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GENERAL TOPOLOGY

$C_p(X)$ is not $G_{\delta\sigma}$: a Simple Proof

by

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Summary. It is known that if $C_p(X)$ is a $G_{\delta\sigma}$-subset of $\mathbb{R}^X$ then X is discrete. We present a simple proof of this.

1. Introduction. Let $C_p(X)$ be the set of continuous real-valued functions on a Tychonov space X, and topologize $C_p(X)$ as a subspace of the full product $\mathbb{R}^X$. It is known by [5, 2] that unless X is discrete, $C_p(X)$ cannot be a G_δ-subset of $\mathbb{R}^X$, or more generally a $G_{\delta\sigma}$-subset, or an F_σ-subset. But it can be an $F_{\sigma\delta}$-subset. The proofs of these results are simple, except the proof in [2] of the result that $C_p(X)$ is not a $G_{\delta\sigma}$-subset of $\mathbb{R}^X$ unless X is discrete. The aim of this note is to provide a simple proof of that fact, inspired by the proof of [4, Lemma A.2.4] and well-known Baire category type approximation methods from infinite-dimensional topology. Let us finally note that the $G_{\delta\sigma}$-theorem was used in [1] to prove that if X is a nondiscrete, countable space such that $C_p(X)$ is Borel then it is of exact multiplicative class. In addition, all such function spaces $C_p(X)$ which are absolute $F_{\sigma\delta}$'s are homeomorphic by [3]. The precise topological structure of the function spaces $C_p(X)$ of higher Borel complexity is a mystery despite various interesting and complicated partial results (see e.g. [1]).

I am indebted to Henryk Michalewski for spotting an inaccuracy in an earlier version of this note.

2. A reduction. *All spaces under discussion are Tychonov.* Let X be a space and $f \in C_p(X)$. Since $C_p(X)$ is a subspace of $\mathbb{R}^X$ with the Tychonov

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product topology, it follows that basic neighbourhoods of f have the form

$$N(f,F,\varepsilon)=\{g\in\mathbb{R}^X:(\forall x\in F)(|f(x)-g(x)|<\varepsilon)\},$$

where $F\subseteq X$ is finite, and $\varepsilon>0$.

LEMMA 2.1. *Suppose that $C_p(X)$ contains a non-empty G_δ-subset of $\mathbb{R}^X$. Then X is the topological sum of a countable space and a discrete space.*

Proof. Let S be a non-empty G_δ-subset of $\mathbb{R}^X$ which is contained in $C_p(X)$. Pick an arbitrary element $f\in S$. Let $\{U_n:n<\omega\}$ be a sequence of open subsets of $\mathbb{R}^X$ such that $\bigcap_{n<\omega}U_n=S$. For every $n<\omega$ there exists a finite subset $F_n\subseteq X$ such that every $g\in\mathbb{R}^X$ such that $g\restriction F_n=f\restriction F_n$ belongs to U_n. Put $F=\bigcup_{n<\omega}F_n$. Then F is countable, and every $g\in\mathbb{R}^X$ with $g\restriction F=f\restriction F$ belongs to $\bigcap_{n<\omega}U_n=S\subseteq C_p(X)$. Now if $h:X\setminus F\to\mathbb{R}$ is an arbitrary function then

$$(*)\qquad (f\restriction F)\cup h$$

belongs to S and hence is continuous on X. This implies that h is continuous on $X\setminus F$ and so, h being arbitrary, it follows that $X\setminus F$ is discrete. It also follows from $(*)$ that F and $Y=X\setminus F$ are closed. Striving for a contradiction, assume that there exists an $x\in(\overline{F}\setminus F)\cup(\overline{Y}\setminus Y)$. Let $h:Y\to\mathbb{R}$ be the constant function with value $f(x)+1$. Since $\xi=(f\restriction F)\cup h$ is continuous on X, and $\xi\restriction F=f\restriction F$, it follows that ξ and f agree on $\overline{F}$. So if $x\in\overline{F}\setminus F$ then $f(x)=\xi(x)=f(x)+1$, yields the desired contradiction. On the other hand, if $x\in\overline{Y}\setminus Y\subseteq F$ then since $\xi\restriction Y$ is the constant function with value $f(x)+1$ and $x\in F$, it follows that $f(x)+1=\xi(x)=(\xi\restriction F)(x)=f(x)$, which yields the same contradiction. □

