

## EXTENDING MONOTONE MAPPINGS

BY

JAN J. DIJKSTRA (TUSCALOOSA, ALABAMA)

AND JAN VAN MILL (AMSTERDAM)

All spaces are assumed to be Tychonoff. A *monotone* map is a closed continuous surjection with connected fibres. If  $A$  and  $B$  are subsets of a space  $X$  then  $A$  is called *locally connected rel*  $B$  if for every  $a \in A$  and every neighbourhood  $U$  of  $a$  in  $X$  there is a neighbourhood  $V$  of  $a$  such that  $V \subset U$  and  $V \cap B$  is connected.

As far as extending monotone maps over compacta the following is known:

PROPOSITION 1. *If  $f : X \rightarrow Y$  is monotone and  $C$  is a compactification of  $X$  such that  $f$  extends to a continuous  $\tilde{f} : C \rightarrow \beta Y$  then  $\tilde{f}$  is monotone.*

PROPOSITION 2. *If  $f : X \rightarrow Y$  is monotone,  $D$  is a compactification of  $Y$  such that  $D \setminus Y$  is locally connected rel  $Y$ , and  $C$  is a compactification of  $X$  such that  $f$  extends to a continuous  $\tilde{f} : C \rightarrow D$  then  $\tilde{f}$  is monotone.*

The first proposition is folklore (see Hart [3, Lemma 2.1]) and the second proposition can be found in Dijkstra [1]. The two propositions have the same conclusion but very dissimilar premises: for instance, if  $Y$  is metric then its Čech–Stone remainder is never locally connected rel  $Y$ . Our first theorem unifies these propositions.

In this paper we will discuss functions  $f : X \rightarrow Y$  and  $\tilde{f} : C \rightarrow D$  such that  $X$  and  $Y$  are dense subsets of  $C$  and  $D$  respectively. Unless stated otherwise, if  $A$  is a subset of  $X$  or  $Y$  respectively, then  $\bar{A}$  and  $\text{int}(A)$  refer to the closure and the interior of  $A$  in  $C$  or  $D$  respectively. Let  $I$  be the interval  $[0, 1]$ . A *zero set*  $A$  in a space  $Y$  is the preimage of 0 for some continuous  $\alpha : Y \rightarrow I$ . A *perfect* map is a closed continuous surjection with compact fibres.

THEOREM 3. *If  $D$  is a compactification of a space  $Y$  then the following statements are equivalent:*

---

1991 *Mathematics Subject Classification*: 54C20, 54C10, 54D35.

The first author is pleased to thank the Vrije Universiteit in Amsterdam for its hospitality and support.

(1) For every space  $X$ , every monotone map  $f : X \rightarrow Y$ , and every compactification  $C$  of  $X$  such that  $f$  extends to a continuous  $\tilde{f} : C \rightarrow D$ , the map  $f$  is monotone.

(2) There are a space  $X$  and a monotone map  $\tilde{f} : \beta X \rightarrow D$  such that  $\tilde{f}(X) \subset Y$ .

(3) For any pair of disjoint zero sets  $A$  and  $B$  in  $Y$  we have  $\bar{A} \cap \bar{B} \cap \text{int}(\overline{A \cup B}) = \emptyset$ .

(4) For any pair of disjoint closed subsets  $A$  and  $B$  of  $Y$  we have  $\bar{A} \cap \bar{B} \cap \text{int}(\overline{A \cup B}) = \emptyset$ .

We obtain Theorem 3 as an immediate corollary of the following more general statement.

**THEOREM 4.** *If  $Y$  is a dense subspace of a space  $D$  then the following statements are equivalent:*

(1) Let  $X$  be a dense subspace of a space  $C$  and let  $\tilde{f} : C \rightarrow D$  be a closed continuous map such that  $f = \tilde{f}|X$  is a monotone map from  $X$  onto  $Y$ . If  $\tilde{f}$  is perfect or if  $C$  is normal then  $\tilde{f}$  is monotone.

(2) There are a space  $X$ , a space  $C$  with  $X \subset C \subset \beta X$ , and a monotone map  $\tilde{f} : C \rightarrow D$  such that  $\tilde{f}(X) \subset Y$ .

(3) For any pair of disjoint zero sets  $A$  and  $B$  in  $Y$  we have  $\bar{A} \cap \bar{B} \cap \text{int}(\overline{A \cup B}) = \emptyset$ .

(4) For any pair of disjoint closed subsets  $A$  and  $B$  of  $Y$  we have  $\bar{A} \cap \bar{B} \cap \text{int}(\overline{A \cup B}) = \emptyset$ .

We need an elementary lemma:

**LEMMA 5.** *If  $f : C \rightarrow Y$  is continuous and  $X$  is a dense subset of  $C$  such that  $f|X : X \rightarrow Y$  is closed then for every  $y \in Y$  we have  $f^{-1}(y) = \overline{f^{-1}(y) \cap X}$ .*

*Proof.* Let  $x$  be an element of  $C$  that is not in  $\overline{f^{-1}(y) \cap X}$ . To prove  $x \notin f^{-1}(y)$ , select a closed neighbourhood  $U$  of  $x$  that is disjoint from  $\overline{f^{-1}(y) \cap X}$ . Since  $f|X$  is closed the set  $V = Y \setminus f(U \cap X)$  is an open neighbourhood of  $y$ . Note that  $f^{-1}(V) \cap \text{int}(U)$  is an open set which is disjoint from  $X$ . Since  $X$  is dense,  $f^{-1}(V)$  and  $\text{int}(U)$  are disjoint. Since  $x \in \text{int}(U)$  we have  $f(x) \neq y$ .

