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On mappings of compact spaces into Cartesian spaces

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Abstract

Eilenberg proved that if a compact space X admits a zero-dimensional map $f:X \to Y$, where Y is m-dimensional, then there exists a map $h:X \to I^{m+1}$ such that $f \times h:X \to Y \times I^{m+1}$ is an embedding. In this paper we prove generalizations of this result for σ-compact subsets of arbitrary spaces. An example of a compact space X and of a zero-dimensional σ-compact subset $A \subset X$ is given such that for any continuous function $f:X \to \mathbb{R}$ which is one-to-one on the set A and any G_δ-subset B of X with $B \supset A$ the restriction $f|B:B \to \mathbb{R}$ has infinite fibers. This example is used to demonstrate that our results are sharp. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In 1935 Eilenberg [3] proved that *if a compact metrizable space X admits a zero-dimensional map $f: X \to Y$, where Y is m-dimensional and metrizable, then there exists a map $h: X \to I^{m+1}$ such that*

$$f \times h: X \to Y \times I^{m+1}$$

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is an embedding. (Here I denotes the unit interval $[0,1]$.) This theorem was generalized later for perfect mappings of general metrizable spaces X and Y by Pasynkov [11]. The arguments of Eilenberg and Pasynkov used different ideas but they were both based on the following classical result due to Urysohn [13]: *every n-dimensional metrizable space is the union of $n+1$ zero-dimensional subspaces*. In [1] an infinite version of Urysohn's Theorem was proved that allows us here to obtain a stronger version of the Eilenberg theorem (we restrict our considerations to compact spaces here although the Pasynkov theorem can be generalized in the same manner). In addition, we present counterexamples to various natural problems that arise in our investigations and that deal with possible generalizations of Lavrentieff's Theorem on extending homeomorphisms over G_δ-sets.

All spaces under discussion are separable and metrizable, and all maps are continuous.

2. Function spaces

We begin with some simple observations on function spaces that will be used later in this paper.

For spaces X and Y, where X is compact, we let $\mathcal{C}(X,Y)$ denote the collection of all maps from X to Y. We endow it with the topology of uniform convergence. If ϱ is an admissible metric for Y then

$$\hat{\varrho}(f_1, f_2) = \max_{x\in X}\bigl\{\varrho(f_1(x), f_2(x))\bigr\}$$

is an admissible metric for $\mathcal{C}(X,Y)$. It is well known and easy to prove that $\hat{\varrho}$ is complete if and only if ϱ is complete.

For a closed subset $A\subseteq X$, we let $\phi_A : \mathcal{C}(X,Y)\to\mathcal{C}(A,Y)$ be the restriction map. If no confusion can arise we sometimes suppress the index A in ϕ_A.

Lemma 1. *Let X and Y be spaces, where X is compact. In addition, let $A\subseteq X$ be closed.*

(a) *If $\mathcal{H}$ is a G_δ-set in $\mathcal{C}(A,Y)$, then $\phi^{-1}(\mathcal{H})$ is a G_δ-set in $\mathcal{C}(X,Y)$.*

(b) *If $Y\in ANR$ and $\mathcal{H}$ is dense in $\mathcal{C}(A,Y)$, then $\phi^{-1}(\mathcal{H})$ is dense in $\mathcal{C}(X,Y)$.*

Proof. For (a) it suffices to observe that ϕ is continuous.

For (b), let $f: X\to Y$ be a map and $\varepsilon>0$. Let $\mathcal{U}$ be the cover of Y consisting of all open sets of diameter less than ε. Since $\mathcal{H}$ is dense in $\mathcal{C}(A,Y)$, A is compact, and Y is an *ANR*, there is an element $h\in\mathcal{H}$ such that $f|A$ and h are $\mathcal{U}$-homotopic. An application of the controlled Borsuk Homotopy Theorem 5.1.3 in [10] shows that h can be extended to a map $\hat{h}: X\to Y$ such that $\hat{h}$ and f are $\mathcal{U}$-close. As a consequence, f is ε-close to a function whose restriction to A belongs to $\mathcal{H}$. □

If we endow a product of two spaces with its max-metric, then the next assertion holds.