Now let X be an arbitrary space such that $C_p(X)$ is a $G_{\delta\sigma}$-subset of $\mathbb{R}^X$. By Lemma 2.1 it follows that $X=F\oplus D$, where F is countable, D is discrete, and $\oplus$ means topological sum. Since $C_p(X)$ is canonically equivalent to the subspace $C_p(F)\times\mathbb{R}^D$ of $\mathbb{R}^F\times\mathbb{R}^D=\mathbb{R}^X$, it follows that $C_p(F)$ is a $G_{\delta\sigma}$-subset of $\mathbb{R}^F$. We may therefore assume in the proof of the $G_{\delta\sigma}$-theorem that without loss of generality we deal with function spaces $C_p(X)$ with X countable.

3. Proof of the $G_{\delta\sigma}$-theorem. Let X be a space, and let $Y\subseteq\mathbb{R}$. By $C_p(X,Y)$ we denote the set of continuous functions $f:X\to Y$, topologized as a subspace of the full product Y^X. Since Y^X is a subspace of $\mathbb{R}^X$, it follows that $C_p(X,Y)$ is simply the subspace $\{f\in C_p(X):f[X]\subseteq Y\}$ of $C_p(X)$.

Let X be a space and $\varepsilon>0$. We put

$$U(f,\varepsilon)=\{g\in C_p(X):(\forall x\in X)(|f(x)-g(x)|<\varepsilon)\}.$$

LEMMA 3.1. *Let X be a countable space, let $f \in C_p(X)$ and let $\varepsilon > 0$. Then*

(1) $U(f,\varepsilon)$ is a G_δ-subset of $C_p(X)$.
(2) $U(f,\varepsilon)$ and $C_p(X)$ are homeomorphic.
(3) $\bigcup_{0<\delta<\varepsilon} U(f,\delta)$ is dense in $U(f,\varepsilon)$.

Proof. That $U(f,\varepsilon)$ is a G_δ-subset of $C_p(X)$ is clear since for every $x \in X$ the set

$$\{g \in C_p(X) : |g(x) - f(x)| < \varepsilon\}$$

is open in $C_p(X)$ and X is countable.

To show that $C_p(X) \approx U(f,\varepsilon)$, first consider the translation $\xi : \mathbb{R}^X \to \mathbb{R}^X$ defined by

$$\xi(g) = g - f \quad (g \in \mathbb{R}^X).$$

This function maps $U(f,\varepsilon)$ onto $U(\underline{0},\varepsilon)$, where $\underline{0}$ is the constant function with value 0. So it suffices to show that this set and $C_p(X)$ are homeomorphic. But by observing that $(-\varepsilon,\varepsilon)$ and $\mathbb{R}$ are homeomorphic, this follows by the remark at the beginning of this section.

It remains to prove (3). It suffices to prove that $\bigcup_{0<\delta<\varepsilon} U(\underline{0},\delta)$ is dense in $V = U(\underline{0},\varepsilon)$. So let $g \in V$, $F \subseteq X$ be finite, and $\gamma > 0$. It suffices to show that $N(g,F,\gamma) \cap \bigcup_{0<\delta<\varepsilon} U(f,\delta) \neq \emptyset$. But this is trivial. Simply observe that $\max|g[F]| = t < \varepsilon$ and the function $h : X \to \mathbb{R}$ defined by $h(x) = \min\{g(x),t\} + \max\{g(x),-t\} - g(x)$ is continuous and belongs to $N(g,F,\gamma) \cap U(\underline{0}, {}^1\!/_2 t + {}^1\!/_2 \varepsilon)$. □

A separable metrizable space X is *complete* if its topology is generated by a complete metric. It is well-known that a subspace Y of a complete space X is complete if and only if Y is a G_δ-subset of X ([6, Theorem 4.7.4]).

If X is countable then $\mathbb{R}^X$ is complete. So for a subspace $A \subseteq \mathbb{R}^X$ the statements 'A is complete' and 'A is a G_δ-subset of $\mathbb{R}^X$' are equivalent.

THEOREM 3.2. *If $C_p(X)$ is a $G_{\delta\sigma}$-subset of $\mathbb{R}^X$ then X is discrete.*

Proof. As observed in the previous section, we may assume without loss of generality that X is countable. So let X be countable and nondiscrete such that $C_p(X) = \bigcup_{i=0}^{\infty} G_i$, where G_i is complete for every i and $G_0 = \emptyset$. We will derive a contradiction.

