BY THEIR FRUITS YE SHALL KNOW THEM: SOME REMARKS ON THE INTERACTION OF GENERAL TOPOLOGY WITH OTHER AREAS OF MATHEMATICS

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1. Introduction

In his letter of invitation to contribute to this *"Handbook of the History of Topology"*, Professor James asked us to discuss the role of general topology in other areas of topology. So this paper is *not* a paper on the history of general topology, it is a paper on the history of its *interactions* with other fields of mathematics. Of the many possibilities, we decided to report on the one hand on the genesis of general topology and on the other hand on infinite-dimensional topology and set theoretic topology.[1] For a much more comprehensive desciption of (parts of) the history of general topology, we refer the reader to [15].

The primary goal in general topology, also sometimes called point set topology, is the investigation and comparison of different classes of topological spaces. This primary goal continues to yield interesting problems and results, which derive their significance from their relevance with respect to this primary goal and from the need of applications. In the history of general topology we distinguish three periods. The first period is *the prehistory* of the subject. It led to the work of Hausdorff, Brouwer, Urysohn, Menger and Alexandroff. The prehistory resulted in a definition of general topology, but it left many questions unanswered. The second period, roughly from the 1920s until the 1960s was *general topology's golden age.* Many fundamental theorems were proved. Many of the results from that period can be viewed as a necessary consequence of the genesis of the subject. However, much work from the golden age was also an investment in the future, an investment that started to yield fruit in the third period lasting from the 1960s until the present. That is why we will call this period *the period of harvesting.*

In this paper we concentrate on the first and the last period: the prehistory and on the period of harvesting. In §2, which deals with the prehistory, we describe in particular the historical background of the concept of an abstract topological space. We discuss the contributions of Georg Cantor, Maurice Fréchet and Felix Hausdorff. That

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[1] These two research areas are familiar to us. Other possibilities to report on could have been: Topological Dynamics, Theoretical Computer Science, Topological Groups, Topological Games, Categorical Topology, Dimension Theory, Topological Algebra, Descriptive Set Theory, etc.

discussion is rather informal; it reflects the informal style of that particular period in history. Because a complete description of that work is out of the question, we concentrate in particular on the background and the genesis of some crucial topological notions. Although we repeatedly went back to the original texts, we also relied heavily on secondary sources. We would like to mention Purkert and Ilgauds [141] and Dauben [50] with respect to Cantor, Taylor [166] for information on Fréchet, and Scholz [152] concerning Hausdorff. We also used Moore [132] and Monna [131] with appreciation.[2]

Sections 4 and 5 of the paper are devoted to the period of harvesting. In that period general topology rather unexpectedly succeeded in solving several difficult problems outside its own area of research, in functional analysis and in geometric and algebraic topology. Also here a survey of all significant results is impossible. There were in that period at least two major developments in general topology that revolutionized the field: the creations of *infinite-dimensional topology* and *set theoretic topology*[3]. It was mainly due to the efforts of Dick Anderson and Mary Ellen Rudin that these fields have played such a dominant role in general topology ever since.

There is a well-known pattern that occurs often in mathematics. An established part of mathematics generates non-trivial questions and possible ways to answer these questions that are new, but of little immediate significance. Research in the area is essentially pursued for its own sake. However, if the mathematics is good, after a longer or shorter period, the theories involved significantly contribute to solve external problems. Hilbert [85] wrote:

> "The final test of every new mathematical theory is its success in answering pre-existent questions that the theory was not designed to answer. By their fruits ye shall know them — that applies also to theories".[4]

And indeed, there is no doubt that the most convincing test for the value of a theory is its external significance.[5] We will show that the genesis and further development of general topology offer many examples that illustrate this pattern. We believe, frankly, that research in general topology is almost exclusively driven by two things: the existence of difficult, challenging problems, and the beauty of many of the results. Of course, not everything that was and is done in general topology is equally important, as is the case in any other field of mathematics. It is, for example, relatively easy to define variations of the axiomatic bases of the various types of spaces and, as in other fields, it is not always

[2]Manheim wrote [118] the first book on the history of general topology and certainly at the time it was a useful contribution. He restricted himself to what we call the prehistory of the field.

[3]Also shape theory was created by Borsuk, see e.g. [31], but this field was much more motivated from algebraic and geometric topology than infinite-dimensional and set theoretic topology.

[4]"Wherefore by their fruits ye shall know them", St. Matthew 7:20.

[5]Hallett [79]: It may take a long time before the external significance of a theory becoms clear. For example, when the Greeks were pursuing mathematics entirely for its own sake, independent of applications, they developed an elaborate theory of conic sections. Only many centuries later Kepler applied this theory to describe the orbits of the planets. External significance is a sufficient condition for quality, it proves the value of a theory afterwards. Obviously, a theory may generate and solve such interesting problems that even without definite proof through external significance, the theory should be considered valuable (Koetsier [103, p. 171]).

easy to say in advance whether certain lines of research are worth pursueing. Yet, most of the areas of research in general topology represent good mathematics. The two examples of infinite-dimensional topology and set theoretic topology illustrate this.

2. The Prehistory of General Topology

2.1. **Developments in 19th century analysis.**

2.1.1. *Weierstrassian analysis.* Cauchy played a major role in the *first revolution of rigour*, that had turned eighteenth century calculus from a collection of formal methods to solve problems, into a coherent deductive theory based upon definitions of the fundamental concepts of convergence, continuity, the derivative and the integral in terms of the notion of limit (Grabiner [78]). Yet, after Cauchy, a further development and refinement of concepts was inevitable. Cauchy primarily used his new conceptual apparatus to give a solid foundation of existing analysis and in his mathematics a function is still always associated with a formula. In the second half of the nineteenth century the conceptual apparatus itself became the object of investigation. This happened in combination with a much more general concept of function: a function became, in principle, a completely arbitrary correspondence between numbers. In particular the discovery that discontinuous functions can be expressed by means of Fourier series – dating from the beginning of the 19th century – contributed considerably to this change. For example, in 1854, in his "Habilitationsschrift", Riemann studied the problem of the representation by means of Fourier series of as large a class of arbitrary functions as possible. This automatically led to the problem of the integrability of highly discontinuous functions. Riemann discovered that a function could possess an infinite number of points of discontinuity in any interval and still be integrable (in the sense of Cauchy-Riemann). It became clear that such highly discontinuous functions could be studied and research partially shifted from the investigation of functions defined by a particular formula or classes of formulas to the investigation on a much more general level: abstracting from particular examples that illustrate those relations, the relations themselves between notions like real number, function, series, convergence, limit, continuity, differentiability, integrability became subject of investigation.

From this perspective Cauchy's work showed weaknesses and a *second revolution of rigour* took place in analysis, that is associated with the name of Weierstrass. It became clear that Cauchy had not sufficiently distinguished between, for example, uniform convergence and non-uniform convergence. It also became clear that he had, essentially, taken the real numbers and their properties, for example their completeness, for granted. A proof of a theorem like "A real function that is continuous in a closed and bounded interval attains its maximum value", which we owe to Weierstrass, would have been out of place in Cauchy's work[6] and the same holds for more fundamental theorems like the Bolzano-Weierstrass Theorem, actually due to Weierstrass alone: "Every infinite

[6] Cauchy's well-known proof of the intermediate value theorem is in the context of Cauchy's work rather exceptional. But also in this case the first completely satisfactory proof was given by Heine [83].

bounded subset of $\mathbb{R}^n$ has a limit point".[7] This theorem was stated for $n = 2$ by WEIERSTRASS in a course of lectures in 1865. In 1874 he gave a general proof (MOORE [132, p. 17]). The theorem is necessary to prove the existence of limits, something that CAUCHY had also, at heart, still taken for granted.

2.1.2. *Volterra, Ascoli.* As early as 1883[8] VOLTERRA had the idea to create a theory of functionals[9], or real-valued "functions of lines", as he called the field. VOLTERRA wrote several papers on the subject.[10] The lines are all real-valued functions defined on some interval $[a, b]$. These functions are viewed as elements of a set for which notions like neighbourhood and limit of a sequence can be defined. VOLTERRA gave definitions for the continuity and the derivative of a function of lines and he tried to build up a line-function theory analogous to RIEMANN's theory of complex functions. These attempts were not motivated by their immediate significance in solving problems in the calculus of variations. HADAMARD wrote about VOLTERRA's motivation:

> "Why was the great Italian geometer led to operate on functions as the infinitesimal calculus operated on numbers [...]? Only because he realised that this was a harmonious way to complete the architecture of the mathematical building".[11]

WEIERSTRASS' teaching was influential also in Italy. In 1884 GIULIO ASCOLI (1843-1896) extended the Bolzano-Weierstrass Theorem to sets of functions as follows. He studied a set $\mathcal{F}$ of uniformly bounded functions on $[a, b]$. In order to prove that a sequence of functions $\{f_n\}$ in $\mathcal{F}$ possesses a convergent subsequence $\{g_n\}$, he needed the assumption that the set $\mathcal{F}$ is equicontinuous. The result is known as Ascoli's Theorem. Equicontinuity of $\mathcal{F}$ means then that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that for all $f \in F$ and for all $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. The proof-idea is that a subsequence $\{g'_n\}$ of $\{f_n\}$ is chosen, such that $\{g'_n(a)\}$ converges. Then a subsequence $\{g''_n\}$ is chosen from $\{g'_n\}$ such that $\{g''_n(b)\}$ converges. Then a subsequence $\{g'''_n\}$ is chosen from $\{g''_n\}$ such that $\{g'''_n((a + b)/2)\}$ converges. Continuing in this way a sequence of converging sequences is generated that correspond to the elements of a set that is dense in $[a, b]$. The "diagonal sequence" then does the job (MOORE [132, p. 81]).

2.1.3. *The Dirichlet-principle and the theorem of Ascoli- Arzelà.* The Italian attempts to extend results from WEIERSTRASS' real analysis to sets of functions and real functions defined on such sets, can certainly be understood as "a harmonious way to complete the architecture of the mathematical building". Yet there were also other reasons. An example is Dirichlet's principle. In 1856–1857 DIRICHLET lectured on potential theory in Göttingen. Modelling conductors, he considered a part Ω of $\mathbb{R}^3$, bounded by a surface S on which a

[7]As far as we know the notion of limit point or accumulation point was first used by WEIERSTRASS.

[8]According to WHITTAKER, see MONNA [131, p. 108].

[9]The term functional was introduced by HADAMARD in 1903 (MONNA [131, p. 108]).

[10]Atti della Reale Accademia dei Lincei, (4), 3, 1887, 97-105, 141-46, 153-58 = Opere matematiche, 1, 294-314, and other papers of the same and later years. We have not seen these papers.

[11]Quoted by SIEGMUND-SCHULTZE [159, p. 377].

continuous function is defined and dealt with the problem of the existence of a function u on Ω that equals f on S and satisfies $\triangle u(x,y,z) = 0$. In order to solve the problem he considered the integral

$$U = \int_\Omega \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] dV,$$

on Ω, which obviously is non-negative for all functions u considered. He concluded that there must be at least one function u on Ω for which the integral reaches a minimum value. One can show that the minimizing function satisfies $\triangle u(x,y,z) = 0$ and DIRICHLET thought he had solved the problem (MONNA [131, pp. 27-30]). In 1871 HEINE criticised DIRICHLET for accepting without proof the existence of a minimizing function (MONNA [131, p. 41]). And indeed this method, which is sometimes called *Dirichlet's principle*, needs further justification, because the existence of a greater lower bound for the values of the integral does not necessarily imply that there exists a function that correponds to that greater lower bound. In a paper from 1889 by CESARE ARZELÀ [14] the author refers to VOLTERRA and his “functions that are dependent on lines” (“funzioni dipendenti dalle linee”) and writes that continuity for such functions had been defined but that the existence of maxima and minima still needed investigation. Expressing the hope that his work will lead to a justification of the “Principio di Riemann-Dirichlet” he proceeded to prove what is nowadays usually called the Theorem of Ascoli-Arzelà. First ARZELÀ generalized ASCOLI's theorem from 1884 and proved that an equicontinuous set $\mathcal{F}$ of uniformly bounded functions on $[a,b]$ has a limit-function. By definition a limit- function f of $\mathcal{F}$ is a function that has the property that for every $\varepsilon > 0$, there are infinitely many functions g in $\mathcal{F}$ for which, for all x,

$$f(x) - \varepsilon < g(x) < f(x) + \varepsilon.$$

Then ARZELÀ turned to continuous real-valued functionals defined on such an equicontinuous set $\mathcal{F}$ of functions – something which ASCOLI had not done – and showed that, if the set $\mathcal{F}$ is closed, i.e. contains all its limit- functions, the lower bound of the set of values of the functional, the upper bound and all values in between are taken.

In 1896 ARZELÀ published a paper in which he applied his results to the Dirichlet principle. He succeeded to prove it only under certain extra conditions (MONNA [131, p. 112]). Nowadays the fundamental Ascoli-Arzelà Theorem in analysis is phrased in terms of compactness, a term introduced by FRÉCHET in 1904. However, in order to understand the background of the ideas of FRECHET, it is necessary to describe the birth of transfinite set theory first.