Lemma 2. *For a compactum X and metric spaces (Y_1,ϱ_1) and (Y_2,ϱ_2), the space $\mathcal{C}(X, Y_1\times Y_2)$ is isometric to the product $\mathcal{C}(X,Y_1)\times\mathcal{C}(X,Y_2)$.*

3. Regularly branched maps

We shall follow the terminology of the article [2]. For every $k \geqslant 0$ and every map $f : X \to Z$ let

$$B_k(f) = \left\{z \in Z\colon |f^{-1}(z)| \geqslant k\right\}.$$

Let X and Z be finite-dimensional. Then $f : X \to Z$ is called *regularly branched* if for every $k \geqslant 0$,

$$\dim B_k(f) \leqslant k \cdot \dim X - (k-1) \cdot \dim Z.$$

A regularly branched map is very special. We will demonstrate this by the following (trivial) observations. If $f : X \to Z$ is regularly branched then it does not increase dimension. This follows easily because $B_1(f) = f(X)$ and so $\dim f(X) \leqslant \dim X$. Secondly, if $\dim X < \frac{1}{2} \dim Z$ and $f : X \to Z$ is regularly branched then f is one-to-one. Also, if $\dim X \leqslant m-1$ then every regularly branched map $f : X \to Z$, where $\dim Z = m$ is $\leqslant m$-to-one because

$$\dim B_{m+1}(f) \leqslant (m+1)(m-1) - m^2 = -1$$

and so $B_{m+1}(f) = \emptyset$. We remark finally that if $\dim X \geqslant \dim Z$ then every map $f : X \to Z$ is regularly branched. For let $k \geqslant 0$ be arbitrary. Then

$$\begin{aligned} k \cdot \dim X - (k-1) \cdot \dim Z &= k \cdot (\dim X - \dim Z) + \dim Z \\ &\geqslant \dim Z \geqslant \dim B_k(f). \end{aligned}$$

We therefore change the definition of a regularly branched map in this special case. Indeed, if $m = \dim Z$ and $\dim X \geqslant \dim Z$ then a map $f : X \to Z$ is called regularly branched provided that

$$\dim f = \dim X - m,$$

where $\dim f = \max\{\dim f^{-1}(y)\colon\ y \in Z\}$.

We now state the following result due to Hurewicz [6], [8, §45, Statement IX], which is the basis for our considerations.

Hurewicz's Theorem. *Let $m \geqslant 1$ and let X be a finite-dimensional compact space. The set $\mathcal{H}(X, \mathbb{R}^m)$ of all regularly branched maps from X into $\mathbb{R}^m$ is a dense G_δ-set in the space $\mathcal{C}(X, \mathbb{R}^m)$ of all maps $f : X \to \mathbb{R}^m$.*

Observe that this theorem implies that if X is compact and m-dimensional then there are many zero-dimensional maps from X into $\mathbb{R}^m$.

Our first result in this section is the following "infinite" Hurewicz Theorem. If $N \subseteq \omega$ then $p_N : I^\omega \to I^N$ denotes the projection.

Theorem 1. *Let $\mathcal{A} = \{A_j\}$ be a countable family of closed finite-dimensional subspaces of a compactum X. Let $\mathcal{H}$ be the set of all maps $f : X \to I^\omega$ with the following property*: *for* *every finite $N \subset \omega$ and every $A \in \mathcal{A}$ the map $(p_N \circ f)|A : A \to I^N$ is regularly branched. Then $\mathcal{H}$ is a dense G_δ-subset of $\mathcal{C}(X, I^\omega)$.*

