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Normal spaces and fixed points of Čech–Stone extensions

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Abstract

We present two examples of nice normal spaces X having the property that for some fixed-point free homeomorphism on X its Čech–Stone extension has a fixed point. One of the spaces presented here is locally countable, locally compact, separable, normal, countably paracompact and weakly zero-dimensional. The other one is hereditarily normal and strongly zero-dimensional. Our construction of this example however requires the Continuum Hypothesis. Since for paracompact spaces with finite covering dimension every fixed-point free homeomorphism has a fixed-point free Čech–Stone extension, these results are "best possible". © 1998 Elsevier Science B.V.

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1. Introduction

Given a (Tychonoff) space X and a fixed-point free homeomorphism $f: X \to X$ it is natural to ask whether its Čech–Stone extension $\beta f: \beta X \to \beta X$ is fixed-point free.

Van Douwen [4] showed (among other things) that a paracompact space X with finite covering dimension has the property that every fixed-point free homeomorphism $f: X \to X$ has a fixed-point free Čech–Stone extension βf.

This result motivated several authors to construct examples of "nice" and "very nice" spaces which do not have this property. One of the important tools in this context is the concept of a coloring of a map.

A *coloring* of a pair (X, f), where X is a space and $f: X \to X$ is continuous and fixed-point free, is a finite closed cover $\mathcal{A}$ of X such that $f[A] \cap A = \emptyset$ for all $A \in \mathcal{A}$. If

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βf is fixed-point free, then the compactness of βX easily implies the existence of a finite (open) cover of βX (and so also of X) such that

$$f[A] \cap A = \emptyset \quad \text{for all } A \in \mathcal{A}.$$

Normality allows to shrink the sets to closed sets using a standard technique from Dimension Theory (see, e.g., [1, I.8.8]).

Using this van Douwen showed in [4] that on the topological sum

$$\bigoplus^{n\in\omega} \mathbb{S}^n$$

of n-spheres, the topological sum of the antipodal maps is fixed-point free but its Čech–Stone extension is not.

Another example is the product space $\{-1,0,1\}^{\omega_1}$ with the product map ι obtained from the function $\iota(x) = -x$ ($x \in \{-1,0,1\}$). Put

$$X = \{-1,0,1\}^{\omega_1} \setminus \{\mathbf{0}\},$$

where $\mathbf{0}$ is the element in $\{-1,0,1\}^{\omega_1}$ all of whose coordinates are zero. Then $\iota^* = \iota \restriction X$ is fixed-point free, but $\beta\iota^*$ is not because $\beta X = \{-1,0,1\}^{\omega_1}$ and $\beta\iota^* = \iota$. For more details see [2].

A space X is called *weakly zero-dimensional* if it has a basis consisting of open and closed sets. In addition, X is called *strongly zero-dimensional* if its Čech–Stone compactification is weakly zero-dimensional. As is well known, this is equivalent to the property that every two disjoint zero-sets in X can be separated by an open and closed set. (This explains our terminology.)

Recently Good [5] presented two new examples of spaces whose Čech–Stone extensions behave bad. Good's first example is first countable, strongly zero-dimensional and subparacompact. Such a space cannot be collectionwise normal, since for collectionwise normal spaces subparacompactness coincides with paracompactness. It is, however, not normal. Good's second example is weakly zero-dimensional, normal, countably paracompact and first countable. It is a combination of van Douwen's example, that we mentioned above, and Dowker's construction of a normal, weakly zero-dimensional space of positive covering dimension.

Here we present two new examples. The first example is locally countable, locally compact (hence first countable), weakly zero-dimensional, separable, normal and countably paracompact. Since Good's example is not locally compact and van Douwen's example is not weakly zero-dimensional, it improves both.

Although the space in the first example has many nice properties, it fails to be strongly zero-dimensional. So far, all known examples of strongly zero-dimensional spaces with "bad" Čech–Stone behaviour are not normal. In the third section we will construct our second example, which will be hereditarily normal and strongly zero-dimensional. This construction however requires the Continuum Hypothesis.

In what follows, ι will always denote an *involution*, i.e., a continuous map which is its own inverse.

2. Refining topologies

Our first example is based on van Douwen's technique for constructing locally compact, submetrizable spaces from [3, §4,5,7]. We will modify the original construction in such a way that an initially given involution ι on the space X under consideration remains continuous. The idea is not to consider points, but to consider pairs of the form $\{x, \iota(x)\}$ $(x \in X)$, i.e., orbits under the involution ι.

For every ordinal number ξ there is a unique limit ordinal η and a unique finite ordinal n such that $\xi = \eta + n$. If n is even, then ξ is an *even* ordinal; otherwise ξ is an *odd* ordinal.