2.2. **Cantor.**

2.2.1. *From Fourier-series to derived sets and transfinite counting.* GEORG CANTOR studied in Berlin under KUMMER, KRONECKER and WEIERSTRASS. In 1869 he became a Privatdozent at the University of Halle. His doctoral thesis and his Habilitationsschrift were on number theory, but soon CANTOR turned to analysis. EDWARD HEINE, one of his colleagues at the University of Halle, had suggested him to study the problem of the uniqueness of the representation of a function by means of a trigonometric series. In the years 1870 through 1872 CANTOR published a series of papers on that matter. In 1870 he published a proof of the theorem saying that if

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \sin nx + b_n \cos nx)$$

converges to $f(x)$ for all x on $(0, 2\pi)$, $f(x)$ cannot be represented by another trigonometric series converging to $f(x)$ for all x on $(0, 2\pi)$. In 1871 he improved the proof and, moreover, showed that the representation remained unique if the requirement of convergence or the convergence to $f(x)$ would be dropped for a finite set of exceptional points. Soon CANTOR's main interest moved from trigonometric series to exceptional sets consisting of infinitely many points. It was this problem that led to the theory of transfinite sets and at the same time to a number of topological results. CANTOR realized that the proof of the theorem could be easily modified to hold if the infinite exceptional set contained a finite number of limit points and that it could even be proved if it contained an infinite number of limit points, provided the set of limit points itself possessed at most a finite number of limit points. The argument was easily extended to higher levels of sets of limit points of sets of limit points. However, how could one describe such complex subsets of the continuum? This question led CANTOR in fact to a definition of the real number system, which was independent of those of WEIERSTRASS, MÉRAY, HEINE and DEDEKIND that were also given in that period. CANTOR [36] started with the set of rational numbers, which he called A. He considered the set of all Cauchy sequences of rational numbers (as they are now called - CANTOR himself called them fundamental sequences) and defined what we would now call an equivalence relation on that set. The set of equivalence classes is called B. The ordering and the elementary operations are then extended from A to the union of A and B. CANTOR now repeats the construction: In precisely the same way by considering Cauchy sequences in $A \cup B$ a set C is generated, then sequences in $A \cup B \cup C$ generate a set D, etc. In this way after λ steps a set L is reached whose elements CANTOR called "numbers of type λ". CANTOR was aware of the fact that he could identify A with a subset of B and he also knew that B, C, D, etc. are isomorphic (although he does not use that terminology), but he avoided the identification. He needed the hierarchy of number sets to identify point sets on a line. In order to do that he first introduced the notion "derived set" of a point set on a line. The first derived set P^1 of a point set P is by definition the set of all limit points of P and recursively: the derived set P^λ of a set P is the first derived set of $P^{\lambda-1}$. CANTOR then called P a set of type ν, iff the ν-th derivative P^ν is finite. The existence of such sets can now be seen by using the above defined hierarchy of number sets, because if we take one point on the line whose coordinate is a number of type ν, we know that this number represents a Cauchy sequence of numbers of type $\nu - 1$, while those numbers all represent Cauchy sequences of numbers of type $\nu - 2$, etc. If in this way, we go all the way back to the rational numbers, we wind up on the line with a point set of type ν.

Applying this new apparatus CANTOR proved the uniqueness theorem for trigonometric series for exceptional sets of type ν where ν is an arbitrary natural number. In the same paper he wrote with respect to the hierarchy of number systems defined by means of Cauchy sequences:

> "[...] the notion of number, in so far as it is developed here, carries within it the germ of a necessary and absolutely infinite extension".[12]

Although he does not mention it in his paper he had at the time already extended that hierarchy beyond the finite levels. And indeed, the question whether there exist sets that are such that P^{ν} is infinite for all finite ν, arises naturally. We know that already in 1870 CANTOR was aware of the possibility to count beyond the finite (PURKERT and ILGAUDS [141, p. 39]).The idea of the transfinite ordinal numbers, was born in this context.

2.2.2. *The birth of the transfinite cardinals.* In 1872 and 1873 the nature of the continuum intrigued CANTOR more and more. In a letter to DEDEKIND, dated November 29, 1873, he wrote that he had tried to find a one-one correspondence between the natural numbers and the real numbers, but that he had failed. Several days later the transfinite cardinal numbers were born; they still would have to go a long way, but the idea was there. On December 7 of the same year CANTOR wrote to DEDEKIND that he had found the proof that the proposed one-to-one correspondence does not exist. CANTOR, who would later use the diagonal method, gave the following simple proof. Let a_1, a_2, a_3, a_4, etc. be the sequence of all real numbers. Consider an interval $[p, q]$. Find in the sequence the first two real numbers that represent an interval $[p_1, q_1]$ inside $[p, q]$. Find then the first two real numbers that represent an interval $[p_2, q_2]$ inside $[p_1, q_1]$, etc. This inevitably leads to a nested sequence of intervals with a non-empty intersection of points that do not occur in the sequence a_1, a_2, a_3, a_4, etc. CANTOR published the proof in 1874 (CANTOR [38, pp. 115–118]) pointing out that the proof implied the existence of transcendental numbers. It is remarkable that at the time, CANTOR and DEDEKIND both considered these results as interesting but not of great importance (PURKERT and ILGAUDS [141, p. 45]). The next problem CANTOR turned to was the question whether a two-dimensional continuum could be mapped one-to-one on the real numbers. In 1877 he found the answer: the unit square, yes, even the n-dimensional unit cube can be mapped one-to-one on the interval $[0, 1]$. The paper was published in 1878. CANTOR immediately realised that the result created a problem for the traditional view that the number of dimensions of a continuum corresponded to the number of parameters needed to describe it. Here we have the beginning of dimension theory. A survey of its further history was given by JOHNSON [94, 95]. See also KOETSIER and VAN MILL [104].

2.2.3. *Transfinite set theory and topological notions.* Those first results from the period 1872–1878 gave CANTOR the idea that the problem of the nature of the different kinds of point sets could be approached systematically. That is what he did in a famous series of six publications under the title "About infinite linear pointmanifolds" ("Über unendliche

[12] "[...] der Zahlenbegriff, soweit er hier entwickelt ist, den Keim zu einer in sich notwendigen und absolut unendlichen Erweiterung in sich trägt" (CANTOR [38, p. 95]).

lineare Punktmannigfaltigkeiten"), that appeared in the years 1879 through 1884. The papers wonderfully show how CANTOR's theory gradually developed; they also show the emergence of several topological notions and results. Right from the start set theory and general topological notions have been intimately connected. A complete discussion of the six publications goes beyond the purpose of this paper. We will mention a few results.

In the first paper CANTOR distinguishes point sets of the first kind – the n-th derivative is empty for a finite n – and points sets of the second kind – by definition those that are not of the first kind. The notion of "density in an interval" is introduced and it is shown that sets of the first kind are never dense in an interval. CANTOR also shows that all sets of the first kind and also some but not all of the second kind are countable. In the second paper CANTOR introduces the sequence: $P^{\infty}, P^{\infty+1}, P^{\infty+2}, \ldots$ etc., where P^{ν} refers to the ν-th derivative of a set P. The fourth paper contains a number of topological results. He calls a set P of $\mathbb{R}^n$ "isolated" if it contains none of its limit-points. He proves: "Every isolated set in $\mathbb{R}^n$ is at most countable" and some related results.

In the fifth paper the transfinite ordinal numbers, viewed as well-ordered sets, are "constructed", and denoted in the now standard way: $\omega, \omega+1, \ldots, \omega \cdot \omega, \ldots$ etc. The ordinal numbers are related to the cardinal numbers by means of the notion of number class. The theory developed in this way generated two fundamental problems: the need to prove that every set can be well-ordered (this would guarantee that all cardinal numbers could be reached by means of ordinal numbers) and the continuum hypothesis. The continuum hypothesis is in the last sentence of the fifth paper. In this text from 1878 CANTOR writes that his investigations point at the conclusion that among the infinite "linear manifolds", i. e. the subsets of $\mathbb{R}$, there would occur only two cardinal numbers. He added: "we postpone a precise investigation to a later occasion" (Cantor [38, p. 133]).

In this fifth paper CANTOR also discusses the question when a subset of $\mathbb{R}^n$ should be called a "continuum". In order to answer that question he defines the notions of a perfect point set and a connected point set. A *perfect* point set is by definition equal to its derivative. A set T is by definition connected if for any two points t and t' of T and for any $\varepsilon > 0$, there exists a finite number of points $t_1, t_2, \ldots, t_n$ of T in such a way that all distances $tt_1, t_1t_2, t_2t_3, \ldots, t_{n-1}t_n, t_nt'$ are all smaller than ε. A subset of $\mathbb{R}^n$ then is defined as a *continuum* iff it is a perfect, connected set (CANTOR [38, p. 194]). In a note CANTOR gave the famous example of a set that is perfect and at the same time dense in no interval:

$$\left\{\frac{c_1}{3} + \frac{c_2}{3^2} + \frac{c_3}{3^3} + \cdots : c_i \in \{0, 2\} \text{ for every } i\right\}.$$

The sixth paper contains a number of topological results on point sets in $\mathbb{R}^n$, which CANTOR undoubtedly obtained while working towards a proof of the continuum hypothesis. Two examples are: "A perfect set is not countable", and "If a set is not countable, it can be split into a perfect set and a countable set".

2.2.4. *The reception of set theory.* After a period in which he hardly wrote anything CANTOR published in 1895 and 1897 in two parts his last important paper on set theory: "Contributions to the foundation of transfinite set theory" ("Beiträge zur Begründung der transfiniten Mengenlehre"). In this paper, primarily devoted to "general" set theory[13], CANTOR's theory got its final form. At the same time the appreciation for set theory among mathematicians was growing slowly. Right from the start CANTOR's set theory had been met by sceptical reactions. In particular KRONECKER was very critical. However, significant external applications made many prominent mathematicians understand the value of set theory. In 1872 HEINE, CANTOR's colleague in Halle, had proved that a real-valued function that is continuous on an interval $[a, b]$ of $\mathbb{R}$ is uniformly continuous. The proof runs roughly as follows. Using the continuity, HEINE first aims at constructing for all small $\varepsilon > 0$ a monotonously increasing sequence $\{x_i\}$ with $a = x_1$ for which

$$|f(x_{i+1}) - f(x_i)| = 3\varepsilon.$$

and for all x with $x_i \leq x \leq x_{i+1} \leq b$, one has

$$|f(x) - f(x_i)| \leq 3\varepsilon.$$

If the sequence cannot be constructed (because the function varies less than 3ε) or the construction stops after a finite number of steps because in the remaining interval the function varies less than 3ε, we are done. If the sequence is infinite, it converges to a number X in the interval. Then there exists also an η for which for all x with $X - \eta \leq x \leq X$ we have

$$|f(x) - f(X)| \leq 2\varepsilon.$$

This, however, contradicts the fact that in the interval $[X - \eta, X]$ there are infinitely many points of the sequence $\{x_i\}$ for which $|f(x_{i+1}) - f(x_i)| = 3\varepsilon$.

In his 1894 doctoral thesis EMILE BOREL (1871–1956) applied Cantorian set theory to problems of analytic continuation in the theory of functions of a complex variable. It was undoubtedly this work which put him on the road to his later contributions to measure theory. However, one of his proofs involved "a theorem interesting in itself [...]: If one has an infinity of sub-intervals on a line (that is a closed interval) such that every point of the line is interior to at least one of them, a finite number of intervals can effectively be determined having the same property" (Quoted and translated by HAWKINS; GRATTAN-GUINNESS [65, p. 175]). We have here the Heine-Borel Covering Theorem. BOREL's proof uses CANTOR's transfinite ordinals. He considers a transfinite sequence $\{(a_\lambda, b_\lambda) : \lambda < \alpha\}$ of open intervals that covers the interval $[a, b]$ from the left to the right and then by transfinite induction on α proves that the collection can be reduced to a finite collection. HALLETT [79] discussed this proof an argued that, although the Heine-Borel Theorem was soon proved without the use of transfinite numbers, BOREL's proof still counts as one of the first applications of transfinite ordinal numbers ouside of set theory. Soon other applications followed. HURWITZ gave an invited lecture at an international congress of mathematicians in Zürich in 1897 on the development of the general theory of analytical functions in which he summarized CANTOR's theory of transfinite ordinal numbers and subsequently applied it to classify analytical functions on the basis of their sets of singular points. Set theoretical methods had arrived in a classical discipline like the theory of

[13] ZERMELO wrote: "viele Hauptsatze der 'allgemeinen' Mengenlehre finden erst hier ihre klassische Begründung" (CANTOR [38, p. 351]).

complex functions. And both the problem of the Continuum Hypothesis and the Well-Ordering Theorem occur in HILBERT's famous list of problems he considered in 1900 to be the most important for mathematical research in the twentieth century.

2.3. **Maurice Fréchet's "Analyse Générale".**

2.3.1. *Tables, chairs, and beer mugs: another revolution of rigour.* In the history of analysis one distinguishes often the "first revolution of rigour", brought about by CAUCHY and the "second revolution of rigour", brought about by WEIERSTRASS. The systematic introduction of the axiomatic method in mathematics (in combination with the language of set theory and first-order predicate logic) could, undoubtedly, also be characterized as a revolution of rigour. There is a famous story told by CONSTANCE REID in her biography of HILBERT:

> "In his docent days HILBERT had attended a lecture in Halle by HERMANN WIENER on the foundations and structure of geometry. In the railway station in Berlin on his way back to Königsberg, under influence of WIENER's abstract point of view in dealing with geometric entities, he had remarked thoughtfully to his companions: 'One must be able to say at all times – instead of points, straight lines and planes – tables, chairs, and beer mugs' (REID [142, p. 57])."