*Proof of Theorem 4.* Statement (2) follows trivially from (1). We shall prove: (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (1).

Assume (2) and let  $A$  and  $B$  be disjoint zero sets in  $Y$  such that for some  $y \in D$  we have  $y \in \bar{A} \cap \bar{B} \cap \text{int}(\overline{A \cup B})$ . Then  $y \in D \setminus Y$  and  $\tilde{f}^{-1}(y)$  is a connected subset of  $C \setminus X$ . If  $W = \text{int}(\overline{A \cup B})$  then  $\tilde{f}^{-1}(W) \setminus \tilde{f}^{-1}(A \cup B)$  is an open subset of  $C$  that is disjoint from  $X$ . Since  $X$  is dense in  $C$  we

have  $\tilde{f}^{-1}(W) \subset \overline{\tilde{f}^{-1}(A \cup B)}$  and  $\tilde{f}^{-1}(y) \subset \overline{\tilde{f}^{-1}(A \cup B)}$ . Since  $\tilde{f}^{-1}(A)$  and  $\tilde{f}^{-1}(B)$  are disjoint zero sets in  $\tilde{f}^{-1}(Y)$  and  $X \subset \tilde{f}^{-1}(Y) \subset C \subset \beta X$  we see that  $\overline{\tilde{f}^{-1}(A)}$  and  $\overline{\tilde{f}^{-1}(B)}$  are a pair of disjoint closed sets in  $C$  that cover  $\tilde{f}^{-1}(y)$ . So  $\tilde{f}^{-1}(y)$  is disjoint from one of them, say  $\overline{\tilde{f}^{-1}(A)}$ . Then  $y$  is not in  $\tilde{f}(\overline{\tilde{f}^{-1}(A)})$ , which contains  $\overline{A'}$ , because  $\tilde{f}$  is closed and surjective. This is a contradiction.

Assume (3) and let  $A$  and  $B$  be disjoint closed sets in  $Y$  such that for some  $y \in D$  we have  $y \in \overline{A} \cap \overline{B} \cap \text{int}(\overline{A \cup B})$ . Then  $y \in D \setminus Y$ . Put  $W = \text{int}(\overline{A \cup B})$  and select a continuous  $\alpha : D \rightarrow [0, 1]$  such that  $\alpha(y) = 1$  and  $\alpha|_{D \setminus W} = 0$ . We now define the continuous map  $\gamma : Y \rightarrow [-1, 1]$  as follows:

$$\gamma = (\alpha|_{A \cup (X \setminus W)}) \cup (-\alpha|_{B \cup (X \setminus W)}).$$

Define the zero sets  $A' = \gamma^{-1}([1/2, 1])$  and  $B' = \gamma^{-1}([-1, -1/2])$  in  $Y$ . Note that  $A' \cup B' = Y \cap \alpha^{-1}([1/2, 1])$ . Let  $O$  stand for the open set  $\alpha^{-1}((1/2, 1])$  and observe that  $O \subset \overline{A' \cup B'}$ . So  $y$  is in the interior of  $\overline{A' \cup B'}$ . We show that  $y \in \overline{A'}$  (and hence  $y \in \overline{B'}$  by symmetry). By assumption,  $y \in \overline{A}$  and since  $O$  is a neighbourhood of  $y$  we have  $y \in \overline{A \cap O}$ . Note that  $A \cap O \subset A'$  and hence  $y \in \overline{A'}$ .

Assume (4) and let  $f : X \rightarrow Y$  be a monotone map such that  $\tilde{f} : C \rightarrow D$  is a closed continuous extension of  $f$  that is not monotone. Assume moreover that  $\tilde{f}$  is perfect or that  $C$  is normal. Let  $y$  be an element of  $D$  with a disconnected fibre. If  $y \in Y$  then  $\tilde{f}^{-1}(y) = f^{-1}(y)$  by Lemma 5. Since  $f$  is monotone this would imply that  $\tilde{f}^{-1}(y)$  is connected and hence we know that  $y \in D \setminus Y$ . Since  $\tilde{f}^{-1}(y)$  is compact or  $C$  is normal we can find a disjoint open cover  $\{U, V\}$  of  $\tilde{f}^{-1}(y)$  in  $C$  such that both  $U$  and  $V$  intersect the fibre. Then  $F = \tilde{f}(C \setminus (U \cup V))$  is a closed subset of  $D$  that does not contain  $y$ . Let  $W$  be a closed neighbourhood of  $y$  in  $D$  that is disjoint from  $F$ . Note that  $\tilde{f}^{-1}(W) \subset U \cup V$ . Define  $A' = U \cap f^{-1}(W) = f^{-1}(W) \setminus V$  and  $B' = V \cap f^{-1}(W) = f^{-1}(W) \setminus U$ .