Lemma 2.1. *Let X be a separable metrizable space with $|X| = \mathfrak{c}$ and such that every uncountable closed subset has cardinality $\mathfrak{c}$. Furthermore let $\iota: X \to X$ be a fixed-point free involution. Then there exists a refinement $\mathcal{T}^*$ of the topology of X, such that $(X, \mathcal{T}^*)$ is locally compact, locally countable, weakly zero-dimensional, separable, normal, countably paracompact, and ω_1-compact. We denote the topological space $(X, \mathcal{T}^*)$ by $\Lambda(X)$. Moreover $\Lambda(X)$ has the following property:*

If $\langle F_n \colon n \in \omega \rangle$ is a sequence of closed sets in $\Lambda(X)$ such that $|\bigcap_{n\in\omega} F_n| \leqslant \omega$ then $|\bigcap_{n\in\omega} \mathrm{cl}_X F_n| \leqslant \omega$. (λ_ω)

Also, $\iota: \Lambda(X) \to \Lambda(X)$ is an involution.

Proof. Let $\langle B_i \colon i \in \omega \rangle$ enumerate a base for X such that $B_0 = X$ and $B_{i+1} = \iota[B_i]$ if i is even. For every $x \in X$ and $j < \omega$ define

$$E(x, j) = \bigcap \{B_i \colon i \leqslant 2j + 1 \text{ and } x \in B_i\}.$$

Obviously, for every $x \in X$ and $j < \omega$ we have $E(\iota(x), j) = \iota[E(x, j)]$. The collection $\langle E(x, j) \colon j \in \omega \rangle$ is a local base at x in X and if $x \in E(y, j)$ and $i \geqslant j$, then $E(x, i) \subseteq E(y, j)$.

Define

$$\mathfrak{K} = \left\{ \langle K_n \colon n \in \omega \rangle \colon \ K_n \text{ is a countable subset of } X, \text{ and } \left| \bigcap_{n\in\omega} \mathrm{cl}_X K_n \right| > \omega \right\}.$$

We will construct $\Lambda(X)$ in such a way that

If $\langle K_n \colon n \in \omega \rangle \in \mathfrak{K}$ then $|\bigcap_{n\in\omega} \mathrm{cl}_{\Lambda(X)} K_n| = \mathfrak{c}$. $(*)$

Since X is hereditarily separable (λ_ω) follows from $(*)$.

Observe that if $K = \langle K_n \colon n \in \omega \rangle \in \mathfrak{K}$ then $\langle \iota[K_n] \colon n \in \omega \rangle \in \mathfrak{K}$. This last sequence will for simplicity be denoted by $\iota[K]$.

Enumerate $\mathfrak{K}$ as $\langle \langle K_{\gamma,n} \colon n \in \omega \rangle \colon \gamma \in \mathfrak{c} \rangle$ in such a way that every element $K \in \mathfrak{K}$ is listed $\mathfrak{c}$ times and such that $K_{\gamma+1} = \iota[K_\gamma]$ if γ is even.

Enumerate X in a one-to-one fashion as $\langle x_\alpha \colon \alpha < \mathfrak{c} \rangle$ and such that $\mathbb{Q} = \langle x_\alpha \colon \alpha < \omega \rangle$ where $\mathbb{Q}$ is a fixed countable dense subset of X which is invariant under ι. Let this enumeration of X be such that $x_{\alpha+1} = \iota(x_\alpha)$ for all even α.

Now define for all even α,

$$X_\alpha = \{x_\beta\colon \beta < \alpha\}.$$

If $\alpha < \mathfrak{c}$ is odd then we define $X_\alpha = X_{\alpha-1}$.

Now we want to define an injection $\varphi : \mathfrak{c} \to \mathfrak{c} \setminus \omega$. We will do this by induction on $\gamma < \mathfrak{c}$. Let $\gamma < \mathfrak{c}$ be even and assume that φ is already defined on γ and is injective. We will define $\varphi(\gamma)$ and $\varphi(\gamma + 1)$. First, let

$$\xi_\gamma = \min\bigl\{\lambda \geqslant \omega\colon \ \lambda \text{ is even, and } \varphi(\beta) < \lambda \text{ for all } \beta < \gamma\bigr\}.$$

Next, let

$$\sigma = \min\left\{\alpha \geqslant \xi_\gamma\colon \bigcup_{n\in\omega} K_{\gamma,n} \subseteq X_\alpha \text{ and } x_\alpha \in \bigcap_{n\in\omega} \mathrm{cl}_X K_{\gamma,n}\right\},$$

and let

$$\rho = \min\left\{\alpha \geqslant \xi_\gamma\colon \bigcup_{n\in\omega} K_{\gamma+1,n} \subseteq X_\alpha \text{ and } x_\alpha \in \bigcap_{n\in\omega} \mathrm{cl}_X K_{\gamma+1,n}\right\}.$$

Observe that σ and ρ are well-defined because every uncountable closed subset of X has cardinality $\mathfrak{c}$.