In an appendix to the book WEYL writes that according to BLUMENTHAL it must have been 1891 and WIENER's paper was on the role of DESARGUES's and PAPPUS's theorems (REID [142, p. 264]). HILBERT's remark contains in a nutshell an important aspect of the abstract, axiomatic point of view: the theory becomes independent of its intended model; whatever names are used for the undefined terms, the axioms completely determine the way in which these terms are related. In HILBERT's "Foundations of Geometry" ("Grundlagen der Geometrie") of 1899 this point of view is applied to geometry. For many mathematicians HILBERT's book represented the future; after more than 2000 years Euclid had been dethroned. Consequently, in the first decade of the twentieth century the axiomatic method was very much in the air. In 1904 ZERMELO published a proof of the Well-Ordering Theorem (MOORE [132, p. 159]). The proof contains the first explicit statement of what was later called the Axiom of Choice. The reactions to the paper were such that ZERMELO found it necessary to secure the proof even further. The result was ZERMELO's axiomatization of set theory (MOORE [132, p. 157]).[14]

The axiomatic method is, on the one hand, a method by means of which an already existing theory can be given its final form. However, the axiomatic method is also a powerful research method. Its basic rule is: "The occurrence of analogy between different areas points at the existence of a more general structure that should be defined explicitly by means of a suitable set of axioms". In France, BOREL used the axiomatic method, LEBESGUE did and also FRÉCHET, who applied it on a problem suggested to him by HADAMARD, whose student he was. At the first International Congress of Mathematicians

[14]So ZERMELO's primary motivation was not the occurrence of the antinomies. By the way, the so-called Russell's paradox was also found by ZERMELO, several years before RUSSELL did so (MOORE [132, p. 89]).

in 1897 HADAMARD lectured briefly on possible future applications of set theory. He remarked that it would be worthwhile to study sets composed of functions. Such sets might have properties different from sets of numbers or points in space. He said:

> "But it is primarily in the theory of partial differential equations of mathematical physics that research of this kind will play, without any doubt, a fundamental role"[15]

and one of the examples that he implicitly referred to was DIRICHLET's principle.

2.3.2. *The genesis of Fréchet's thesis.* In 1904 and 1905 FRÉCHET published a series of short papers on "abstract sets" or "abstract classes" in the "Comptes Rendus", that layed the groundwork of his thesis: "Sur quelques points du calcul fonctionnel", Rendiconti del Circolo Matematica di Palermo, 1906, pp 1-74. That thesis is one of FRÉCHET's most important contributions to mathematics. We will concentrate on the early papers in order to get an idea of the genesis of the thesis. In his first paper [70] the analogy between WEIERSTRASS' theorem: "A real function continuous in a closed and bounded interval attains its maximum value" and the Dirichlet principle is given as the motivation to develop a general theory of continuous real functions (FRÉCHET said "opérations fonctionnelles") on arbitrary sets that encompasses both theorems. FRÉCHET did not mention the Italians and it is possible that he only heard about their work in 1905. In his first paper FRÉCHET introduced an abstract axiomatic theory of limits. The theory refers to a set or class C of arbitrary elements and concerns infinite sequences of elements $A_1, A_2, A_3, \dots$ of C that may or may not possess a limit element B in C. FRÉCHET's axioms are

(i) If a sequence has a limit B, then all infinite subsequences have the same limit, and
(ii) If $A_i = A$ for all i, then the limit of the sequence equals A.

In terms of this axiomatically defined notion of limit FRÉCHET can then define the notions of a closed subset of C, a compact subset of C and of a continuous real function on a subset of C.

(iii) A subset E of C is by definition *closed* if every limit element of a sequence of elements of E belongs to E.
(iv) A subset E of C is *compact* if for all sequences E_n consisting of non-empty closed subsets of C, that are such that E_{i+1} is a subset of E_i for all i, the intersection of all the E_n's is non-empty.
(v) The continuity of a function F on C is also defined in terms of sequences: F is *continuous* on a subset E of C if for all sequences $\{A_i\}$ in E that have a limit B in E, the sequence $\{F(E_i)\}$ has the limit $F(B)$.

FRÉCHET then, without further proof, phrases the generalisation of WEIERSTRASS' theorem as follows: "If E is a closed and compact set in C and U is a continuous functional operation on C, then the values of U are bounded and U assumes an absolute maximum value at some point A of E." In his next note [73] in the "Comptes Rendus" FRÉCHET

[15] "Mais c'est principalement dans la théorie des équations aux dérivées partielles de la physique mathématique que les études de cette espèce joueraient, sans nul doute, un role fondamental" (quoted by TAYLOR [166, p. 259]).

answers in the negative the question whether the derived set of a set E is necessarily closed. The counterexample that he gives consists of all real polynomials in the set of all real functions on an interval; a function f is the limit of a sequence of polynomials if there is pointwise convergence. This was a problem for FRÉCHET, because he felt that in order to get interesting generalisations of existing theorems he would need the property that the derivative of a set is always closed. In the third note [72] the new ideas are applied to the space E^* of infinitely many dimensions, the elements of which are all real sequences $\{a_i\}$. Sequence A is the limit of a sequence $\{A_i\}$ of sequences iff for all p the sequence of pth coordinates of the A_i converges to the pth coordinate of A[16]. A set of points A in E^* is bounded iff there are fixed numbers M_i such that for all points in the set for all i the absolute value of the $i$the coordinate is smaller than M_i. FRÉCHET defines a condensation point ("point de condensation") of a set A as a limit-point that remains a limit-point of the set if one removes in an arbitrary way a countable infinite number of points from the set. FRÉCHET then states, without actually giving proofs, that he succeeded in proving several theorems. Three examples are: "The necessary and sufficient condition for a subset of E^* to be compact is that it is bounded", "The derived set of a subset of E^* is closed" and "Every uncountable and bounded subset of E^* possesses at least one condensation point".

The desire to develop a general theory in which the derived set of a set E is necessarily closed, continued to bother FRÉCHET. There exists an interesting letter (probably from 1904) concerning this point from HADAMARD to FRÉCHET in which HADAMARD suggests the use of an abstract notion of nearness or neighbourhood ("voisinage") (quoted by TAYLOR [166, pp. 245-246]). HADAMARD wrote:

> "Would it be good if you started, in general, from the notion of neighbourhood and not from that of limit?"[17]

In [71] FRÉCHET had decided to introduce a generalised notion of "voisinage", assuming that in the classes of arbitrary elements to each couple of arbitrary elements there corresponds a real number (A, B) for which

(i) $(A, B) \geq 0$,
(ii) $(A, B) = 0$ iff $A = B$,
(iii) If (A, C) and (B, C) are infinitely small, then so (A, B).

[The last requirement means: if (A, C) and (B, C) are sequences converging to zero, then so (A, B).] FRÉCHET defines the notion of limit in terms of this abstract notion of distance: $\{A_n\}$ converges to A iff (A_n, A) converges to 0. Without proof FRÉCHET states that always when the limit can be defined by means of a suitable "écart": I) Every derived set is closed, II) A functional operation that is continuous on a compact set is uniformly continuous. FRÉCHET also refers to ASCOLI and ARZELÀ, remarking that these theorems and the ones in his earlier notes can be seen as generalisations of the Italian work.

[16] So FRÉCHET considers the space that we now call s, see §4.

[17] "Feriez vous bien de partir, en général, de la notion de voisinage et non de celle de limite? [...]" (TAYLOR [166, p. 246]).

2.3.3. *Fréchet's 1906 thesis.* Fréchet's 1906 thesis is based on the papers from the period 1904-1905. We will only discuss the thesis very briefly.[18] In the thesis an abstract class with sequential limits that satisfy the two requirements from his first 1904 note is called "une classe (L)". We shall call them *L(imit)-classes*. In the second chapter of the first part of the thesis FRÉCHET introduces an abstract notion of distance, which he calls "voisinage". An abstract set is "une classe (V)", or, as we will say, a *V(oisinage)- class* if there exists a real-valued binary function (A, B) on the set which satisfies:

(i) $(A, B) = (B, A) \geq 0$,
(ii) $(A, B) = 0$ iff $A = B$,
(iii) There exists a positive real function $f(\varepsilon)$, defined for positive ε, for which
$$\lim_{\varepsilon \to 0} f(\varepsilon) = 0,$$
such that, whenever $(A, B) \leq \varepsilon$ and $(B, C) \leq \varepsilon$, then $(A, C) \leq f(\varepsilon)$.

A V-class can be turned into an L-class by means of the definition: $\{A_n\}$ converges to A iff (A_n, A) converges to 0.

It is remarkable that at one point in the thesis FRÉCHET replaces the third V-class axiom by a triangle inequality:

(iii^a) for all A, B, C we have $(A, C) \leq (A, B) + (B, C)$.

Here is what we nowadays call a metric space (following HAUSDORFF [80, p. 211]). It is called a"classe (E)" by FRÉCHET, because here he uses the term "écart" instead of "voisinage". For such écart-classes or E-classes FRÉCHET proves the theorem: If a subset G of an E-class is such that every continuous functional operation on G is bounded on G and attains on G its least upper bound, then G is closed and compact. The E-class was actually introduced because FRÉCHET could not prove this theorem for V- classes.[19] The thesis also contains a generalization of the Heine-Borel Covering Theorem: If E is a closed and compact subset of a V-class then every countable covering $\mathcal{M}$ of E contains a finite number of sets that also cover E. In the second part of his thesis FRÉCHET applies the abstract theory to concrete examples.

2.4. **Hausdorff's definition of a Hausdorff space.**

2.4.1. *Hilbert.* While FRÉCHET was developing a theory of abstract spaces, others were doing similar things. Before FRÉCHET started working on abstract spaces, HILBERT in 1902 briefly wrote about the possibility to characterize the notion of manifold in an abstract way, while in 1906 in Hungary, independent of FRÉCHET, FRIGYES RIESZ (1880–1956) also attempted to give as general as possible a characterization of the notion of space. In HILBERT's proposal the notion of neighbourhood is central. Hilbert wrote:

> "The plane[20] is a system of things that are called points. Every point A determines certain sub-systems of points to which the point itself belongs

[18] For a more extensive discussion we refer to TAYLOR [166]).

[19] In 1908 HAHN succeeded in doing so and in 1917 CHITTENDEN [44] turned it into a metrisation theorem.

[20] As will be clear later, HILBERT uses the notion of plane in a generalised sense.

> and that are called neighbourhoods of the point A. The points of a neighbourhood can always be mapped by means of a one-to-one correspondence on the points of a certain Jordan-area in the number plane. The Jordan-area is called the image of that neighbourhood. Every Jordan-area, that contains (the image of) A, and is contained in an image, is also image of a neighbourhood of A. If different images of a neighbourhood are given, then the resulting mapping of the two corresponding Jordan- areas on each other is continuous. If B is any point in a neighbourhoofd of A, this neighbourhood is also a neighbourhood of B. To any two neighbourhoods of A always corresponds such neighbourhood of A, that the two neighbourhoods have in common. When A and B are any two points of the plane, there exists always a neighbourhood of A that contains at the same time B. These requirements contain, it seems to me, for the case of two dimensions, the sharp definition of the notion that RIEMANN and HELMHOLTZ denoted as "multiply extended manifold" and Lie as "number manifold", and on which they based their entire investigations. They also offer the foundation for a rigourous axiomatic treatment of the analysis situs."[21]

The quotation, which contains everything that Hilbert wrote about the subject, dates from 1902, that is from before FRÉCHET started his topological work. HILBERT never continued the line of research that the quotation suggested. He left the further "rigourous axiomatic treatment of the analysis situs" to others. The axioms define an abstract notion of space and the basic concept is the concept of neighbourhood. Some of the axioms only concern the set theoretic properties of neighbourhoods. The other properties of the neighbourhoods are, however, fixed by means of axioms concerning the (continuous) one-one correspondences that are postulated to exist between neighbourhoods and Jordan-areas in the number-plane.