Both  $A'$  and  $B'$  are saturated closed subsets of  $X$ . This can be seen as follows: if  $b \in Y$  such that  $f^{-1}(b)$  intersects for instance  $A'$  then  $f^{-1}(b) \subset A' \cup B'$  since  $A' \cup B' = f^{-1}(W)$  is saturated. Since  $f$  is monotone,  $f^{-1}(b)$  is connected and hence  $f^{-1}(b) \subset A'$ . Since  $f$  is a closed map we see that  $A = f(A')$  and  $B = f(B')$  are disjoint closed subsets of  $Y$ , whose union is  $W \cap Y$ . Observe that  $\text{int}(W) \subset \overline{A \cup B}$ . So  $y$  is in the interior of  $\overline{A \cup B}$  and by assumption (4),  $y \notin \overline{A}$  or  $y \notin \overline{B}$ . By symmetry we may assume that  $y$  is outside  $\overline{A}$ .

Let  $x$  be an element of  $U$  such that  $\tilde{f}(x) = y$ . Then  $U \cap \tilde{f}^{-1}(W) \setminus \tilde{f}^{-1}(\overline{A})$  is a neighbourhood of  $x$  and hence  $P = U \cap \tilde{f}^{-1}(W) \setminus \overline{A'}$  is a neighbourhood of  $x$ . Since  $A' = U \cap \tilde{f}^{-1}(W) \cap X$  we infer that  $P$  does not

intersect  $X$ —a contradiction. So we may conclude that  $U$  is disjoint from  $\tilde{f}^{-1}(y)$ , which contradicts our assumption that  $\{U, V\}$  separates  $\tilde{f}^{-1}(y)$ . The proof is complete.

REMARKS. We say that  $D$  is a *monotone extension* of  $Y$  if  $Y$  is a dense subset of  $D$  and the pair  $(Y, D)$  satisfies the conditions (1)–(4) in Theorem 4. If  $D$  is moreover compact then we call it a *monotone compactification* of  $Y$ .

Consider Theorem 3. It may be surprising that the criterion expressed by statement (3) does only depend on  $Y$  and  $D$  and that the domain of the monotone map does not seem to matter. In this context observe that

(5) The extension of the identity  $\tilde{v}: \beta Y \rightarrow D$  is monotone

is one of many statements that imply (2) and follow from (1).

If we substitute  $D = \beta Y$  in Theorem 3 then (3) is obviously satisfied and Proposition 1 follows. If  $D \setminus Y$  is locally connected rel  $Y$  and  $A$  and  $B$  are disjoint closed sets in  $Y$  such that  $y \in \bar{A} \cap \bar{B} \cap \text{int}(\overline{A \cup B})$  then we can find a neighbourhood  $U \subset \text{int}(\overline{A \cup B})$  of  $y$  in  $D$  such that  $U \cap Y$  is connected. Then  $U \cap A$  and  $U \cap B$  are both nonempty, which means that  $A$  and  $B$  separate the connected set  $U \cap Y$ . So  $D$  is a monotone extension of  $Y$  and Proposition 2 also follows from the theorem.

EXAMPLE 1. As an illustration to Theorem 3 we give a simple example of a monotone compactification that is not covered by Proposition 1 or 2. Let  $I = [0, 1]$  and define the following subspaces of  $I \times I$ :

$$D = (\{0\} \cup \{1/n : n \in \mathbb{N}\}) \times I \quad \text{and} \quad Y = D \setminus \{(0, 0)\}.$$

We verify that  $D$  is a monotone compactification of  $Y$  and hence Theorem 3 guarantees that for every space  $X$  which is the preimage of  $Y$  under a perfect monotone map and every compactification  $C$  of  $X$  the remainder  $C \setminus X$  is a continuum.

Let  $A$  and  $B$  be disjoint closed subsets of  $Y$  such that  $\bar{A} \cap \bar{B} \cap \text{int}(\overline{A \cup B}) \neq \emptyset$ . Then  $\bar{A} \cap \bar{B} \cap \text{int}(\overline{A \cup B}) = \{(0, 0)\}$  and we can find an  $\varepsilon > 0$  such that  $([0, \varepsilon] \times [0, \varepsilon]) \cap Y \subset A \cup B$ . We may assume that  $(0, \varepsilon)$  is in  $A$  and hence not in  $B$ . Since  $B$  is closed there is an  $N > 1/\varepsilon$  such that  $(1/n, \varepsilon) \in A$  for every  $n \geq N$ . Since for every  $n \geq N$ ,  $\{1/n\} \times [0, \varepsilon]$  is a connected subset of  $A \cup B$  we have  $\{1/n\} \times [0, \varepsilon] \subset A$  for  $n \geq N$ . Consequently,  $([0, 1/N] \times [0, \varepsilon]) \cap Y \subset A$  and hence  $(0, 0) \notin \bar{B}$ , which is a contradiction.

EXAMPLE 2. Consider condition (1) in Theorem 4. A natural question is whether the mild restriction that  $\tilde{f}$  be perfect or  $C$  be normal is really necessary. The following example shows that the answer is yes.

Let  $L$  be the “long halfline,” i.e. the space  $[0, \omega_1) \times [0, 1)$  with the topology generated by the lexicographic order. Let  $\alpha L = L \cup \{\omega_1\}$  be the compactification of  $L$ . Let  $X = Y = L \times [0, 1)$ ,  $C = (\alpha L \times I) \setminus \{(\omega_1, 1)\}$ , and let  $D$  be

the one-point compactification  $Y \cup \{\infty\}$  of  $Y$ . We take for the monotone map  $f : X \rightarrow Y$  the identity and  $\tilde{f} : C \rightarrow D$  and  $\bar{f} : \alpha L \times I \rightarrow D$  are the extensions of  $f$ . It is obvious that  $\{\infty\}$  is locally connected rel  $Y$  so  $D$  is a monotone compactification of  $Y$ . The fibre  $\tilde{f}^{-1}(\infty) = (L \times \{1\}) \cup (\{\omega_1\} \times [0, 1))$  has two components so  $\tilde{f}$  is not monotone.