Since every X_α is invariant under ι, we obtain

$$\bigcup_{n\in\omega} K_{\gamma+1,n} = \iota\left[\bigcup_{n\in\omega} K_{\gamma,n}\right] \subseteq \iota[X_\sigma] = X_\sigma.$$

In addition,

$$\iota(x_\sigma) \in \iota\left[\bigcap_{n\in\omega} \mathrm{cl}_X K_{\gamma,n}\right] = \bigcap_{n\in\omega} \mathrm{cl}_X K_{\gamma+1,n}.$$

Since $\iota(x_\sigma) = x_{\sigma+1}$ if σ is even, and $\iota(x_\sigma) = x_{\sigma-1}$ if σ is odd, we conclude that $\rho \leqslant \sigma + 1$. By a similar argument we obtain $\sigma \leqslant \rho + 1$.

Let us first consider the case that σ is odd. Since ξ_γ is even, we have $\sigma - 1 \geqslant \xi_\gamma$. Since σ is odd, $X_{\sigma-1} = X_\sigma = \iota[X_\sigma]$, as well as

$$x_{\sigma-1} = \iota(x_\sigma) \in \bigcap_{n\in\omega} \mathrm{cl}_X K_{\gamma+1,n}.$$

So we conclude that $\rho \leqslant \sigma - 1$, and so, $\sigma = \rho + 1$. It follows similarly that if ρ is odd then $\rho = \sigma + 1$. So the minimum of ρ and σ is even, and if $\sigma \neq \rho$, say $\sigma < \rho$ then $\sigma + 1 = \rho$, and vice versa.

If $\sigma \neq \rho$ then we simply define $\varphi(\gamma) = \sigma$ and $\varphi(\gamma + 1) = \rho$. If $\sigma = \rho$ then we define $\varphi(\gamma) = \sigma$ and $\varphi(\gamma + 1) = \sigma + 1$. Clearly, $\varphi \restriction (\gamma + 2)$ is an injection.

This completes the definition of φ.

For each $\alpha \in \mathfrak{c} \setminus \omega$ it is possible to choose a sequence $s_\alpha : \omega \to X_\alpha$ such that:

- $s_\alpha(i) \in E(x_\alpha, i)$ for all $i \in \omega$.
- Infinitely many terms of s_α are in $\mathbb{Q}$.
- If α is even, then $s_{\alpha+1}(i) = \iota(s_\alpha(i))$.

- If there is a γ such that $\varphi(\gamma) = \alpha$, then each $K_{\gamma,n}$ contains infinitely many terms of s_α and all terms of s_α lie in $\mathbb{Q} \cup \bigcup_{n\in\omega} K_{\gamma,n}$. Notice that γ is unique since φ is injective. If there is no γ such that $\varphi(\gamma) = \alpha$, then there are no additional restrictions.

Since $s_\alpha(i) \in X_\alpha$ for $\alpha \in \mathfrak{c}\setminus\omega$ and $i \in \omega$, we can construct collections $\langle L(x_\alpha, j)\colon j \in \omega\rangle$ of subsets of X with transfinite recursion on α as follows

$$L(x_\alpha, j) = \begin{cases} \{x_\alpha\} & \text{if } \alpha \in \omega, \\ \{x_\alpha\} \cup \bigcup_{i\geqslant j} L(s_\alpha(i), i) & \text{if } \alpha \in \mathfrak{c} \setminus \omega. \end{cases}$$

By transfinite induction one can show that $L(x_{\beta+1}, j) = \iota[L(x_\beta, j)]$ for all even β.

Since $\langle L(x, j)\colon j \in \omega\rangle$ is a decreasing family, with the property that if $x \in L(y, j)$ for $x, y \in X$ and $j \in \omega$, then $L(x, i) \subseteq L(y, j)$ for some $i \in \omega$, we may define the topology for $\Lambda(X)$ by declaring $\langle L(x, j)\colon j \in \omega\rangle$ to be a local base at $x \in X$.

One easily sees that this topology is finer than the original topology, that it is locally compact and locally countable and that $\mathbb{Q}$ is a dense subset of $\Lambda(X)$; it also follows that $\Lambda(X)$ is weakly zero-dimensional.

Straight from [3, §4] we get that $(*)$ holds, and so does (λ_ω), therefore $\Lambda(X)$ is normal, countably paracompact and ω_1-compact.