2.4.2. *Riesz.* RIESZ' approach to the problem and also his motivation are quite different. In his 1907 paper (a German translation of a Hungarian paper that appeared in 1906) RIESZ

[21]"Die Ebene ist ein System von Dingen, welche Punkte heißen. Jeder Punkt A bestimmt gewisse Teilsysteme von Punkten, zu denen er selbst gehört und welche Umgebungen des Punktes A heißen. Die Punkte einer Umgebung lassen sich stets umkehrbar eindeutig auf die Punkte eines gewissen Jordanschen Gebietes in der Zahlenebene abbilden. Das Jordansche Gebiet wird ein Bild jener Umgebung genannt. Jedes in einem Bilde enthaltene Jordansche Gebiet, innerhalb dessen der Punkt A liegt, ist wiederum ein Bild einer Umgebung von A. Liegen verschiedenen Bilde einer Umgebung vor, so ist die dadurch vermittelte umkehrbar eindeutige Transformation der betreffenden Jordanschen Gebiete aufeinander eine stetige. Ist B irgendein Punkt in einer Umgebung von A, so ist diese Umgebung auch zugleich eine Umgebung von B. Zu irgend zwei Umgebungen eines Punktes A gibt es stets eine solche Umgebung des Punktes A, die beiden Umgebungen gemeinsam ist. Wenn A und B irgend zwei Punkte unserer Ebene sind, so gibt es stets eine Umgebung von A, die zugleich den Punkt B enthält. Diese Forderungen enthalten, wie mir scheint, für den Fall zweier Dimensionen die scharfe definition des Begriffes, den RIEMANN und HELMHOLTZ als "mehrfach ausgedehnte Mannigfaltigkeit" und LIE als "Zahlenmannigfaltigkeit" bezeichneten und ihren gesamten Untersuchungen zugrunde legten. Auch bieten sie die Grundlage für eine strenge axiomatische Behandlung der Analysis situs." (HILBERT [84, pp. 165-166]).

distinguishes our subjective experience of time and space from the mathematical continua by means of which we describe them. His goal is to give as general a characterisation as possible of mathematical continua and to show the precise relation between our subjective experience and mathematical continua. In a footnote Riesz criticises the way in which philosophers have dealt with notions like continuous and discrete and he repeats Russell's remark about the followers of Hegel: "the Hegelian dictum (that everything discrete is also continuous and vice versa) has been tamely repeated by all his followers. But as to what they meant by continuity and discreteness, they preserved a discrete and continuous silence; [...]" (Riesz [143]). The relation of our subjective experience of space and time and mathematical continua is described by Riesz as follows. Mathematical continua possess certain properties of continuity, coherence and condensation. On the other hand, our subjective experience of time is discrete and consists of countable sequences of moments. Systems of subsets of a mathematical continuum can be interpreted as a physical continuum when subsets with common elements are interpreted as undistinguishable and subsets without common elements as distinguishable. Riesz [143, p. 111] is an interesting paper in which Riesz, who had read Fréchet's work and appreciated it, developed a different theory of abstract spaces, based on the notion of "Verdichtungsstelle", i.e. "condensation point" or, as we will translate "limit point". In his theory Riesz succeeded in deriving the Bolzano-Weierstrass Theorem and the Heine-Borel Theorem. We will not discuss this paper. We will restrict ourselves to a shorter paper that was presented by Riesz in 1908 at the International Congress of Mathematicians in Rome. In that paper, "Stetigkeit und Abstrakte Mengenlehre", (Riesz [144]) concentrates on the characterisation of mathematical continua. We will briefly describe some of the ideas that Riesz describes in the paper. As we said, Riesz' basic notion is the notion of limit point (Verdichtungsstelle). Riesz did consider Frechet's restriction to limit points of countable sequences as too severe. That is why in his theory limit points satisfy the following three axioms:

(i) Each element that is a limit point of a subset M is also a limit point of every set containing M.
(ii) When a subset is divided into two subsets, each limit point is a limit point of at least one of the subsets.
(iii) A subset consisting of only one element does not have a limit point.

A mathematical continuum is for Riesz, by definition, any set for which a notion of limit point is defined that satisfies these three axioms. However, in order to be able to develop some theory on the basis of the axioms Riesz is forced to add a fourth axiom:

(iv) Every limit point of a set is uniquely determined through the totality of its subsets for which it is a limit point.

Riesz uses the examples of $\mathbb{R} \setminus \{0\}$ and $\mathbb{R} \setminus [0, 1]$, that exhibit as far as their limit points are concerned precisely the same structure, to show that the four axioms are not enough to characterize properties of "continuity". That is why in his paper Riesz suggests to add the notion of "linkage" (Verkettung). For any pair of subsets of a manifold it should be defined whether they are linked or not. Such a linkage structure must satisfy the following three axioms:

(i) If subsets S_1 and S_2 are linked, then every pair of sets that contain S_1 and S_2 are also linked.
(ii) If subsets S_1 and S_2 are linked and S_1 is split into two subsets, at least one of the two is linked to S_2.
(iii) Two sets that each contain only one element cannot be linked.

Although he believed that the notions of limit point and linkage could be used to develop an abstract theory of sets in the sense of HADAMARD's proposal of 1897, RIESZ himself did not continue this work. In later publications FRÉCHET used some of RIESZ' ideas.

2.4.3. *Hausdorff.* The different attempts to give an abstract definition of space culminated in the work of FELIX HAUSDORFF (1868–1942). In 1912 HAUSDORFF, professor at the university of Bonn, taught a class on set theory. Chapter 6 of his notes[22] deals with "Point sets" ("Punktmengen") and is called "Neighbourhoods" ("Umgebungen"). HAUSDORFF writes:

> "Point sets on a straight line (linear), in the plane (planar), in space (spatial), in general in an n-dimensional space $r = r_n$. A *point* is defined by a system of n real numbers $(x_1, x_2, \ldots, x_n)$ and vice versa, that we think as orthogonal coordinates. As *distance* of two points we define
>
> $$x \cdot y = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \geq 0.$$
>
> The *neighbourhood* U_x of a point x is the collection of all points y for which $x \cdot y < \rho$ (ρ a positive number; the inner area of a "Sphere" with radius ρ).
>
> For the sake of illustration we will usually take the plane $r = r_2$; if the individual numbers of dimensions cause deviations, we will especially emphasize them. The neighbourhoods have the following properties:
> (α) Every U_x contains x and is contained in r.
> (β) For two neighbourhoods of the same point $U'_x \supseteq U_x$ or $U_x \supseteq U'_x$ holds.
> (γ) If y lies in U_x, then there also exists a neighbourhood U_y, that is contained in U_x ($U_x \supseteq U_y$).
> (δ) If $x \neq y$, then there exist two neighbourhoods U_x, U_y without a common point: $(\theta(U_x, U_y) = 0)$.
>
> The following considerations are based initially only on these properties. They hold very generally, if r is a point set $\{x\}$, if to the points x correspond point sets U_x with these 4 properties. Such a system is, for example, the following: one defines as a neighbourhood of x the system of points where
>
> $$|x_1 - y_1| < \rho, \quad y_2 = x_2;$$
>
> a neighbourhood is then a horizontal segment (without endpoints) of length 2ρ. Or: one defines as a neighbourhood the system
>
> $$|x_1 - y_1| < \rho, \quad |x_2 - y_2| < \rho$$

[22] HAUSDORFF, manuscript 1912b, par. 6, Archive Bonn University.

i.e., the inner area of a square with side-length 2ρ, whose centre is x, etc."[23]

According to SCHOLZ [152], HAUSDORFF was led to the four axioms in the spring or the summer of 1912 by a logical analysis of the foundations of complex analysis. In this context it is remarkable that at the same time, WEYL was applying HILBERT's ideas from 1902 in his work his on Riemann surfaces (WEYL [184]). SCHOLZ argues that both were independently influenced by HILBERT. In 1914 HAUSDORFF's "Grundzüge der Mengenlehre" appeared, one of the first textbooks on set theory. Above we saw how set theory was born from point set theory in $\mathbb{R}^n$ and that CANTOR's first papers show a mixture of point set theory and more abstract considerations. Hausdorff carefully distinguishes general set theory from point set theory. The first seven chapters of his book are devoted to general set theory. It is remarkable that he did not include ZERMELO's axiomatization. In the first chapter, after mentioning the antinomies, he writes why not:

> "E. ZERMELO undertook the subsequently necessary attempt to limit the borderless process of set-creation by suitable restrictions. Because so far these extremely shrewd investigations can not yet claim to be finished and an introduction of the beginner in set theory in this way would be connected with great difficulties, we will permit here the naive notion of set, at the same time, however, we will in fact stick to the restrictions that cut off the road to that paradox."[24]

In chapter 7 of his book, HAUSDORFF addresses the question of the position of point set theory within the system of general set theory. Point set theory here means abstract point set theory. He briefly discusses three possible approaches to turn a set that is so far treated

[23]"Punktmengen auf einer Geraden (linear), in der Ebene (ebene), im Raume (räumliche), allgemein in einem n-dimensionalen Raume $r = r_n$. Ein Punkt x ist durch ein System von n reellen Zahlen $(x_1, x_2, \ldots, x_n)$ und umgekehrt definiert, die wir als rechtwinklige Coordinaten denken. Als *Entfernung* zweier Punkte definieren wir $x \cdot y = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \geq 0$. Unter einer Umgebung U_x des Punktes x verstehen wir den Inbegriff aller Punkte y für die $x \cdot y < \rho$ (ρ eine positive Zahl; Inner[e]s einer "Kugel" mit Radius ρ). Wir werden zur Veranschaulichung in der Regel die Ebene $r = r_2$ nehmen; sollten die Einzelnen Dimensionenzahlen Abweichungen hervorrufen, so werden die besonders hervorgehoben werden. Die Umgebungen haben folgende Eigenschaften: (α) Jedes U_x enthält x und ist in r enthalten. (β) Für zwei Umgebungen desselben Punktes ist $U'_x \supseteq U_x$ oder $U_x \supseteq U'_x$. (γ) Liegt y in U_x, so giebt es auch eine Umgebung U_y, die in U_x enthalten ist ($U_x \supseteq U_y$). (δ) Ist $x \neq y$, so giebt es zwei Umgebungen U_x, U_y ohne gemeinsamen Punkt ($\theta(U_x, U_y) = 0$). Die folgenden Betrachtungen stützen sich zunächst nur auf diese Eigenschaften. Sie gelten sehr allgemein, wenn r eine Punktmenge $\{x\}$ ist, wenn Punkten x Punktmengen U_x zugeordnet sind mit diesen 4 Eigenschaften. Ein solches System ist z. B. folgendes: man definiere als ein Umgebung von x das System der Punkte, wo $|x_1 - y_1| < \rho$, $y_2 = x_2$; eine Umgebung ist dann eine horizontale Strecke (ohne Randpunkte) von der Länge 2ρ. Oder: als Umgebung werde das System $|x_1 - y_1| < \rho, |x_2 - y_2| < \rho$ definiert, d. h. das Inner eines Quadrates von der Seitenlänge 2ρ, dessen Mittelpunkt x ist, u.s.w."

[24]"Den hiernach notwendigen Versuch, den Prozeß der Uferlosen Mengenbildung durch geeignete Forderungen einzuschränken, hat E. ZERMELO unternommen. Da indessen diese äußerst scharfsinnigen Untersuchungen noch nicht als abgeschlossen gelten können und da eine Einführung des Anfangers in die Mengenlehre auf diesem Wege mit großen Schwierigkeiten verbunden sein dürfte, so wollen wir hier den naiven Mengenbegriff zulassen, dabei aber tatsächlich die Beschränkungen innehalten, die den Weg zu jenem Paradoxon abschneiden." (HAUSDORFF [80, p. 2]).

purely as a system of its elements without considering relations between the elements, into a space. His goal is obviously to define a very general notion of space that encompasses not only the $\mathbb{R}^n$, but also Riemann surfaces, spaces of infinitely many dimensions and spaces the elements of which are curves or functions (HAUSDORFF [80, p. 211]). He gives two advantages of such a general notion: it simplifies theories considerably and it prevents us from illegitemately using intuition (die Anschauung). The first possibility is to base point set theory on the notion of the distance (Entfernung) of two elements, that is a function that associates with each pair of elements of a set a particular value. HAUSFDORFF remarks that on the basis of the notion of distance the notion of a converging sequence of points and its limit can be defined. Moreover, on the basis of the notion of distance, one can also associate with each point of a set subsets of the space called neigbourhoods of the point.

However, one can also turn a set into a space by circumventing the notion of distance and starting from a function $f(a_1, a_2, a_3, \ldots, a_k, \ldots)$ which maps certain sequences of elements (the converging sequences) of the set M on elements of M (the limits of the sequences). Thirdly, one can also start with the notion of neighbourhood. Formally one then maps every element of a set M on certain subsets of M that are called the neighbourhoods of the element. Which of the three "spatial" notions one chooses as the most fundamental is for HAUSDORFF to a certain extent a matter of taste (HAUSDORFF [80, p. 211]). Neigbourhoods and limits can be defined in terms of distances. By means of neighbourhoods one can define limits, but in general no distances. By means of limits one can define neither neighbourhoods nor distances. Hausdorff writes; "Thus the distance theory seems to be the most special and the limit theory the most general; on the other hand the limit theory creates immediately a relation with the countable (with sequences of elements), which the neighbourhood theory avoids." (HAUSDORFF [80, p. 211]).

As a good teacher he now first gives an example. He defines metric spaces by means of the well-known three axioms. $\mathbb{R}^n$ with the Euclidean distance is an example of a metric space. HAUSDORFF concentrates on the four properties of the spherical neighbourhoods that he had already given in his 1912 lectures. He writes:

> "A topological space is a set E in which the elements (points) x are mapped on certain subsets U_x, that we call neighbourhoods of x, in accordance with the following neighbourhood axioms [...]."[25]

He then gives the four axioms that occur already in his 1912 lectures and he shows that the spherical neighbourhoods in $\mathbb{R}^n$ satisfy the axioms.

HAUSDORFF's generalization of the notion of space represented a major contribution to the unification of mathematics. Geometry and analysis had been separate disciplines. Axiomatization ended that. HAUSDORF succeeded in picking a set of axioms that was, on the one hand, general enough to handle abstract spaces and, on the other hand, restrictive enough to yield an interesting theory. He succeeded in giving a theory of topological and

[25]Unter einem topologischen Raum verstehen wir eine Menge E, worin den elementen (Punkten) x gewisse Teilmengen U_x zugeordnet sind, die wir Umgebungen von x nennen, und zwar nach Maßgabe der folgenden Umgebungsaxiome [...]" (HAUSDORFF [80, p. 213]).

metric spaces that encompassed the previous results and generated many new notions and theorems.