It remains to show that  $\bar{f}$  is closed. Let  $F$  be a closed subset of  $C$  and let  $G$  denote the closure of  $F$  in  $\alpha L \times I$ . If  $\infty \in \tilde{f}(F)$  then  $\tilde{f}(F)$  equals  $\bar{f}(G)$  and hence is compact. If  $\infty \notin \tilde{f}(F)$  then  $\alpha L \times \{1\}$  and  $\{\omega_1\} \times [0, 1)$  are disjoint from  $F$ . Since  $[0, 1)$  is Lindelöf and  $\omega_1$  has uncountable cofinality there is a neighbourhood  $U$  of  $\omega_1$  in  $\alpha L$  such that  $F \cap (U \times [0, 1)) = \emptyset$ . So  $F$  is disjoint from  $U \times I$  and hence  $F$  and  $\bar{f}(F)$  are compact.

Let  $D$  be a compactification of  $Y$ . Theorem 3 answers the question when all “compactifications” with range  $D$  of monotone maps onto  $Y$  are monotone. We now turn to the question when we can guarantee the *existence* of monotone “compactifications” onto  $D$  of monotone maps onto  $Y$ . Before presenting a criterion we discuss an illuminating example.

EXAMPLE 3. Put  $D = I$  and  $Y = I \setminus \{1/n : n \in \mathbb{N}\}$ . Consider the following closed subspace of  $Y \times I$ :

$$X = (\{0\} \times I) \cup \bigcup_{n=1}^{\infty} \left( \left( \frac{1}{2n}, \frac{1}{2n-1} \right) \times \{0\} \right) \cup \left( \left( \frac{1}{2n+1}, \frac{1}{2n} \right) \times \{1\} \right).$$

The map  $f : X \rightarrow Y$  is simply the restriction of the projection. Since  $X$  is closed in  $Y \times I$  and  $I$  is compact we find that the projection  $f$  is perfect. Note that every fibre of  $f$  is either a singleton or an interval so  $f$  is monotone.

Assume now that  $C$  is a compactification of  $X$  such that  $f$  extends to a monotone  $\tilde{f} : C \rightarrow D$ . Since  $\tilde{f}$  is monotone,  $\tilde{f}^{-1}((0, 1/n])$ ,  $n \in \mathbb{N}$ , is a decreasing sequence of continua in  $C$ . Consequently,

$$K = \bigcap_{n=1}^{\infty} \overline{\tilde{f}^{-1}((0, 1/n])}$$

is a continuum that is obviously contained in  $\tilde{f}^{-1}(0) = \{0\} \times I$ . Note that both  $(0, 0)$  and  $(0, 1)$  are in  $K$  but that  $\{0\} \times (0, 1)$  is an open locally compact subspace of  $X$  and hence also open in  $C$ . This means that  $K$  and  $\{0\} \times (0, 1)$  are disjoint so the continuum  $K$  equals  $\{(0, 0), (0, 1)\}$ , a contradiction. We may conclude that  $f$  does not have a monotone “compactification” whose range is  $D$ .

We say that a space  $Y$  has *ordered neighbourhood bases* if every  $y \in Y$  has a neighbourhood basis that is linearly ordered by the inclusion relation. First countable spaces are obvious examples of such spaces.

**THEOREM 6.** *If  $Y$  is a dense subspace of a space  $D$  and  $Y$  has ordered neighbourhood bases then the following statements are equivalent:*

(1) *For every space  $X$  and every monotone map  $f : X \rightarrow Y$  there exists a space  $C$  such that  $X$  is dense in  $C$  and  $f$  extends to a monotone and perfect  $\tilde{f} : C \rightarrow D$ .*

(2) *For every closed subspace  $X$  of  $Y \times I$  such that the projection  $f : X \rightarrow Y$  is monotone there exists a space  $C$  such that  $X$  is dense in  $C$  and  $f$  extends to a monotone  $\tilde{f} : C \rightarrow D$ .*

(3) *Every  $y \in Y$  has a neighbourhood  $U$  in  $D$  such that  $U$  is a monotone extension of  $Y \cap U$ .*

(4) *There exists an open  $O$  in  $D$  that is a monotone extension of  $Y$ .*

**PROOF.** Statement (2) follows trivially from (1). We shall prove: (4) $\Rightarrow$ (1), (3) $\Rightarrow$ (4), and  $\neg$ (3) $\Rightarrow$  $\neg$ (2).