We observed that $L(x_{\beta+1}, j) = \iota[L(x_\beta, j)]$ (for β even), and so, since ι is its own inverse, we get that ι is continuous. $\square$

Construction 2.2. We start with the spaces $\mathbb{T}^n = \mathbb{S}^n \times [0, 1]$, the product of $\mathbb{S}^n$, the n-dimensional sphere in $\mathbb{R}^{n+1}$, and $[0, 1]$, the unit interval. Now let $\iota_n : \mathbb{T}^n \to \mathbb{T}^n$ be defined by $\iota_n\big((x, y)\big) = (-x, y)$, i.e., the antipodal map on $\mathbb{S}^n$ and the identity on $[0, 1]$. This map is clearly fixed-point free.

Using Lemma 2.1 we can construct for each n the "nice" space $\Lambda_n = \Lambda(\mathbb{T}^n)$ corresponding to $\mathbb{T}^n$ and ι_n such that ι_n is continuous in this new topology.

Let Δ be the topological sum $\bigoplus_{n\in\omega} \Lambda_n$. This Δ will be our space. In a straightforward way the properties of the Λ_n imply that Δ is locally compact, locally countable, separable, normal, countably paracompact, ω_1-compact, weakly zero-dimensional and submetrizable.

Define a fixed-point free involution $j : \Delta \to \Delta$ by

$$j(x) = \iota_n(x) \quad \text{if } x \in \Lambda_n.$$

We show by contradiction that (Δ, j) is not finitely colorable. Suppose $\{A_1, \dots, A_k\}$ is a finite closed cover of Δ such that $j[A_m] \cap A_m = \emptyset$ for all $1 \leqslant m \leqslant k$. Now let $A^*_m = A_m \cap \Lambda_k$ for every $1 \leqslant m \leqslant k$. Then $\{A^*_1, \dots, A^*_k\}$ is a closed coloring of Λ_k. Since for all $1 \leqslant m \leqslant k$ the sets A^*_m and $\iota_k[A^*_m]$ are closed and disjoint, we get from (λ_ω) that

$$\left|\mathrm{cl}_{\mathbb{T}^k} A^*_m \cap \mathrm{cl}_{\mathbb{T}^k} \iota_k[A^*_m]\right| \leqslant \omega.$$

Hence also

$$\left|\bigcup_{m=1}^{k} \left(\mathrm{cl}_{\mathbb{T}^k} A^*_m \cap \mathrm{cl}_{\mathbb{T}^k} \iota_k[A^*_m]\right)\right| \leqslant \omega.$$

Therefore there is a $y \in [0,1]$ such that (x, y) is not in this set for all $x \in \mathbb{S}^k$. So with $B_m = \mathrm{cl}_{\mathbb{T}^k} A^*_m \cap (\mathbb{S}^k \times \{y\})$, we find that $\{B_m\colon\ 1 \leqslant m \leqslant k\}$ is a closed cover of a k-dimensional sphere with k sets such that $B_m \cap \iota_k[B_m] = \emptyset$ for all $1 \leqslant m \leqslant k$. But this contradicts the Liusternik–Shnirel'man Theorem.

We mentioned many times that this example is weakly zero-dimensional. It fails to be strongly zero-dimensional as can been seen from Theorem 8.1 of [3].

3. An example under CH which is normal and strongly zero-dimensional

In this section we sketch the construction under the Continuum Hypothesis of an example of a hereditarily separable, hereditarily (collectionwise) normal, countably compact, strongly zero-dimensional space H with a fixed-point free involution $\iota : H \to H$ such that $\beta\iota$ has a fixed point. The interest in this example comes from its normality and its strong zero-dimensionality, since all previous examples failed to be both normal and strongly zero-dimensional.

Construction 3.1. We follow a construction of Hajnal and Juhász [6] who constructed using the Continuum Hypothesis a hereditarily separable, hereditarily (collectionwise) normal, countably compact subgroup of $\{0,1\}^{\omega_1}$ which is not Lindelöf and has no convergent sequences. The key concept in their construction is the concept of an ω-HFD (ω-hereditarily finally dense) subspace.

Their construction can be repeated for any prime number p, thus obtaining an example of a hereditarily separable, hereditarily normal, countably compact subgroup of p^{ω_1} which is not Lindelöf. In particular we can construct this Hajnal–Juhász-group G in $3^{\omega_1} = \{-1,0,1\}^{\omega_1}$. From the construction it follows that G is G_δ-dense and so is $H = G \setminus \{\mathbf{0}\}$.

By the Hernández–Sanchis Theorem [7, Corollary 11], the compact topological group 3^{ω_1} is the Čech–Stone compactification of its G_δ-dense subset H. Observe that 3^{ω_1} is zero-dimensional, whence H is strongly zero-dimensional.

Now if we define $\alpha : \{-1,0,1\} \to \{-1,0,1\}$ via $\alpha(x) = -x$ and let $\iota : H \to H$ be the product map of the α's restricted to H, then ι is a fixed-point free involution. Since $\beta H = 3^{\omega_1}$ we can easily see that the Čech–Stone extension $\beta\iota$ has a fixed point. This completes the example.

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