CLAIM 1. *$C_p(X)$ does not have a dense complete subspace.*

This is [2, Theorem 1]. We repeat the proof for completeness sake. There is a function $g \in \mathbb{R}^X \setminus C_p(X)$. Define $\Phi : \mathbb{R}^X \to \mathbb{R}^X$ by $\Phi(f) = f + g$. Then Φ is a homeomorphism of $\mathbb{R}^X$ onto itself and $\Phi[C_p(X)]$ is a dense subspace of $\mathbb{R}^X \setminus C_p(X)$. Thus $\mathbb{R}^X \setminus C_p(X)$ also contains a dense complete subspace, which violates the Baire Category Theorem since $\mathbb{R}^X$ is complete. This proves the claim.

To complete the proof of the $G_{\delta\sigma}$-theorem, we shall by induction on i define a sequence $\{f_i : i \geqslant 0\}$ in $C_p(X)$ and a decreasing sequence of positive real numbers $\{\varepsilon_i : i \geqslant 0\}$ such that

(1) f_0 is the constant function with value 0, and $\varepsilon_0 = 2^0$,
(2) $U(f_0, 1/4\varepsilon_0) \supseteq U(f_1, \varepsilon_1) \supseteq U(f_1, 1/4\varepsilon_1) \supseteq U(f_2, \varepsilon_2) \supseteq \ldots$,
(3) $\varepsilon_i \leqslant 2^{-i}$ for every i,
(4) $\varepsilon_{i+1} < 3^{-i-1} \cdot \varepsilon_i$ for every i,
(5) $U(f_i, \varepsilon_i) \cap G_i = \emptyset$ for every i.

Assume that we have proved the existence of such sequences. Then for every i we have by (2) and (3) that

$$\sup_{x \in X} |f_i(x) - f_{i+1}(x)| \leqslant 1/4\varepsilon_i \leqslant 2^{-i-2}.$$

So we conclude that the sequence $(f_i)_i$ converges uniformly to a continuous function $f : X \to \mathbb{R}$. It follows moreover from (4) that for $x \in X$, $i \geqslant 0$ and $n \geqslant 2$ we have

$$\begin{aligned} |f_i(x) - f_{n+i}(x)| &\leqslant |f_i(x) - f_{i+1}(x)| \\ &\qquad + |f_{i+1}(x) - f_{i+2}(x)| + \ldots + |f_{n+i-1}(x) - f_{n+i}(x)| \\ &\leqslant 1/4\varepsilon_i + \sum_{j=1}^{n-1} 3^{-j} \cdot \varepsilon_i. \end{aligned}$$

This implies that

$$\sup_{x \in X} |f_i(x) - f(x)| \leqslant 1/4\varepsilon_i + \sum_{j=1}^{\infty} 3^{-j} \cdot \varepsilon_i = 3/4\varepsilon_i < \varepsilon_i$$

for every i, i.e. $f \in \bigcap_{i=0}^{\infty} U(f_i, \varepsilon_i) \subseteq C_p(X) \setminus \bigcup_{i=1}^{\infty} G_i = \emptyset$, which is a contradiction.

It remains to perform the induction. Assume that f_i and ε_i have been determined. By Lemma 3.1 it follows that $U(f_i, 1/4\varepsilon_i)$ is a G_δ-subset of $C_p(X)$ which in addition is homeomorphic to $C_p(X)$. This implies by Claim 1 that $U(f_i, 1/4\varepsilon_i)$ does not contain a dense complete subspace. Observe that $G_{i+1} \cap U(f_i, 1/4\varepsilon)$ is a G_δ-subset of G_{i+1} and hence is complete. So $G_{i+1} \cap U(f_i, 1/4\varepsilon)$ is not dense in $U(f_i, 1/4\varepsilon)$. So there are by Lemma 3.1(3) an element $f_{i+1} \in U(f_i, \delta)$ for some $0 < \delta < 1/4\varepsilon_i$, a finite set $F \subseteq X$ and a $\gamma > 0$ such that

$$N(f_{i+1}, F, \gamma) \cap (G_{i+1} \cap U(f_i, 1/4\varepsilon_i)) = \emptyset.$$

Now let ε_{i+1} be a positive real number smaller than

$$\min\{\gamma, {}^1\!/_4\varepsilon_i - \delta, 2^{-(i+1)}, 3^{-i-1} \cdot \varepsilon_i\}.$$

It is clear that f_{i+1} and ε_{i+1} satisfy the inductive hypotheses. □

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