Assume (4) and let  $f : X \rightarrow Y$  be monotone. Extend  $f$  to  $\bar{f} : \beta X \rightarrow \beta D$ . Put  $U = \bar{f}^{-1}(O)$  and  $C' = \bar{f}^{-1}(D)$ . Note that  $\bar{f}|U$  is a perfect map from  $U$  onto  $O$ . Since  $O$  is a monotone extension of  $Y$  we see that  $\bar{f}|U$  is monotone. Consider the closed subspace  $F = C' \setminus U$  of  $C'$  and the closed map  $p = \bar{f}|F$  from  $F$  onto  $G = D \setminus O$ . Let  $C$  be the adjunction space  $C' \cup_p G$  and let  $\pi : C' \rightarrow C$  be the quotient map. Then we can define a function  $\tilde{f} : C \rightarrow D$  such that  $\tilde{f} \circ \pi = \bar{f}|C'$ . The map  $\tilde{f}$  obviously extends  $f$  and is closed and continuous. If  $y \in O$  then  $\tilde{f}^{-1}(y)$  is a fibre of the map  $\bar{f}|U$  and hence a continuum. If  $y \in D \setminus O = G$  then  $\tilde{f}^{-1}(y)$  is a singleton. So we may conclude that  $\tilde{f}$  is monotone and perfect.

Assume (3). If we define

$\mathcal{U} = \{U : U \text{ an open subset of } D \text{ such that}$

$U \text{ is a monotone extension of } Y \cap U\}$

then  $O = \bigcup \mathcal{U}$  is an open set in  $D$  that contains  $Y$ . Let  $g : V \rightarrow O$  be a perfect extension of the identity on  $Y$  such that  $Y \subset V \subset \beta Y$ . Note that by Theorem 4,  $g|g^{-1}(U)$  is monotone for each  $U \in \mathcal{U}$  and hence  $g$  is monotone. So  $O$  is a monotone extension of  $Y$ .

Assume that condition (3) is false, i.e. there is a  $y \in Y$  such that no neighbourhood  $U$  in  $D$  is a monotone extension of  $Y \cap U$ . Let  $\{V_\alpha : \alpha < \kappa\}$  be a neighbourhood basis for  $y$  in  $Y$  where  $\kappa$  is some regular cardinal and  $V_\alpha \subset V_\beta$  for  $\beta < \alpha < \kappa$ . Define  $\tilde{V}_\alpha = D \setminus (\overline{Y \setminus V_\alpha})$  for each  $\alpha$  and note that  $\{\tilde{V}_\alpha : \alpha < \kappa\}$  is a neighbourhood basis for  $y$  in  $D$  because  $Y$  is dense and  $D$  is regular.

We construct by induction for each  $\alpha < \kappa$  an ordinal  $\gamma(\alpha) < \kappa$ , a point  $y_\alpha \in D \setminus Y$ , an open subset  $U_\alpha$  of  $D$ , and disjoint closed subsets  $A_\alpha$  and  $B_\alpha$  of  $Y$  such that

- (i)  $\gamma(\beta) < \gamma(\alpha)$  for  $\beta < \alpha$ ,
- (ii)  $y_\alpha \in U_\alpha \cap \bar{A}_\alpha \cap \bar{B}_\alpha$ ,
- (iii)  $U_\alpha \subset (\overline{A_\alpha \cup B_\alpha}) \cap \tilde{V}_{\gamma(\alpha)} \setminus \tilde{V}_{\gamma(\alpha+1)}$ .

Let  $\alpha < \kappa$ . If  $\alpha$  is a successor ordinal then we assume that  $\gamma(\alpha)$  has already been selected, if  $\alpha = 0$  then we put  $\gamma(\alpha) = 0$ , and if  $\alpha$  is a limit ordinal then we put  $\gamma(\alpha) = \sup_{\beta < \alpha} \gamma(\beta)$ . We can find a  $y_\alpha \in \tilde{V}_{\gamma(\alpha)}$  and disjoint closed subsets  $A_\alpha$  and  $B_\alpha$  of  $Y$  such that  $y_\alpha \in \bar{A}_\alpha \cap \bar{B}_\alpha \cap \text{int}(\overline{A_\alpha \cup B_\alpha})$ . Select a  $\gamma(\alpha + 1) > \gamma(\alpha)$  such that  $y_\alpha \notin \tilde{V}_{\gamma(\alpha+1)}$ . Then define

$$U_\alpha = \text{int}(\overline{A_\alpha \cup B_\alpha}) \cap \tilde{V}_{\gamma(\alpha)} \setminus \tilde{V}_{\gamma(\alpha+1)}.$$

Note that the  $U_\alpha$ 's are pairwise disjoint. Put  $O = \bigcup_{\alpha < \kappa} U_\alpha$  and define the subset  $X$  of  $Y \times I$  by

$$X = ((Y \setminus O) \times I) \cup \bigcup_{\alpha < \kappa} ((A_\alpha \cap U_\alpha) \times \{0\}) \cup ((B_\alpha \cap U_\alpha) \times \{1\}).$$

Let  $f : X \rightarrow Y$  be the projection. Since  $\{A_\alpha \cap U_\alpha, B_\alpha \cap U_\alpha : \alpha < \kappa\}$  is a pairwise disjoint open covering of  $O \cap Y$ , we see that  $X$  is closed in  $Y \times I$  and that every fibre of  $f$  is a singleton or an interval. Since  $X$  is closed we find that  $f$  is perfect by the compactness of  $I$ . Consequently,  $f$  is monotone.

Let  $C$  be a space such that  $X$  is dense in  $C$  and  $f$  extends to a monotone  $\tilde{f} : C \rightarrow D$ . Define the following closed subsets of  $C$ :

$$A = \overline{(Y \times \{0\}) \cap X} \quad \text{and} \quad B = \overline{(Y \times \{1\}) \cap X}.$$

We show that  $\tilde{f}^{-1}(O)$  is contained in  $A \cup B$ . If  $x \in \tilde{f}^{-1}(O)$  and  $V$  is a neighbourhood of  $x$  that is contained in  $\tilde{f}^{-1}(O)$  then we can find a  $z \in V \cap X$ . Since  $f(z) \in O$  we have  $z \in (Y \times \{0, 1\}) \cap X$ . Hence  $z$  is in  $A \cup B$  and so is  $x$ .

