

Since the integrand is bounded below by $-\log(\operatorname{diam}(C))$ we have by Fubini that $I(\mu_\theta)$ is a measurable function of θ and that

$$\int_S I(\mu_\theta) \, d\sigma(\theta) = \int_C \int_C \int_S \log \frac{1}{|p_\theta(x-y)|} \, d\sigma(\theta) \, d\mu(x) \, d\mu(y),$$

where σ stands for the Lebesgue measure on S. If we parametrize S in such a way that $\theta = 0$ corresponds with the direction of the vector $x - y$ then we find

$$\begin{aligned} \int_S \log \frac{1}{|p_\theta(x-y)|} \, d\sigma(\theta) &= \int_0^\pi \log \frac{1}{|x-y| \sin\theta} \, d\theta \\ &= \pi \log \frac{1}{|x-y|} + \int_0^\pi \log(\csc\theta) \, d\theta. \end{aligned}$$

Consequently

$$\int_S I(\mu_\theta) \, d\sigma(\theta) = \pi I(\mu) + \int_0^\pi \log(\csc\theta) \, d\theta < \infty.$$

This result implies that $I(\mu_\theta) < \infty$ for all θ's outside some set of measure zero and the proposition is proved. □

Corollary 7. *If* γ *is a pseudo-capacity and if* A *is a countable* G_δ*-set (in particular* *a countable compactum) in S then there is a planar Cantor set C such that $\gamma(p_\theta(C)) = 0$ for every $\theta \in A$ and $p_\theta(C)$ is a finite union of intervals for every other direction θ.*

If we substitute $A = \{0, \pi/2\}$ and $\gamma = \text{cap}$ into this corollary then the resulting C projects onto capacity zero sets on the x and y-axes and onto finite unions of intervals in all other directions. If we define K as the union of the horizontal and vertical projections of C seen as subsets of $\mathbf{R}$ then $\text{cap}\, K = 0$. Since $K \times K$ contains C its projections in every direction other than horizontal or vertical contain intervals. This result implies Example 2.

Observe that Theorem 3 applies to for instance Hausdorff measures just as well as it applies to capacities. In fact, since a set with vanishing logarithmic capacity has Hausdorff dimension zero (see [4, Theorem 3.13]) Theorem 3 implies:

Corollary 8. *If A and B are disjoint σ-compacta in S such that A is countable then there is a planar Cantor set C such that the Hausdorff dimension of $p_\theta(C)$ is one for every $\theta \in B$ and zero for every θ in some G_δ-set that contains A.*

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