2.5. **L. E. J. Brouwer.** Above we sketched the genesis of the notion of topological space as it was finally defined by HAUSDORFF. His book was very influential. For years it was an important source for many mathematicians. Yet our story, which is so far restricted to the genesis of the notion of topological space, is very one-sided. In order to do some more justice to the actual development, the contributions of BROUWER must be mentioned. BROUWER's approach to general topology is totally different from HAUSDORFF's. Also their views of mathematics were completely different. HAUSDORFF was a great supporter of the axiomatic method. BROUWER rejected the axiomatic method and argued that mathematics ought to be founded in intuition.[26]

In the first decade of this century ARTHUR SCHOENFLIES had attempted to give a thorough set-theoretic foundation of topology.[27] In SCHOENFLIES' work a central result is JORDAN's Theorem: a closed *Jordan curve*, i.e. the one-to-one continuous image of a circle, divides the plane into two domains with the image as their common boundary. A *domain* is an open connected set. At certain points SCHOENFLIES work is quite subtle. For example, he distinguishes between simple closed curves and closed curves that are not simple by means of the notion of accessibility. By definition a point P on the boundary of a domain is *accessible* if it can be reached from an arbitrary point in the domain by a finite or an infinite polygonal path in the domain. A *closed curve* is here by definition a bounded closed point set that divides the plane into two domains with the curve as their common boundary (SCHOENFLIES [151, pp. 118–120]). Closed curves that are such that all their points are accessible from the two domains are called *simple* by SCHOENFLIES. An important result that he proved is the following: simple closed curves are closed Jordan curves. In his early work BROUWER relied on SCHOENFLIES' results. However, in the winter of 1908–1909 he discovered suddenly that SCHOENFLIES' results were not reliable. In [33], entitled "Zur Analysis Situs", he gave a series of devastating counterexamples. BROUWER does not criticise SCHOENFLIES' theory of simple closed curves, but attacks his more general theory of closed curves. In the paper he gave the sensational example of a closed curve that splits the plane into three domains of which it is the common boundary. It is also the first example of an indecomposable continuum. SCHOENFLIESS' general theory of closed curves and domains had to be rejected entirely. Soon BROUWER produced several other highly original papers. We will mention only two: ("Beweis der Invarianz der Dimensionenzahl", submitted in June 1910 and published in 1911 (BROUWER [34]) and his paper [35]. The first paper marks, according to FREUDENTHAL, the onset of a new period in topology. Although the paper is very short and merely contains a simple proof of the invariance of dimension, "it is in fact much more than this – the paradigm of an entirely new and highly promising method, now known as *algebraic topology.* It exhibits the ideas of simplicial mapping, barycentric extension, simplicial approximation, small modification, and, implicitly, the mapping degree and its invariance under homotopic change, and the

[26]See also KOETSIER and VAN MILL [104].

[27]For a fuller treatment we refer to JOHNSON [94, 95].

concept of homotopy class." (FREUDENTHAL [64, p. 436])[28]. In the second paper Brouwer proved the basic theorem on fixed points: every continuous transformation of the n-simplex into itself possesses at least one fixed point. Although BROUWER's results were reached from a totally different philosophical position his results could be easily incorporated in and considerably enriched the axiomatic framework created by HAUSDORFF. This led to much further work.

2.6. **Functional analysis.** BROUWER's work shows how problems in point set topology led to algebraic topology. We will, however, use the example of BROUWER's fixed point theorems to illustrate another way in which results from general topology penetrated other areas of mathematics.

Between HILBERT's, FRÉCHET's and RIESZ' first attempts and the publication of HAUSDORFF's book the number of "sets with a spatial character", deviating from Euclidean space, had grown, and with it the potential value of abstract characterisations of the notion of space. For example, between 1904 and 1910 HILBERT had published his six famous "communications" on the foundations of the theory of integral equations. The space ℓ^2 had gradually become the object of investigation. The proof of the isomorphism of ℓ^2 and L^2, the space of quadratic Lebesgue-integrable functions, led to the notion of Hilbert space. In 1910 RIESZ introduced the normed linear function space L^p. That work meant also the start of modern operator theory.[29]

In his Lwów dissertation of 1920 BANACH introduced the notion of a "Banach space" (the name is FRÉCHET's). In the 1920s and 1930s the Polish school carefully applied set-theoretic methods to functional analysis and proved fundamental theorems like the Hahn-Banach Theorem and the Banach Fixed-Point Theorem. With BANACH's "Théorie des opérations linéaires" of 1932 functional analysis was established as one of the central fields in modern analysis. Banach's student J. P. SCHAUDER (Studia Mathematica 2, 1930, pp. 170–179) and SCHAUDER and J. LERAY (Ann. de l' École Normale Superieure 51, 1934, pp. 45–78) carried over BROUWER's topological notions into infinite-dimensional spaces and generalized his fixed point theorem in order to establish the existence of solutions of non-linear differential equations. This work was of great importance for the development of non-linear functional analysis in the 1950s.

3. INTERMEZZO: THE GOLDEN AGE

The next phase in the history of general topology, its golden age, lasted roughly from the 1920s until the 1960s. Among the main themes in this period were dimension theory, paracompactness, compactifications and continuous selections. The important results and the way in which they are related can all be found in a number of classical textbooks (see below). About this phase we will be very brief.

[28]For a more extensive treatment of BROUWER's work in dimension theory we refer to JOHNSON [94, 95]. See also KOETSIER and VAN MILL [104]. For BROUWER's topological work as a whole we refer to FREUDENTHAL [64].

[29]For a more extensive survey of the history of functional analysis at the beginning of the century we refer to SIEGMUND-SCHULTZE [159].

Dimension theory was fully developed (see HUREWICZ and WALLMAN [90] for a beautiful survey of dimension theory until 1941). In the late 1940's and early 1950s paracompactness, introduced by DIEUDONNÉ [54], was the leading theme in general topology. STONE [164] proved that metrizable spaces are paracompact and NAGATA, SMIRNOV and BING published/proved their metrization theorems in [136], [160] and [28] respectively. The work on compactifications in the 1950's culminated in the publication of the beautiful book [76] by GILLMAN and JERISON. MICHAEL [123, 124, 125] developed his theory of continuous selections. For more information, see e.g. HU [88], DUGUNDJI [61], NAGATA [138, 137], ENGELKING [68, 67] and ARHANGEL'SKIĬ and PONOMAREV [13].

We will turn now to the third period that we distinguish in the history of general topology: the period of harvesting. We will concentrate in the sections 4 and 5 on two major areas of research that developed out of the golden age, infinite-dimensional topology and set theoretic topology, and show how these solved difficult problems outside of general topology.

The style of §2 was rather informal, in keeping with the pioneering works of the area. In the sections 4 and 5 we attempt to describe some complex results from the front line of mathematics. In order to do so we will use the much more compressed, conceptual style of modern mathematics.

4. The Period of Harvesting: Infinite-dimensional topology

4.1. **The beginning.** As usual, let a *separable Hilbert space* be the set

$$\ell^2 = \left\{ x \in \mathbb{R}^\infty : \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$$

endowed with the norm

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}. \tag{4.1}$$

The metric derived from (4.1) is complete and hence ℓ^2 is a complete linear space.[30]

By using convexity type arguments, KLEE [102] proved that $\ell^2 \setminus \{\text{pt}\}$ and ℓ^2 are homeomorphic. We say that points can be *deleted* from ℓ^2. In fact, he proved in that even arbitrary compact sets can be deleted from any infinite-dimensional normed linear space. This result demonstrates a striking difference between finite-dimensional and infinite-dimensional normed linear spaces. For a finite-dimensional linear space is equivalent to some $\mathbb{R}^n$ and no point can be deleted from $\mathbb{R}^n$, since $\mathbb{R}^n \setminus \{\text{pt}\}$ is not contractible.

KLEE's results were later substantially simplified by the approach of BESSAGA [22, 23] who proved, among other things, that if an infinite-dimensional linear space admits a C^k-differentiable norm (except at 0) which is not complete, then the deleting homeomorphisms can in fact be chosen to be diffeomorphisms of class C^k.

[30] A *linear space* in this article is a real topological vector space.

Motivated by the results of Klee, Anderson [9] studied in 1967 the question which sets can be deleted from another classical linear space, namely the countable infinite topological product of real lines $\mathbb{R}^\infty$ (see §2.3.2). This space is denoted by s. Its topology is generated by the following complete metric:

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}. \tag{4.2}$$

So s is a locally convex complete metrizable linear space. Such a space is called a *Fréchet space* in the literature. Unfortunately, s has an unpleasant defect: its topology is not *normable* in the sense that there is no norm $\|\cdot\|$ on it so that the metrics

$$\rho(x,y) = \|x - y\|$$

and d in (4.2) are equivalent. This is clear once one realizes that every neighbourhood of the origin contains a nontrivial (linear) subspace of $\mathbb{R}^\infty$.

The linear structure on s is therefore very different from the linear structure on a normed linear space, and so the methods of Klee and Bessaga do not apply if one wishes to prove results on the possibility of deleting sets. But by using a completely different method, Anderson [9] showed that from s one can delete sets as easily as from ℓ^2. In fact, he got the following remarkable result[31]:

Theorem 4.1.1. *Let X be any separable metrizable space. Then every σ- compact set can be deleted from $X \times s$.*

A new field in topology was born: it was called *infinite-dimensional topology.*

Anderson was motivated by purely intrinsic topological questions. Soon however it turned out quite unexpectedly that his methods could be used to solve a classical open problem, posed by Fréchet [74, pp. 94–96] in 1928. In 1932 in [19, p. 233], Banach stated that Mazur had solved the problem, but this claim turned out to be incorrect. Subsequently it was understood that the question was still open.

To put the question into perspective, let us first make a few remarks. The spaces s and ℓ^2 are both natural generalizations of the finite-dimensional Euclidean spaces $\mathbb{R}^n$, but their linear structures are notably different. There does not exist a homeomorphism $h\colon s \to \ell^2$ which is *linear*, i.e. has the property that

$$h(\lambda x + \mu y) = \lambda h(x) + \mu h(y)$$

for all $x, y \in s$, and $\lambda, \mu \in \mathbb{R}$. The question therefore naturally arises whether s and ℓ^2 are (topologically) homeomorphic at all. The question of Fréchet and Banach is much more eleborate, it asks whether all infinite-dimensional Fréchet spaces are homeomorphic.

The question had a long history when Anderson considered it in 1966. By several ad hoc methods, homeomorphy of many linear spaces had already been established. The first relevant result is due to Mazur [121] who proved in 1929 that all spaces L^p and ℓ^p for $1 \le p < \infty$ are homeomorphic to ℓ^2. Then Kadec in a series of papers developed an interesting “renorming technique” for separable Banach spaces and finally proved in 1965

[31]As usual, a space is σ-compact if it can be written as a union of countably many compact subspaces.

that all infinite-dimensional separable Banach spaces are homeomorphic (see KADEC [97]). KADEC's proof used the result of BESSAGA and PEŁCZYŃSKI [25] that a separable Banach space containing a linear subspace homeomorphic to ℓ^2 is in fact itself homeomorphic to ℓ^2. This result combined with another result of BESSAGA and PEŁCZYŃSKI [26] showed that the homeomorphy of s and ℓ^2 would imply the positive answer to Fréchet's question, i.e. the homeomorphy of all separable infinite-dimensional Fréchet spaces. The proofs of these interesting results combine techniques from functional analysis, especially the geometry of Banach spaces, with various ingeneous arguments from general topology.

This final, but crucial, open problem was solved in the affirmative by ANDERSON [7] using the results from his previous paper [9]. He proved that s and ℓ^2 are homeomorphic and hence settled the question of Fréchet and Banach in the affirmative.

4.2. **The Hilbert cube Q.** Let Q denote the product $\prod_{n=1}^{\infty}[-1,1]_n$ of countably many copies of $[-1,1]$. The topology on Q is the Tychonoff product topology. Alternatively, its topology is generated by the metric

$$d(x,y)=\sum_{n=1}^{\infty} 2^{-n}\cdot|x_n-y_n|.$$

So Q is a compact metrizable space. Geometrically one should think of it as an infinite-dimensional brick the sides of which get shorter and shorter. This can be demonstrated in the following way. Let $x(n)\in Q$ be the point having all coordinates 0 except for the n-th coordinate which equals 1. So $x(n)$ is the "endpoint" of the n-th axis in Q. In addition, let y be the "origin" of Q, i.e., the point all coordinates of which are 0. Intuitively, each $x(n)$ has distance 1 from y and hence $x(n)$ and y are far apart. However, the appearance of the factor 2^{-n} in the definition of d implies that

$$d(x(n),y)=2^{-n},$$

whence the sequence $(x(n))_n$ converges to y in Q.

It can be shown that Q is homeomorphic to the subspace

$$\{x\in\ell^2:(\forall n\in\mathbb{N})(|x_n|\le\tfrac{1}{n}\}$$

of ℓ^2.

The first paper in infinite-dimensional topology is in fact KELLER's paper [100] from 1931. In that paper it is shown that all infinite-dimensional compact convex subsets of ℓ^2 are homeomorphic to Q, and also that Q is *topologically homogeneous*, i.e. for all $x,y\in Q$ there exists a homeomorphism $f\colon Q\to Q$ with $f(x)=y$. This last result is at first glance very surprising since the finite-dimensional analogues $\mathbb{I}^n$ of Q are not homogeneous. For $n=1$ this is a triviality, and for larger n this boils down to the Brouwer Invariance of Domain Theorem.

A familiar construction in topology is that of the *cone* over a locally compact space X, it is the one-point compactification of the product $X \times [0,1)$. The compactifying point is called the *cone point* of the cone which itself is denoted $\text{cone}(X)$.[32]

Now, it is clear that for each n the cone over $\mathbb{I}^n$ is homeomorphic to $\mathbb{I}^{n+1}$ and so naively one would expect, by taking the "limit" as n goes to infinity, that $\text{cone}(Q) \approx Q$. That this is indeed true follows from KELLER's first result because we can realize $\text{cone}(Q)$ as a compact convex subset of ℓ^2.