Proof. For a finite set $N \subset \omega$, let

$$\mathcal{H}_{j;N} \subset \mathcal{C}(X, I^\omega)$$

be the set of all maps $f : X \to I^\omega$ such that $p_N \circ f|A_j : A_j \to I^N$ is regularly branched. Since $\mathcal{C}(X, I^\omega)$ is homeomorphic to the product $\mathcal{C}(X, I^N) \times \mathcal{C}(X, I^{\omega\setminus N})$ (Lemma 2), the set $\mathcal{H}_{j;N}$ is a dense G_δ-subset of $\mathcal{C}(X, I^\omega)$ in view of Hurewicz's Theorem and Lemma 1. But

$$\mathcal{H} = \bigcap_{j,N} \mathcal{H}_{j;N}.$$

So it remains to apply the Baire Theorem to the complete space $\mathcal{C}(X, I^\omega)$. □

Corollary 1. *For every separable metrizable space X of finite dimension n, there exist family $\{h_1, h_2, \ldots\}$ of maps from X to I such that for all pairwise distinct $j_1, \ldots, j_{2n+1}$ in $\mathbb{N}$ the map*

$$h_{j_1} \times \cdots \times h_{j_{2n+1}} : X \to I^{2n+1}$$

is an embedding.

Proof. According to a result of Hurewicz, [4, Theorem 1.7.2], the space X has an n-dimensional compactification cX. An application of Theorem 1 finishes the proof. □

Corollary 2. *For every n-dimensional σ-compact subset $B \subseteq X$ of a compact space X there exits a map $h : X \to I^{2n+1}$ which is one-to-one on B. Moreover, the set of all these maps h is a dense G_δ-subset of $\mathcal{C}(X, I^{2n+1})$.*

Proof. Let $B = \bigcup_{i=1}^\infty B_i$, where each B_i is compact. We may assume without loss of generality that $B_i \subseteq B_{i+1}$. By the proof of Theorem 1 the set of all maps $h : X \to I^{2n+1}$ which are regularly branched on each B_i is a dense G_δ-subset of $\mathcal{C}(X, I^{2n+1})$. Since $\dim B_i < \frac{1}{2}(2n+1)$, it follows every such h has the property that $h|B_i$ is one-to-one for every i. This is clearly as required. □

In view of the classical Nöbeling–Pontryagin Theorem that every n-dimensional space can be embedded in I^{2n+1}, the question naturally arises whether Corollary 2 can be improved to the effect that the functions h in Corollary 2 restrict to embeddings on B. If h is such that $h|B$ is an embedding then the Lavrentieff Theorem below implies that $h|S$ is an embedding for some G_δ-subset $S \subseteq X$ which contains B. So the non-existence of such a G_δ-set has as trivial corollary that the map h is not an embedding. This leads us to another natural question. If the map h is such as in Corollary 2, does there exist a

G_δ-subset S of X which contains B and on which h is also one-to-one? Both our questions will be answered in the negative in the remaining part of this section.

Lavrentieff's Theorem. *Let $X_0 \subset X$ and $Y_0 \subset Y$ be subsets of complete metric spaces, and let $f : X_0 \to Y_0$ be a homeomorphism. Then f can be extended to a homeomorphism $\bar{f} : \overline{X}_0 \to \overline{Y}_0$ between G_δ-subsets of X and Y.*

Let us recall that a space X is called a *Baire space*, if for each sequence U_n of open dense in X subsets its intersection $\bigcap_n U_n$ is dense in X. Clearly, a space is not Baire if it has a non-empty open subset of the first category.

Levi's Theorem [9]. *Let f be a map from a complete space X onto a Baire space Y. Then there exists a G_δ-set $X_0 \subset X$ such that $f|X_0$ is a homeomorphism and $f(X_0)$ is a dense G_δ-subset of Y.*

By an *interval* we mean a non-degenerate subinterval of $\mathbb{R}$.