Since  $\tilde{f}$  is closed  $\tilde{f}(A)$  is closed in  $D$ . For each  $\alpha < \kappa$ ,  $A_\alpha \cap U_\alpha$  is a subset of  $\tilde{f}(A)$ . Since  $y_\alpha \in \bar{A}_\alpha \cap \bar{U}_\alpha$  we have  $y_\alpha \in \tilde{f}(A)$ . So  $\tilde{f}^{-1}(y_\alpha) \cap A \neq \emptyset$  and by symmetry  $\tilde{f}^{-1}(y_\alpha) \cap B \neq \emptyset$ .

Let  $U$  and  $V$  be disjoint open sets in  $C$  such that  $(y, 0) \in U$ ,  $(y, 1) \in V$ , and  $U \cap B = V \cap A = \emptyset$ . Since  $f$  is perfect we have  $\tilde{f}^{-1}(y) = f^{-1}(y) = \{y\} \times I$ . Note that this fact implies that  $F = \tilde{f}(A \setminus U) \cup \tilde{f}(B \setminus V)$  is a closed subset of  $D$  that does not contain  $y$ . Since  $\sup_{\alpha < \kappa} \gamma(\alpha) = \kappa$  there is an  $\alpha < \kappa$  with  $\tilde{V}_{\gamma(\alpha)} \cap F = \emptyset$ . So  $y_\alpha \in D \setminus F$  by (ii) and (iii). Since  $y_\alpha \in U_\alpha \subset O$  we have  $\tilde{f}^{-1}(y_\alpha) \subset A \cup B$ . Consequently,  $\tilde{f}^{-1}(y_\alpha) \subset U \cup V$  and since  $\tilde{f}^{-1}(y_\alpha) \cap A \neq \emptyset$  and  $\tilde{f}^{-1}(y_\alpha) \cap B \neq \emptyset$  we have  $\tilde{f}^{-1}(y_\alpha) \cap A \cap U \neq \emptyset$  and  $\tilde{f}^{-1}(y_\alpha) \cap B \cap V \neq \emptyset$ . Since  $U$  and  $V$  are disjoint open sets they separate the fibre  $\tilde{f}^{-1}(y_\alpha)$  and hence  $\tilde{f}$  is not monotone, a contradiction.

REMARKS. Note that the condition that  $Y$  has ordered neighbourhood bases is only used to prove (2) $\Rightarrow$ (3). Without any restrictions on  $Y$  we have (3) $\Leftrightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (2). One can think of other conditions that would make Theorem 6 true. For instance, if  $Y$  is an ordered space then the proof can easily be adapted. The question is whether the implications (1) $\Rightarrow$ (3) or (2) $\Rightarrow$ (3) are true in general.

Proposition 1 implies that every monotone map can be “compactified” to a monotone map by using the Čech–Stone compactifications. For separable metric spaces that result is not very satisfactory, especially since we were motivated to look at monotone maps by a problem formulated in Dijkstra and Mogilski [2], which concerns extendibility of cell-like decompositions of Hilbert space. To address the metric case we have the following

**THEOREM 7.** *If  $f : X \rightarrow Y$  is a monotone map between separable metric spaces then there exist metric compactifications  $C$  and  $D$  of  $X$  and  $Y$  such that  $f$  extends to a monotone  $\tilde{f} : C \rightarrow D$ .*

**Proof.** The proof uses the Wallman compactification whose definition we now recall. We call a closed basis  $\mathfrak{W}$  for the topology of a space  $X$  a *Wallman basis* for  $X$  if  $\mathfrak{W}$  is closed under finite intersections and if  $\mathfrak{W}$  is normal (i.e. if  $A$  and  $B$  are disjoint members of  $\mathfrak{W}$  then there are  $V, W \in \mathfrak{W}$  such that  $V \cup W = X$  and  $V \cap B = A \cap W = \emptyset$ ). If  $\mathfrak{W}$  is a Wallman basis for  $X$  then the underlying set for the *Wallman compactification*  $\omega(\mathfrak{W})$  of  $X$  relative  $\mathfrak{W}$  is the set of  $\mathfrak{W}$ -ultrafilters. If  $W \in \mathfrak{W}$  then  $\overline{W} = \{\mathcal{F} \in \omega(\mathfrak{W}) : W \in \mathcal{F}\}$ . The collection  $\{\overline{W} : W \in \mathfrak{W}\}$  functions as a closed basis for the topology on  $\omega(\mathfrak{W})$ . Since  $\mathfrak{W}$  is normal  $\omega(\mathfrak{W})$  is Hausdorff and if  $\mathfrak{W}$  is countable then  $\omega(\mathfrak{W})$  is metrizable. We shall use the following well-known fact: if  $f : X \rightarrow Y$  is a map and  $\mathfrak{X}$  and  $\mathfrak{Y}$  are Wallman bases on  $X$  and  $Y$  respectively such that  $f^{-1}[\mathfrak{Y}] \subset \mathfrak{X}$  then  $f$  extends to a map  $\bar{f} : \omega(\mathfrak{X}) \rightarrow \omega(\mathfrak{Y})$ . See Walker [5] for more information about Wallman compactifications.