Since Q is contractible, the cone point in $\text{cone}(Q)$ has arbitrarily small neighbourhoods with contractible boundaries. This is not surprising since every point on the boundary of $\mathbb{I}^n$ has the same property. However, points in the interior of $\mathbb{I}^n$ do not have this property. But since Q is homogeneous, *every* point of Q has arbitrarily small neighborhoods with contractible boundaries. This is again a striking difference with the finite-dimensional situation.

At the time KELLER made his fundamental observations, they apparently did not get the credit they deserved for they did not play any significant role for approximately thirty-five years. Maybe, but this is speculation on the part of the authors of the present paper, in the thirties KELLER's results were thought of as mere curiosities. Infinite-dimensional topology took approximately thirty-five more years to finally come to real existence. In that process, the work of ANDERSON was vital.

4.3. **Homeomorphism extension results in Q-manifolds.** In [9], ANDERSON also proved results on the possibility of extending homeomorphisms in Q. It was known already that if X is any countable closed subset of Q then any homeomorphism $f\colon X \to X$ can be extended to a homeomorphism of Q (see KELLER [100], KLEE [101] and FORT [69]). In the subsequent paper [8], ANDERSON introduced the fundamental concept of a Z-set in Q and proved that any homeomorphism between such sets can be extended to a homeomorphism of Q.

Before we present the definition of a Z-set, we make some remarks. Let K denote the familiar CANTOR middle-third set in $\mathbb{I}$. It is known that it is characterized by the following topological properties: K is a compact, metrizable, zero-dimensional space[33] without isolated points. So it follows easily that $K \times K \approx K$ from which it follows that K contains a nowhere dense closed copy of itself, say X. It also contains a copy of itself having nonempty interior, namely K itself. Consider any homeomorphism $\varphi\colon X \to K$. Then it cannot be extended to a homeomorphism $\bar{\varphi}\colon K \to K$ for obvious reasons. One of them being that K is fat, having nonempty interior, and X is small, having empty interior.

A space homeomorphic to K is called a CANTOR *set*.

Similar remarks apply to other spaces as well. It is known that any homeomorphism φ between CANTOR sets in $\mathbb{R}^2$ can be extended to a homeomorphism of $\mathbb{R}^2$. However, Antoine's necklace X is a CANTOR set in $\mathbb{R}^3$ whose complement is not simply connected and so no homeomorphism $\varphi\colon X \to Y$, where Y is a CANTOR subset of the x-axis of $\mathbb{R}^3$,

[32]There are other constructions of cones that work for general spaces. One then considers the product $X \times \mathbb{I}$ and identifies the set $X \times \{1\}$ to a single point. But this cone is in general not metrizable.

[33]Here a space is called zero-dimensional if it has a base consisting of open and closed sets.

can be extended to a homeomorphism of $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$. This phenomenon occurs in Q as well: in [187] WONG constructed a wild Cantor set in Q.

So for a homeomorphism extension theorem, one needs a class of tame subspaces which are flexible enough to perform the required constructions. For Q this class was identified by ANDERSON. He called a closed subset A of Q a *Z- set*[34] provided that for every nonempty homotopically trivial open subset $U \subseteq Q$ the set $U \setminus A$ is nonempty and homotopically trivial as well. He proved in [8] the following fundamental homeomorphism extension theorem:

Theorem 4.3.1. *If $\varphi\colon A \to B$ is a homeomorphism between Z-sets in Q then there exists a homeomorphism $\bar{\varphi}\colon Q \to Q$ extending φ.*

In the proof important ideas of KLEE [101] were exploited.

Later, BARIT [20] observed that if the homeomorphism φ satisfies $d(\varphi, \mathrm{id}) < \varepsilon$ for some $\varepsilon > 0$ then the extension $\bar{\varphi}$ can be chosen to satisfy the same smallness condition.

The final result on the possibility of extending homeomorphisms in manifolds modeled on Q is due to ANDERSON and CHAPMAN [11]. Let X be a space and let $f\colon X \to X$ be a function. If $\mathcal{U}$ is an open cover of X then we say that f is *limited by* $\mathcal{U}$ provided that for every $x \in X$ there exists $U \in \mathcal{U}$ containing both x and $f(x)$. Here is the ANDERSON-CHAPMAN Homeomorphism Extension Theorem from 1971:

Theorem 4.3.2. *Let M be a manifold modeled on Q and let $A, B \subseteq M$ be Z-sets. If $\varphi\colon A \to B$ is a homeomorphism and $\mathcal{U}$ is an open cover of M such that φ is limited by it, then there exists a homeomorphism $\bar{\varphi}\colon M \to M$ extending φ which is also limited by $\mathcal{U}$.*

This is a purely topological result belonging to general topology and at the time ANDERSON and CHAPMAN proved it, they could not have foreseen what potential this theorem turned out to have. We will report on this later.

4.4. **Identifying Hilbert cubes.** In 1964, ANDERSON [6] proved that the Q is homeomorphic to any countably infinite product of dendrons.[35] In particular, one gets the curious result that if T denotes

$$(\mathbb{I} \times \{0\}) \cup (\{\tfrac{1}{2}\} \times \mathbb{I})$$

then $T \times Q$ and Q are homeomorphic. For a published proof of ANDERSON's result, see WEST [181]. So the Hilbert cube surfaces at various places, not only as convex objects such as in KELLER's Theorem cited above. The result started the game of identifying Hilbert cubes. It was a very fascinating game. The tools were from general topology with special emphasis on geometric methods.

The hyperspace 2^X of a compact space X is the space consisting of all nonempty closed subsets of X with topology generated by the Hausdorff metric d_H defined by

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq D_\varepsilon(B) \text{ and } B \subseteq D_\varepsilon(A)\};$$

[34] One of the authors of the present paper once asked ANDERSON why he chose this terminology. He replied that he had no idea.

[35] A dendron is a uniquely arcwise connected Peano continuum.

here $D_\varepsilon(A)$ means the open ball about A with radius ε. Hyperspaces were first considered in the early 1900's in the work of Hausdorff and Vietoris. In 1939 Wojdysławski [186] asked whether for every Peano continuum X the hyperspace 2^X is homeomorphic to Q. At the time of the conjecture this was a rather bold question because the only nontrivial Hilbert cubes that were identified at that time were Keller's infinite-dimensional compact and convex subsets of ℓ^2. In [153] Schori and West proved that $2^{\mathbb{I}}$ is homeomorphic to Q and in [49] Curtis and Schori completed the picture by showing that 2^X is homeomorphic to Q if and only if X is a Peano continuum. This was a spectacular result at that time and fully demonstrated the power and potential of infinite-dimensional topology.

4.5. **Hilbert cube manifolds.** In the early seventies, Chapman began the study of spaces modeled on Q, the so called Hilbert cube manifolds or Q- manifolds. Certain delicate finite-dimensional obstructions turned out not to appear in Q-manifold theory. In some vague sense, Q-manifold theory is a "stable" PL n-manifold theory.

We already mentioned the important homeomorphism extension result Theorem 4.3.2. Using this result, and several ingeneous geometric constructions, Chapman developed the theory of Q-manifolds. It was known from previous work that if P is a polyhedron then $P \times Q$ is a Q-manifold. Chapman [41] proved the converse, namely that all Q-manifolds are of this form, a result that turned out to be of fundamental importance later.

Some truly spectacular results were the result of Chapman's efforts. In 1974 he used Q-manifold theory to prove the invariance of Whitehead torsion. This is the statement that any homeomorphism between compact polyhedra is a simple homotopy equivalence. A map $f\colon X \to Y$ of compact polyhedra is a *simple homotopy equivalence* if it is homotopic to a finite composition

$$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \xrightarrow{f_n} X_n \xrightarrow{f_{n+1}} Y,$$

where each X_i is a compact polyhedron and each f_i is either an elementary expansion or an elementary collapse. Thus a simple homotopy equivalence is a homotopy equivalence of a very special nature. It is one which can be resolved into a finite number of elementary moves. More specifically, Chapman proved in [42]:

Theorem 4.5.1. *A map $f\colon X \to Y$ between compact polyhedra is a simple homotopy equivalence if and only if $f \times \{\mathrm{id}\}\colon X \times Q \to Y \times Q$ is homotopic to a homeomorphism.*

There is also a version of this result for noncompact polyhedra.

All of Chapman's results quoted here can also be found in his book [**?**].

4.6. **West's Theorem.** Borsuk [**?**, Problem 9.1] asked whether every compact ANR has the homotopy type of a compact polyhedron. For simply connected spaces, this question was answered by De Lyra [117] in the affirmative. For non-simply connected spaces Borsuk's problem stayed a mystery for a long time.

The problem was laid to rest by West [182] who showed, using among other things the technique in Miller [128], that for every compact ANR X there are a compact Q-manifold M and a cell-like map from M onto X.

A *cell-like map* between compacta is one for which point-inverses have the shape of a point; a cell-like map between ANR's is a *fine homotopy equivalence* as proved by HAVER [81] and TORUŃCZYK [173]. As we have seen above, the Q-manifold M is homeomorphic to $P \times Q$ for some compact polyhedron P and so X has the same homotopy type as P.

4.7. **Edwards' Theorem.** In 1974, EDWARDS [**?**, Chapter XIV] improved WEST's result by showing that $X \times Q$ is a Q-manifold if and only if X is a locally compact ANR. This provides an elegant proof of WEST's Theorem: for a compact ANR X there is by CHAPMAN's result a compact polyhedron P such that $X \times Q$ and $P \times Q$ are homeomorphic; clearly then X and P have the same homotopy type.

In the proof of EDWARDS' result and in TORUŃCZYK's work, which we shall describe momentarily, a crucial role was played by shrinkable maps. A continuous surjection $f\colon X \to Y$ between compact spaces is said to be *shrinkable* if one can find for every $\varepsilon > 0$ a homeomorphism φ of X onto itself such that $d(f \circ \varphi, f) < \varepsilon$, and $\operatorname{diam}(\varphi(f^{-1}(y)) < \varepsilon$ for all $y \in Y$. So a shrinkable map f is map whose fibers can be uniformly shrunk to small sets by a homeomorphism that looking from Y does not change f too much.

Bing's Shrinking Criterion form [29] characterizes shrinkable maps as uniform limits of homeomorphisms (so-called *near homeomorphisms*). Thus, in order to prove two compact spaces homeomorphic it suffices to produce a shrinkable map between them.

As an example consider $\operatorname{cone}(Q)$. We observed above that from KELLER's Theorem it follows that $\operatorname{cone}(Q) \approx Q$. But this follows also trivially from Bing's Shrinking Criterion. Since one-point compactifications are unique, it follows that we can also think of $\operatorname{cone}(Q)$ as the space obtained from $Q \times [0,1]$ by identifying the set $Q \times \{1\}$ to a single point. The decomposition map is easily seen to be shrinkable, hence a near homeomorphism, and so $\operatorname{cone}(Q) \approx Q \times [0,1] \approx Q$, as desired (for details, see [126, Theorem 6.1.11]).

It is easy to see that if X and Y are compact ANR's and $f\colon X \to Y$ is a near homeomphism then f is cell-like. So the method of shrinkable maps only works for cell-like maps.

4.8. **Toruńczyk's Theorems (part 1).** In 1980, TORUŃCZYK [174] published a remarkable result. He was able to topologically characterize the Q-manifolds among the locally compact ANR's. From EDWARDS's Theorem it was already known that if X is a locally compact ANR then $X \times Q$ is a Q-manifold. TORUŃCZYK studied the question when the projection

$$\pi\colon X \times Q \to X$$

is shrinkable, and came to an astounding conclusion. This map is shrinkable if and only if X has the following property: given $n \in \mathbb{N}$ and two maps $f, g\colon \mathbb{I}^n \to X$ and $\varepsilon > 0$ there exist maps $\xi, \eta\colon \mathbb{I}^n \to X$ such that

$$\xi[\mathbb{I}^n] \cap \eta[\mathbb{I}^n] = \emptyset$$

while moreover

$$d(f, \xi) < \varepsilon \quad \text{and} \quad d(\eta, g) < \varepsilon.$$

For obvious reasons this property is called the *disjoint cells-property.* So one arrives at the following conclusion, which is called Toruńczyk's Theorem:

Theorem 4.8.1. *Let X be a locally compact* ANR*. Then X if a Q- manifold if and only if X satisfies the disjoint-cells propery.*

As in the case of Edwards' Theorem, the Bing Shrinking Criterion and the Z-set Unknotting Theorem 4.3.2 were crucial in the proof of this result.

TORUŃCZYK's remarkable theorem ended the game of identifying Hilbert cubes. For in order to prove that a given space X is homeomorphic to Q, all one needs to prove is that it is an AR and satisfies the disjoint cells-property. Observe that both properties are trivially *necessary* for a space to be homeomorphic to Q. It is fascinating that these two properties that are stated in simple topological terms are also *sufficient.* In order to demonstrate the power of his topological characterization of Q, TORUŃCZYK [174] presented a very short and elegant proof of the CURTIS-SCHORI-WEST Hyperspace Theorem.

4.9. **The Taylor Example.** The above results emphasized the close relationships between infinite-dimensional topology and AR and ANR-theory. As we said above, certain delicate finite-dimensional obstructions turned out not to appear in Q-manifold theory. However, certain delicate *infinite-dimensional* obstructions *do* appear in infinite-dimensional topology. The first result demonstrating this was the result of TAYLOR [167] which we shall describe briefly.