Theorem 2 (Example 1). *There exist a one-dimensional compactum X and a zero-dimensional F_σ-set $A \subset X$, such that for an arbitrary map $f : X \to \mathbb{R}$ and a G_δ-set $B \subset X$ containing A, the map $f|B$ is not one-to-one. Moreover, the map $f|B$ has an infinite preimage.*

Proof. For X we take the product $C \times I$, where C is the familiar Cantor subset of I. Let $A = C \times Q$, where Q is the set of all rationals in I. Striving for a contradiction. assume that there are a G_δ-set $B \supset A$ and a map $f : X \to \mathbb{R}$ such that $f|B$ is one-to-one. For $c \in C$, let

$$I_c = \{c\} \times I, \quad B_c = B \cap I_c, \quad f_c = f|I_c, \quad K_c = f_c(I_c) = f(I_c).$$

Clearly, B_c is a dense G_δ-subset of I_c. Since $f|B_c$ is one-to-one, we have

Claim 1. *$f_c(J)$ is an interval for every interval $J \subseteq I_c$.*

Claim 2. *Let F be a closed subset of K_c. Then* $\dim F = \dim f_c^{-1}(F)$.

Proof. If $\dim F = 0$ then $\dim f_c^{-1}(F) = 0$ according to Claim 1. Now assume that F is an interval, but $f_c^{-1}(F)$ is zero-dimensional. Let L be the perfect kernel of $f_c^{-1}(F)$. In other words, L consists of all condensation points of $f_c^{-1}(F)$. Then L is compact and $f_c^{-1}(F) \setminus L$ is countable. Hence $f(L) = F$, because F is connected. The set L is clearly a nowhere dense subset of I_c. There consequently is an interval $(a, b) \subset I_c$ such that $(a, b) \cap L = \emptyset$ while moreover either $f_c(a)$ or $f_c(b)$ is an interior point of F. Hence by Claim 1, $F \cap f_c((a, b))$ is an uncountable set. But $f_c^{-1}(F) \cap (a, b) \subseteq f_c^{-1}(F) \setminus L$ is a countable set. We arrived at a contradiction. □

Claim 3. *The set $D_c = f_c(B_c)$ is a Baire space.*

Proof. Assume that there is a subinterval $F \subseteq K_c$ such that $D_c \cap F \subseteq \bigcup \mathcal{G}$, where $\mathcal{G}$ is a countable family nowhere dense subsets of K_c. By Claim 2 there is an interval $J \subseteq f_c^{-1}(F)$. Then

$$B_c \cap J \subseteq \bigcup_{G \in \mathcal{G}} f_c^{-1}(G).$$

By another application of Claim 2 it follows that the sets $f_c^{-1}(G)$ are nowhere dense in I_c. So $B_c \cap J$ is a first category subset of J. But on the other hand, $B_c \cap J$ is a dense G_δ-subset of J. This violates the Baire Category Theorem. □

Since a family of pairwise disjoint intervals in $\mathbb{R}$ is countable, there are two intervals K_{c_1} and K_{c_2} with $c_1 \neq c_2$ such that $K = K_{c_1} \cap K_{c_2}$ is an interval. Let $D_i = K \cap f_{c_i}(B_{c_i}),\ i = 1, 2$. In view of Claim 3 the sets D_i are Baire spaces. By Levi's Theorem, D_i consequently contains a dense absolute G_δ-subset (the role of the map f from Levi's Theorem is played by the map $f_{c_i}|B_{c_i} \cap f_{c_i}^{-1}(K)$). This is a contradiction since by assumption $D_1 \cap D_2 = \emptyset$.

So, we proved that $f|B$ is not a one-to-one map using only that an intersection of two sets of type K_c is an interval. But, in fact, there is a set $C_0 \subseteq C$ of the cardinality $\mathfrak{c}$ such that $L = \bigcap\{K_c\colon\ c \in C_0\}$ is an interval. Repeating the previous argument we can find a dense G_δ-subset D of L such that $(f|B)^{-1}(t)$ is infinite for any $t \in D$ (D is contained in the intersection of a countably infinite collection $f_c(B_c)$'s). □

Remark 1. It is easy to see that assuming (MA$+\neg$CH) we can find $t \in I$ such that $(f|B)^{-1}(t)$ is uncountable and being an absolute G_δ-set has cardinality $\mathfrak{c}$. The next example gives us a similar result with no additional set-theoretic assumption.