Let  $f : X \rightarrow Y$  be a monotone map between separable metric spaces. Select a countable Wallman basis  $\mathfrak{C}_0$  for  $X$ . Expand  $f[\mathfrak{C}_0]$ , which is a countable collection of closed subsets of  $Y$ , to a countable Wallman basis  $\mathfrak{D}_0$  for  $Y$ . Next, expand  $f^{-1}[\mathfrak{D}_0] \cup \mathfrak{C}_0$  to a countable Wallman basis  $\mathfrak{C}_1$ . Continuing this back-and-forth process we find an increasing sequence  $(\mathfrak{C}_n)_{n=0}^\infty$  of countable Wallman bases for  $X$  and an increasing sequence  $(\mathfrak{D}_n)_{n=0}^\infty$  of countable Wallman bases for  $Y$  such that  $f[\mathfrak{C}_n] \subset \mathfrak{D}_n$  and  $f^{-1}[\mathfrak{D}_n] \subset \mathfrak{C}_{n+1}$  for each  $n \geq 0$ . So  $\mathfrak{C} = \bigcup_{n=0}^\infty \mathfrak{C}_n$  and  $\mathfrak{D} = \bigcup_{n=0}^\infty \mathfrak{D}_n$  are countable Wallman bases for  $X$  and  $Y$  respectively with the properties  $f^{-1}[\mathfrak{D}] \subset \mathfrak{C}$  and  $f[\mathfrak{C}] = \mathfrak{D}$ . If we define the metric compactifications  $C = \omega(\mathfrak{C})$  and  $D = \omega(\mathfrak{D})$  then  $f$  extends to a continuous  $\tilde{f} : C \rightarrow D$ .

Let  $y$  be an element of  $D$  with a disconnected fibre. If  $y \in Y$  then  $\tilde{f}^{-1}(y) = \overline{f^{-1}(y)}$  by Lemma 5. Since  $f$  is monotone this would imply that



$\tilde{f}^{-1}(y)$  is connected and hence we may assume that  $y \in D \setminus Y$ . Write  $\tilde{f}^{-1}(y)$  as a disjoint union of two nonempty compacta  $A$  and  $B$ . Select from  $\mathfrak{C}$  two disjoint elements  $F$  and  $G$  such that  $\bar{F}$  is a neighbourhood of  $A$  and  $\bar{G}$  is a neighbourhood of  $B$ . Then  $P = \tilde{f}(C \setminus \text{int}(\bar{F} \cup \bar{G}))$  is a closed set that does not contain  $y$ . Let  $W$  be an element of  $\mathfrak{D}$  such that  $\bar{W}$  is a neighbourhood of  $y$  that is disjoint from  $P$ . Note that  $f^{-1}(W) \subset F \cup G$ . Define  $F' = F \cap f^{-1}(W)$  and  $G' = G \cap f^{-1}(W)$  and note that both sets are in  $\mathfrak{C}$ . Also, both  $F'$  and  $G'$  are saturated subsets of  $X$ . This can be seen as follows: if  $b \in Y$  such that  $f^{-1}(b)$  intersects for instance  $F'$  then  $f^{-1}(b) \subset F' \cup G'$  because  $F' \cup G' = f^{-1}(W)$  is saturated. Since  $f$  is monotone,  $f^{-1}(b)$  is connected and hence  $f^{-1}(b) \subset F'$ . Note that  $U = f(F')$  and  $V = f(G')$  are disjoint elements of  $\mathfrak{D}$  and that their union is  $W$ . Since  $D = \omega(\mathfrak{D})$  we see that  $\bar{U}$  and  $\bar{V}$  are also disjoint so that one of them does not contain  $y$ , say  $y \notin \bar{U}$ . Then  $\tilde{f}^{-1}(y)$  is disjoint from  $\tilde{f}^{-1}(\bar{U})$  and hence disjoint from  $\bar{F}'$ . Consequently,  $A = \tilde{f}^{-1}(y) \cap \bar{F}'$  is empty, which is a contradiction.

**COROLLARY 8.** *Let  $C$  and  $D$  be separable metric spaces, let  $\tilde{f} : C \rightarrow D$  be a closed and continuous map, and let  $X$  and  $Y$  be dense subsets of  $C$  and  $D$  respectively such that  $f = \tilde{f}|_X$  is a monotone map from  $X$  onto  $Y$ . Then there is a  $G_\delta$ -subset  $G$  of  $D$  such that  $Y \subset G$  and  $\tilde{f}|_{\tilde{f}^{-1}(G)} : \tilde{f}^{-1}(G) \rightarrow G$  is monotone.*

So every extension of a monotone map over metric compactifications restricts to a perfect monotone extension over completions.