It is an example of a cell-like map from a compactum X to Q which is not a shape equivalence. The space X is the inverse limit of a sequence of compact polyhedra with special properties. That the desired polyhedra exist follows from work of ADAMS [2] and TODA [169]. ADAMS' proof uses complex K-theory.

The TAYLOR Example was widely used in infinite- dimensional topology, shape theory and ANR-theory to obtain all sorts of counterexamples. DAVERMAN and WALSH [52] used it to get an example of a cell-like map $f\colon X \to Y$ between compacta whose non-degeneracy set is contained in a strongly countably dimensional set and which is not a shape equivalence. They also obtained from the TAYLOR Example new examples of locally contractible continua which are not ANR's. It was also used in 1979 to answer BORSUK's problem [**?**, Problem V.12.16] in the negative for the construction of an upper semi-continuous decomposition of Q into copies of itself, whose decomposition space is not an ANR[36]. And it was used to give a negative answer to KURATOWSKI's question [113] from 1951 whether a space with the compact extension property is necessarily an AR; a space X is said to have the *compact extension property* if for every space Y and every *compact* subset A of Y every continuous map from A to X has a continuous extension over X[37]. For the use of the TAYLOR Example in shape theory, see MARDEŠIĆ and SEGAL [120].

4.10. **Dranišnikov's Example.** Through the work of EDWARDS and WALSH [180] it was known in 1981 that the following two fundamental problems in dimension theory are equivalent:

[36]The construction can be found in Topology and its Applications 12 (1981), 315–320.

[37]The construction can be found in Proc. Amer. Math. Soc. 97 (1986), 136–138.

(1) Does there exist an infinite-dimensional compactum with finite cohomological dimension? (This problem is due to ALEXANDROV [5].)
(2) Does there exist a cell-like map $f\colon X \to Y$, where X is a finite dimensional compactum but Y is infinite-dimensional? (This problem, known as the *cell-like dimension raising mapping problem*, grew out of manifold theory and the work of KOZLOWSKI [107]. The first attempt to solve it by proving that every infinite-dimensional compactum contains sets of arbitrarily large finite dimension was shown to lead nowhere by WALSH [179].)

The problem was solved by DRANIŠNIKOV [59] in 1988: there exists an infinite-dimensional compactum with cohomological dimension 3. Essential in his construction is that there is a generalized cohomology theory for which the EILENBERG-MACLANE complex $K(\mathbb{Z}, 3)$ behaves like a point. For 2 dimensions, such an approach does not work. But there does exist an infinite-dimensional compactum with cohomological dimension 2, as was shown by DYDAK and WALSH [62]. Their work is based on the validity of the SULLIVAN conjecture.

As in the case of the TAYLOR Example, the DRANIŠNIKOV Example was also used by various authors to obtain counterexamples to a variety of questions. We will mention only one such application of special interest in infinite-dimensional topology. To put this result into perspective, we will first make some remarks. It was known since 1951 from the work of DUGUNDJI [60] that every locally convex linear space is an AR. Whether the local convexity assumption could be dropped was a fascinating question in infinite-dimensional topology and ANR-theory for a very long time. It was settled in the negative by CAUTY [39] in 1994 by a very interesting method and an essential use of the DRANIŠNIKOV Example. More specifically, he proved that there exists a necessarily non-locally convex linear space L which is not an AR but which is a closed linear subspace of a linear space which is an AR.

4.11. **Toruńczyk's Theorems (part 2).** So far, we mainly concentrated on (locally) compact spaces. As is to be expected, there are also results for complete spaces. Recall that infinite-dimensional topology started with the investigation of completely metrizable linear spaces. In [175] TORUŃCZYK characterized the topology of Hilbert spaces in much the same way as he characterized the topology of the Hilbert cube. In this characterization the disjoint-cells property is replaced by the discrete approximation property; this property states that for every open cover $\mathcal{U}$ of the space X and every map f from the topological sum $\bigoplus_{n=1}^{\infty} \mathbb{I}^n$ to X there is another map g from $\bigoplus_{n=1}^{\infty} \mathbb{I}^n$ to X that is $\mathcal{U}$ close to f and is such that the family $\{g[\mathbb{I}^n] : n \in \mathbb{N}\}$ is discrete. The characterization reads:

Theorem 4.11.1. *A separable space is a manifold modeled on ℓ^2 if and only if it is a completely metrizable separable ANR with the discrete approximation property.*

As a consequence ℓ^2 is characterized as the only separable completely metrizable AR with the discrete approximation property.

TORUŃCZYK has also a similar characterization of manifolds modeled on arbitrary Hilbert spaces, see [175] for more details.

4.12. **Epilogue.** We saw that ANDERSON, interested in questions in general topology, created a new field in topology called *infinite-dimensional topology* and was at the beginning unaware of its potential. But good mathematics inevitably led to good results in various other disciplines, mostly in algebraic and geometric topology. The highlights of infinite-dimensional topology are the theorems of ANDERSON on the homeomorphy of ℓ^2 and s, of CHAPMAN on the invariance of Whitehead torsion, of WEST on the finiteness of homotopy types of compact ANR's and of TORUŃCZYK on the topological characterization of manifolds modeled on various infinite-dimensional spaces.

A large collection of open problems is WEST's paper [183]. The subjects that are being touched upon range from absorbing sets and function spaces to ANR theory. We mention two particularly prominent problems:

(1) Let $\alpha\colon Q \to Q$ be an involution with a unique fixed-point. Is α conjugated to the standard involution β on Q defined by $\beta(x) = -x$?
(2) For $n \geq 3$, let $\mathcal{H}_n$ be the group of all homeomorphisms on $\mathbb{I}^n$ endowed with the compact-open topology. Is $\mathcal{H}_n$ homeomorphic to ℓ^2?

5. The Period of Harvesting: Set theoretic topology

In the sixties, general topology renewed its interaction with set theory. In 1878, CANTOR's work [37] had created set theory and topology as we saw in §2.2. They developed as diverse, complex and independent fields. Soon after their renewed interaction spectacular results surfaced, also in parts of topology where traditionally geometric and algebraic tools were used, or tools from analysis. It is about those results that we wish to report here.

We saw that in [37] CANTOR wrote down the *Continuum Hypothesis* (abbreviated CH) that would have a profound effect on set theory in the 20th century. The CH states that the first uncountable cardinal is $\mathfrak{c}$, the cardinality of the real line (the continuum). The work of GÖDEL [77] and COHEN [46] has shown that CH is consistent with and independent from the 'usual' Zermelo-Fraenkel Axioms of Set Theory. The methods used in these proofs, and especially COHEN's forcing, had a profound effect on the development of a new field in topology called *set theoretic topology.* In that development, the work of MARY ELLEN RUDIN was vital.

In our report below we will almost exclusively concentrate on independence results in topology, that is, results that are independent from and consistent with the 'usual' Zermelo-Fraenkel Axioms of Set Theory. So we will ignore important parts of general topology. Also some problems are being discussed whose solution is very strongly of a set theoretic nature without being an independence result. None of the results mentioned has its roots in general topology.

5.1. **Souslin's problem.** Suppose that S is a connected, linearly ordered topological space without a first or last element. If S is separable then S is isomorphic to $\mathbb{R}$. What happens if one relaxes the separability condition to the condition that any pairwise disjoint collection of nontrivial intervals of X is countable? This is SOUSLIN's Problem from [162]. It was posed in 1920 and has fascinated topologists and set theorists ever since.

The requirement that pairwise disjoint collections of intervals (or more general open sets) are countable is called the *countable chain condition* (abbreviated ccc).

A counterexample to Souslin's question, a ccc connected linearly ordered space without first or last element that is not homeomorphic to the real line, is called a *Souslin line*, and *Souslin's Hypothesis* (SH) is the statement that no Souslin lines exist. JECH [92] and TENNENBAUM [168] used Cohen's forcing method to show that Souslin lines can exist and JENSEN [93] proved that they also exist in Gödel's Constructible Universe, the same universe GÖDEL used to establish the consistency of the Continuum Hypothesis. In [161], SOLOVAY and TENNENBAUM developed the forcing method further and proved the consistency of Souslin's Hypothesis. Their proof established the consistency of a powerful combinatorial principle, which we shall discuss briefly.

The principle, called *Martin's Axiom* (MA), states that no compact Hausdorff space that satisfies the ccc is the union of fewer than $\mathfrak{c}$ nowhere dense sets. Under CH 'fewer than $\mathfrak{c}$' means countable and so MA holds by the Baire Category Theorem. However, MA is also consistent with the negation of CH and it is this combination, MA + ¬CH, that proved to be very powerful indeed.

SOLOVAY and TENNENBAUM [161] showed that under MA + ¬CH there are no Souslin lines, thereby proving that SH is undecidable. Ever since this result, Martin's Axiom played a prominent role in set theory and set theoretic topology, as the rest of our story will tell.

5.2. **Alexandroff's problem.** Most mathematicians in geometric topology are only interested in metrizable spaces, and metrizable manifolds in particular. But there are also mathematically important objects that are not always metrizable, for example, CW-complexes, linear spaces, topological groups and manifolds. By a *manifold* we mean a locally Euclidean Hausdorff space. Manifolds are certainly mathematically important, with or without differential or algebraic structure.

Let M be a manifold. If $A \subseteq M$ is closed then one certainly wants to be able to extend every continuous real valued function $f\colon A \to \mathbb{R}$ to a continuous function $\bar{f}\colon M \to \mathbb{R}$. By the Tietze-Urysohn Theorem, this is equivalent to M being normal. In the process of constructing new continuous functions from old ones (think of homotopies) it is also extremely pleasant if M has the following property: for every closed subset $A \subseteq M$ there is a sequence $\langle U_n \rangle_n$ of open subsets of M such that $A = \bigcap_{n<\omega} U_n$. General topologists say that spaces with this property are *perfect*. If one wants to generalize some of the existing theory on metrizable manifolds to nonmetrizable ones, it becomes clear quite quickly that in many instances it is inevitable to restrict oneself to manifolds that are both normal and perfect, i.e. manifolds that are *perfectly normal*. The question then naturally arises whether there is a perfectly normal manifold which is not metrizable. This question was asked by ALEXANDROFF [4] in 1935 and also by WILDER [185] in 1949.

It seems very unlikely that a set theoretic statement like CH has anything to do with manifolds, let alone with Alexandroff's problem. In [150] however, RUDIN and ZENOR constructed assuming CH an example of a perfectly normal nonmetrizable manifold. Later, KOZLOWSKI and ZENOR [108] even constructed such a manifold that is analytic, again under CH. These provisional solutions to Alexandroff's problem very strongly suggested a positive answer to it.

In [149], Rudin proved that under MA + $\neg$CH, all perfectly normal manifolds are metrizable; as a consequence, she came to the remarkable conclusion that Alexandroff's problem is undecidable.

5.3. **Dowker's problem.** In [30], Borsuk proved his famous homotopy extension theorem for metrizable spaces. Actually, his result is true for spaces X for which the product $X \times \mathbb{I}$ is normal. This generalisation is due to Dowker and was first published in Hurewicz and Wallman [90]. A necessary condition for $X \times \mathbb{I}$ to be normal is that X is normal. So it is natural to ask whether this condition is also sufficient. This is Dowker's Problem. Dowker [58] and Katětov [98] independently gave necessary and sufficient conditions for a space X to have the property that its product with $\mathbb{I}$ is normal.

Theorem 5.3.1. *Let X be a space. Then $X \times \mathbb{I}$ is normal if and only X is normal and for every decreasing sequence of closed subsets $\langle D_n \rangle_n$ of X with $\bigcap_{n<\omega} D_n = \emptyset$ there exists a sequence $\langle U_n \rangle_n$ of open subsets of X such that $D_n \subseteq U_n$ for every n while moreover $\bigcap_{n<\omega} U_n = \emptyset$.*

A normal space X for which $X \times \mathbb{I}$ is not normal is called a *Dowker space* in the literature. So if one wishes to construct a Dowker space, all one needs to do is to construct a normal space X having a sequence of closed subsets $\langle D_n \rangle_n$ of X with $\bigcap_{n<\omega} D_n = \emptyset$ such that if $U_n \subseteq X$ is open and $D_n \subseteq U_n$ for every n then $\bigcap_{n<\omega} U_n \neq \emptyset$. It is surprising that this condition is such a complicated one.

In 1955, Rudin [145] constructed the first example of a Dowker space assuming the existence of a Souslin line. That was a major breakthrough at that time, but as turned out later, had an unpleasant drawback since, as we saw above, SH is undecidable.

But in 1971 it was shown that the solution to Dowker's problem does not depend on set theory: the first example of a *real* (= using no axioms beyond ZFC) example of a Dowker space was constructed again by Rudin [147]. This Dowker space was the only ZFC example of such a space for about twenty years. Balogh [17] constructed another such example only in 1994 (see also his subsequent paper [18]). This very interesting example is 'small' while the original Dowker space is 'large'. It is certainly not the final word on Dowker spaces since it is still unknown whether there can be a first countable Dowker space in ZFC, or one of cardinality ω_1. Using pcf theory, Kojman and Shelah [105] constructed a Dowker subspace of Rudin's Example in [147] of size $\aleph_{\omega+1}$. This is a 'real' example of a small Dowker space since its cardinality is decided in ZFC, while Rudin's and Balogh's are not.

Ironically, Borsuk's Theorem that started all this research, turned out to hold also without the assumption of normality of the product with $\mathbb{I}$, see Morita [133] and Starbird [163].

5.4. **Whitehead's problem.** Whitehead asked whether every compact arcwise- connected abelian topological group is isomorphic to a product of circles. This is a very natural problem for a topologist. We first translate it into purely algebraic language to turn it into a very natural problem for an algebraist as well. If A and B are abelian groups then a surjective homomorphism $f\colon A \to B$ is said to *split* if there is a homomorphism $g\colon B \to A$ with $f \circ g$ is equal to the identity on B. An abelian group G is *Whitehead* if for every abelian group B, every surjective homomorphism $f\colon B \to G$ with kernel isomorphic to $\mathbb{Z}$ splits. It is clear that all free groups are Whitehead and WHITEHEAD asked whether all Whitehead groups are free. It is a consequence of Pontrjagin duality that both problems we attributed here to Whitehead are equivalent.

SHELAH [154, 155] showed that Whitehead's problem in undecidable by showing that under $V = L$ all Whitehead groups are free while under MA + $\neg$CH there exists a Whitehead group which is not free.

The fact that Whitehead's problem can be formulated both into algebraic and topological language is not an exception for a problem that turns out to be dependent upon one's set theory. These problems can often be translated into several mathematical languages and can therefore be attacked from several directions. There are for example numerous problems in Boolean algebras that can be translated into topology and vice versa. Sometimes such a translation helps.

It is questionable of course whether Whitehead's problem discussed above is a 'real' topological problem. We took the liberty of mentioning it because it is such a good example of our point that problems of set theoretic nature can often be attacked from different angles.

5.5. **Choquet's problem.** A BA (= Boolean Algebra) will be identified with its universe. A BA B is called

complete/countably complete/weakly countably complete

if for any two subsets P and Q such that $p \wedge q = 0$ for $r \in P$ and $q \in Q$

without further condition/with $|P| = \omega$ or $|Q| = \omega$/with $|P| = |Q| = \omega$

there is an $s \in B$ which *separates* P and Q, i.e. $p \le s$ fpr $p \in P$ and $q \le s'$ for $q \in Q$. Consider the following statements:

- FB every weakly countably complete BA is a homomorphic image of a countably complete BA;
- BE every countably complete BA is a homomorphic image of a complete BA;
- FE every weakly countably complete BA is a homomorphic image of a complete BA.

The earliest statement we are aware of where one of these statements is considered is LOUVEAU [116]. Here he attributes the question (or conjecture) of whether FE holds to CHOQUET, and proves that under CH the restriction of FE to algebras of size $\le \mathfrak{c}$ holds. The question of whether BE holds was raised by KOPPELBERG [106], who was apparantly unaware of LOUVEAU's paper. She proved that the restriction of BE to algebras of cardinality $\le \mathfrak{c}$ holds under CH. The question of whether FB holds was raised by VAN DOUWEN, MONK and RUBIN [130], who also repeated the question of whether BE holds.

By Stone duality, all these question can be formulated in topological language. They were all solved by topologists. It was shown that FB is not a theorem in ZFC under MA + $\mathfrak{c} = \omega_2$, hence neither is FE.[38] The problem of whether BE holds turned out to be difficult. It was finally shown in Dow and Vermeer [57] that BE is not a theorem of ZFC. The algebra in question is B, the algebra of Borel sets of the unit interval. They showed that if B is the quotient of some complete Boolean algebra then there is a lifting of the quotient of B modulo the meager sets back into B. An appeal to a result of Shelah [158] that such a lifting need not exist finishes the proof.

5.6. **Binary operations on $\beta\omega$.** Let $\beta\omega$ denote the the Čech-Stone compactification of the discrete space ω. As is well-known, the points of this space can be thought of as ultrafilters in $\mathcal{P}(\omega)$. Thinking about the points in $\beta\omega$ in this way, it is easy to extend various binary operations on ω to binary operations on $\beta\omega$. As an example, let us consider ordinary addition on ω.

For $A \subseteq \omega$ and $n \in \omega$ we set

$$A - n = \{k \in \omega : k + n \in A\}.$$

For $p, q \in \beta\omega$ put

$$p + q = \{A \subseteq \omega : \{n \in \omega : A - n \in p\} \in q\}.$$

Then $+$ is a well-defined binary operation on $\beta\omega$ which extends the ordinary addition on ω and moreover is associative and right-continuous (this is due to Glazer, see [47]). So $(\beta\omega, +)$ is a compact right topological semigroup. By a result of Wallace [177, 178] (see also Ellis [66]), the compactness of $\beta\omega$ implies the existence of a point $p \in \beta\omega$ for which $p + p = p$, i.e. a so- called *idempotent*.

Glazer (see [47]) used the existence of idempotents in the semigroup $(\beta\omega, +)$ to give a particularly simple *topological* proof of Hindman's Theorem from [86]: *If the natural numbers are divided into two sets then there is a sequence drawn from one of these sets such that all finite sums of distinct numbers of this sequence remain in the same set.*

This statement was known for some years as the Graham-Rothschild Conjecture.

Several other results from classical number theory can be proved as well by similar methods. In [21] Bergelson, Furstenberg, Hindman and Katznelson again used the semigroup $(\beta\omega, +)$ to present an elementary proof of van der Waerden's Theorem from [176]: *if the natural numbers are partitioned into finitely many classes in any way whatever, one of these classes contains arbitrarily long arithmetic progressions.*

5.7. **Strong homology.** Let $Y^{(k+1)}$ be the topological sum of countably many copies of the $(k+1)$-dimensional Hawaiian earring. The calculation of the strong homology of $Y^{(k+1)}$ is of interest in the question of whether strong homology satisfies the additivity axiom (of Milnor [129]). In [119], Mardešić and Prasolov translated the calculation of the (k-dimensional) strong homology of $Y^{(k+1)}$ into a condition of set theory. They proved that this condition holds under CH, and hence that the (k-dimensional) strong homology of $Y^{(k+1)}$ can be nontrivial. But, as was shown in Dow, Simon and Vaughan [56], there

[38] The construction can be found in Trans. Amer. Math. Soc. 259 (1980), 121–127.

are also models of set theory in which it does not hold, and therefore in such models the (k-dimensional) strong homology of $Y^{(k+1)}$ is trivial.

5.8. **Banach spaces.** In Banach space theory, many results from general topology are applied. The completeness of the real line gives the Hahn-Banach Theorem, Baire's Category Theorem is essential in the proof of the open mapping theorem and the uniform boundedness principle, while Tychonoff's compactness theorem proves the Alaoglou theorem. Etcetera. It is therefore not surprising that set theoretic topology turned out to have very interesting applications in Banach space theory. It is about two of those results that we wish to report here.

For a compact space K we let $M(K)$ denote the space of all finite real-valued regular Borel (or, Baire) measures on K (with $\|\mu\| = |\mu|(K)$, where $|\mu|$ is the total variation of μ).

Pełczyński [140] proved the following result:

Theorem 5.8.1. *Let α be an infinite cardinal number, X a Banach space, and $\ell^1_\alpha \hookrightarrow X$ an isometric imbedding. Then the space $M(\{0,1\}^\alpha)$ admits an isometric imbedding in the dual X^* of X. In particular,*

$$L^1(\{0,1\}^\alpha) \hookrightarrow X^* \quad \textit{and} \quad \ell^1_{2^\alpha} \hookrightarrow X^*.$$

The question naturally arises whether the converse to this theorem holds, i.e. whether from $M(\{0,1\}^\alpha) \hookrightarrow X^*$ it follows that $\ell^1_\alpha \hookrightarrow X$. Pełczyński conjectured that this is true, and verified the conjecture for $\alpha = \omega$, [140]. The answer to Pełczyński's Conjecture is fascinating. For cardinals $\alpha > \omega_1$ it is true, as was shown by Agryros [3]. So there only remains the cardinal ω_1. For that cardinal number the question is undecidable. Under MA + $\neg$CH, Pełczyński's Conjecture is true for $\alpha = \omega_1$ as was also shown by Agryros [3]. But under CH, Haydon [82] constructed a counterexample of a particular nice form since it is of the form $C(K)$ for a certain compact Hausdorff space K. The space K is an inverse limit of an ω_1-sequence of Cantor sets with certain specific properties. Independenly, a similar space was also constructed by Kunen [111] motivated by topological questions. In addition, it also surfaced in the work of Talagrand [165]. So Pełczyński's Conjecture turned out to boil down partly to the construction under CH of a very peculiar compact Hausdorff space K. It is precisely in such constructions where set theoretic topology plays such a prominent role and where its techniques are fundamental.

Another application of set theoretic topology to Banach space theory is the following one. If X is a Banach space and $A \subseteq X$ then $\overline{\text{convex}}(A)$ denotes the closed convex hull of A. If X is separable, then for every uncountable subset $A \subseteq X$ there exists an element $a \in A$ such that $a \in \overline{A \setminus \{a\}}$, in particular, $a \in \overline{\text{convex}}(A \setminus \{a\})$. Davis and Johnson asked whether the latter property could hold in a non-separable Banach space. It was solved in the affirmative by Shelah [156] under the combinatorial principle $\diamondsuit$. But this example is not of the form $C(K)$ for some compact Hausdorff space K. But such a space exists even under the weaker hypothesis CH, as was shown by Kunen [109].

5.9. **Epilogue.** Our overview of set theoretic topology is very much less than complete as a description of what happened in that area (see our remarks at the beginning of the introduction). We have for example not mentioned several very important areas in set theoretic topology such as cardinal functions, S- and L-spaces, the Normal Moore Space Conjecture, βX (including $\beta\omega$), the set theoretic aspects of topological groups, etc. In addition, we could have talked much more about its relation with set theory and we said very little about Boolean Algebras.

For more information on set theoretic topology we refer the reader to RUDIN's book [148], the Handbook of Set Theoretic Topology [112] and the book on Open Problems in Topology [127].

NOTES

In this section we will give some additional information on the material presented in §§4 and 5 that we find useful. No attempt has been made to be complete.

In §3 we already mentioned the following books for among other things information on some the results obtained in the golden age of general topology: HU [88], HUREWICZ and WALLMAN [90], GILLMAN and JERISON [76], DUGUNDJI [61], NAGATA [138, 137], ENGELKING [68, 67] and ARHANGEL'SKIĬ and PONOMAREV [13]. To this list we can add the following books for additional information and later developments, some of which we already mentioned in the other sections: ARHANGEL'SKIĬ [12], BESSAGA and PEŁCZYŃSKI [**?**], BORSUK [**?**, 31], COMFORT and NEGREPONTIS [48], BALCAR and ŠTĚPÁNEK [16], DEVLIN and JOHNSBRÅTEN [53], BOURBAKI [32], HU [89], AARTS and NISHIURA [1], JUHÁSZ [96], KECHRIS [99], KUNEN [110], KURATOWSKI [114, 115], KURATOWSKI and MOSTOWSKI [**?**], MARDEŠIĆ and SEGAL [120], NADLER [135], SHELAH [157], TODORČEVIĆ [171], DAVERMAN [51], AULL and LOWEN [15], CHAPMAN [**?**], RUDIN [148], the Handbook of Set Theoretic Topology [112], the book on Topics in General Topology [134], the book on Open Problems in Topology [127], the book on Recent progress in General Topology [91] and CHIGOGIDZE [43].

Notes on §4. For a different proof that all infinite-dimensional separable Banach spaces are homemorphic, see BESSAGA and PEŁCZYŃSKI [24].

For different proofs of Anderson's Theorem that $s \approx \ell^2$, see ANDERSON and BING [10] and [126, chapter 6].

As we observed, KELLER proved that the Hilbert cube is homogeneous. This result was later generalized by FORT [69] who proved that the infinite product of compact manifolds is homogeneous if and only if none or infinitely many of the factors have a boundary.

For a proof that any homeomorphism φ between CANTOR sets in $\mathbb{R}^2$ can be extended to a homeomorphism of $\mathbb{R}^2$, see KURATOWSKI [114, 115]. For the cited result about Antoine's necklace, the reader can consult e.g. DAVERMAN [51, Corollary 5A].

Let X be a space, and let $A \subseteq X$ be closed. Nowadays we call A a Z-set in X if for every $\varepsilon > 0$ and every continuous function $f\colon Q \to X$ there exists a continuous function $g\colon Q \to X \setminus A$ such that $d(f, g) < \varepsilon$. This definition is easier to work with than the original one and is equivalent to it in the special case $X = Q$ (but this is not entirely trivial). For detailed proofs of the Z-set homeomorphism extension results in Q-manifolds, see BESSAGA and PEŁCZYŃSKI [**?**], CHAPMAN [**?**] and VAN MILL [126, chapter 6].

See Cohen [45] for more information on the concept of a simple homotopy equivalence.

For a detailed description of the Dranišnikov Example, see also Chigogidze [43].

As we remarked, the remaining open problems in infinite- dimensional topology (see West [183]) deal among other things with problems in absorbing sets (see e.g. [27] and [55]), function spaces (see e.g. [40]) and ANR-theory.

For general information on hyperspaces see Nadler [135].

Notes on §5. For a simple proof that metrizable spaces are paracompact, see Rudin [146].

Many of the set theoretic things that we merely touched upon in this section can be found in great detail in Kunen [110].

For more information on SH and many related topics, see Todorčević [170].

For more information on Whitehead's Problem, see Ecklof [63].

For more information on the role of topology in Banach spaces and measure theory, see Negrepontis [139], Mercourakis and Negrepontis [122] and Fremlin [75].

For more information on ultrafilters and combinatorial number theory, see Hindman [87].

A recent book on general/set theoretic topology is Todorčević [172]

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