*Proof.* Note that  $\tilde{f}$  is surjective because it is closed and its range contains  $Y$ . Let  $\hat{C}$  and  $\hat{D}$  be metric compactifications of  $C$  and  $D$  such that  $\tilde{f}$  extends to a continuous  $\hat{f} : \hat{C} \rightarrow \hat{D}$ . If we define  $\check{X} = \hat{f}^{-1}(Y)$  then by Lemma 5,  $\check{f} = \hat{f}|_{\check{X}}$  is a perfect monotone map from  $\check{X}$  to  $Y$ . With Theorem 7 we find metric compactifications  $X'$  and  $Y'$  of  $\check{X}$  and  $Y$  respectively such that  $\check{f}$  extends to a monotone  $f' : X' \rightarrow Y'$ . According to Lavrentiev [4] there exist  $G_\delta$ -sets  $A \subset \hat{C}$ ,  $A' \subset X'$ ,  $B \subset \hat{D}$ ,  $B' \subset Y'$ , and homeomorphisms  $\alpha : A \rightarrow A'$  and  $\beta : B \rightarrow B'$  such that  $\check{X} \subset A$ ,  $\check{X} \subset A'$ ,  $Y \subset B$ ,  $Y \subset B'$ , and  $\alpha|_{\check{X}}$  and  $\beta|_Y$  are identity mappings. Let  $G'' = B' \setminus f'(X' \setminus A')$  and note that  $G''$  is a  $G_\delta$ -set in  $Y'$  that contains  $Y$ . Define the  $G_\delta$ -sets  $F' = f'^{-1}(G'')$ ,  $\hat{F} = \alpha^{-1}(F')$ , and  $\hat{G} = \beta^{-1}(G'')$ . Since  $\hat{f}|_X = \beta^{-1} \circ f' \circ \alpha|_X$  and  $X$  is dense we have  $\hat{f}|_{\hat{F}} = \beta^{-1} \circ f' \circ \alpha|_{\hat{F}}$ . Since  $f'|_{F'}$  is perfect and monotone so is  $\hat{f}|_{\hat{F}}$ . Put  $F = \hat{F} \cap C$  and  $G = \hat{G} \cap D$ . Consider the map  $g = \tilde{f}|_F = \hat{f}|_F$  from  $F$  to  $G$ .

It is obvious that  $\tilde{f}^{-1}(G) = F$  and that  $g$  is closed<sup>4</sup> and surjective. It remains to verify that  $g$  has connected fibres. If  $y \in Y$  then  $g^{-1}(y)$  is connected by Lemma 5. Let  $y \in G \setminus Y$  and  $x \in \hat{F}$  such that  $\hat{f}(x) = y$ . Select a sequence  $x_1, x_2, \dots$  in  $X$  that converges to  $x$ . If  $x \notin C$  then  $\{x_n : n \in \mathbb{N}\}$

is closed in  $C$ . Since  $\tilde{f} : C \rightarrow D$  is closed we see that  $\{f(x_n) : n \in \mathbb{N}\}$  is closed in  $D$ . This contradicts the fact that  $f(x_1), f(x_2), \dots$  is a sequence in  $Y$  that converges to  $y \in D \setminus Y$ . So we may conclude that if  $y \in G \setminus Y$  then  $g^{-1}(y) = \hat{f}^{-1}(y)$ . Since  $\hat{f}$  is monotone,  $g$  is monotone.

EXAMPLE 4. In view of Theorems 3 and 7 it is natural to ask the following question: does every separable metric space  $Y$  have a metric compactification  $D$  with the property that whenever  $f : C \rightarrow D$  is a map with compact metric domain such that  $f|_X : X \rightarrow Y$  is monotone for some dense  $X \subset C$ , then  $f$  is monotone as well?

The answer is easily seen to be no. Consider a metric compactification  $D$  of the natural numbers  $\mathbb{N}$  and let  $\tilde{\iota} : \beta\mathbb{N} \rightarrow D$  be the extension of the identity. Since  $|D \setminus \mathbb{N}| \leq c$  and  $|\beta\mathbb{N} \setminus \mathbb{N}| = 2^c$  we can pick a  $y \in D \setminus \mathbb{N}$  with nontrivial fibre. Pick a subset  $A$  of  $\mathbb{N}$  such that both  $\bar{A}$  and its complement in  $\beta\mathbb{N}$  intersect  $\tilde{\iota}^{-1}(y)$ . Let  $B_1$  be the closure of  $A$  in  $D$  and let  $B_2$  be the closure of  $\mathbb{N} \setminus A$  in  $D$ . If  $C$  is the topological sum of  $B_1$  and  $B_2$  then the natural map from  $C$  to  $D$  is an extension of the identity that is not monotone.

#### REFERENCES

- [1] J. J. Dijkstra, *Strongly negligible sets outside Fréchet manifolds*, Bull. London Math. Soc. 19 (1987), 371–377.
- [2] J. J. Dijkstra and J. Mogilski, *Countable dimensionality and dimension raising cell-like maps*, Topology Appl. 80 (1997), 73–79.
- [3] K. P. Hart, *The Čech–Stone compactification of the real line*, in: Recent Progress in General Topology, M. Hušek and J. van Mill (eds.), North-Holland, Amsterdam 1992, 317–352.
- [4] M. Lavrentiev, *Contributions à la théorie des ensembles homéomorphes*, Fund. Math. 6 (1924), 149–160.
- [5] R. C. Walker, *The Stone–Čech Compactification*, Springer, Berlin, 1974.

Department of Mathematics  
The University of Alabama  
Box 870350  
Tuscaloosa, Alabama 35487-0350  
U.S.A.  
E-mail: jdijkstr@ua1vm.ua.edu

Faculteit Wiskunde en Informatica  
Vrije Universiteit  
De Boelelaan 1081a  
1081 HV, Amsterdam, The Netherlands  
E-mail: vanmill@cs.vu.nl