Theorem 3 (Example 2). *There exist a set $X \subset I^2$ homeomorphic to the rationals $\mathbb{Q}$ and a one-to-one map $f : X \to Y$ onto the set Y of all rationals points of I with the following property*:

If Z is a G_δ-subset of I^2 containing X such that there is an extension $\bar{f} : Z \to I$ of f then there exists a point $t \in I$ such that $|\bar{f}^{-1}(t)| = \mathfrak{c}$.

Proof. Let $p : I^2 \to I$ be the projection onto the first factor. There exists a countable dense set $X \subset I^2$ such that $p|X : X \to Y$ is a one-to-one correspondence. We claim that $f = p|X$ is the desired map. Observe that X is homeomorphic to $\mathbb{Q}$ being a countable space with no isolated points. Let $\bar{f} : Z \to I$ be an extension of f over a G_δ-set $Z \subseteq I^2$. Then clearly $\bar{f} = p|Z$.

Let $W = I^2 \setminus Z$ and let $J \subseteq I$ be some non-empty open interval.

Claim 1. *The set $T^J = \{t \in I\colon\ \{t\} \times J \subseteq W\}$ is of the first category in I.*

Proof. There exists a countable family $\mathcal{F}$ of closed subsets of I^2 such that $W = \bigcup \mathcal{F}$. Let $\mathcal{E}$ be the collection of all closed subintervals of J with rational endpoints. If $t \in T^J$ then $\{t\} \times J \subset \bigcup \mathcal{F}$ and hence the Baire Category Theorem implies that for some $E \in \mathcal{E}$ and for some $F \in \mathcal{F}$ we have $\{t\} \times E \subset F$.

For $E \in \mathcal{E}$ and $F \in \mathcal{F}$ put

$$A(E, F) = \bigl\{t \in I\colon \{t\} \times E \subset F\bigr\}.$$

Since E and F are closed, it is clear that $A(E, F)$ is closed. In addition, if $A(E, F)$ contains an interval then F contains a product of two intervals, which is a contradiction since $\dim F \leqslant 1$. So $A = \bigcup\{A(E, F)\colon E \in \mathcal{E}, F \in \mathcal{F}\}$ is of the first category in I. But since $T^J \subset A$, the same applies to T^J. □

Let $\mathcal{J}$ be the family of all non-empty open intervals in I with rational endpoints.

Claim 2. *The set $T = \bigcup\{T^J\colon J \in \mathcal{J}\}$ is of the first category in I.*

Proof. This is clear from Claim 1. □

Now let $t \in I \setminus T$. Then $Z \cap (\{t\} \times I)$ is a dense G_δ-set in $\{t\} \times I$. It consequently has cardinality $\mathfrak{c}$. But $Z \cap (\{t\} \times I) = \bar{f}^{-1}(t)$, and so we are done. □

Corollary 3. *There exist a zero-dimensional σ-compact subset $A \subset C \times I$ and a dense G_δ-subset $\mathcal{H} \subset \mathcal{C}(C \times I, I)$ such that for each $f \in \mathcal{H}$ the map $f|A : A \to I$ is a one-to-one map. But there does not exist a map $g : C \times I \to I$ such that $g|A$ is an embedding.*

Proof. We let A be the set found in Theorem 2. The first part of the assertion follows from Corollary 2 (for $n = 0$). The second part follows from Theorem 2 and Lavrentieff's Theorem. □

Corollary 4. *There exists a one-to-one map $f : X \to I$ of a countable subset $X \subset I^2$ such that any extension $\bar{f} : P \to I$ of f over a Polish space P containing X has a point whose preimage has cardinality $\mathfrak{c}$.*

Proof. Let X and f be such as in Theorem 3. Assume that there exist an embedding $i : X \to P$ into some Polish space P and a map $\bar{f} : P \to I$ such that $\bar{f} \circ i = f$. By Lavrentieff's Theorem there exist a G_δ-set $Z \subset I^2$ containing X and an embedding $\bar{i} : Z \to P$ extending i. Then $\bar{f} \circ \bar{i}$ is an extension of f over a G_δ-subset of I^2 and therefore has a fibre of cardinality $\mathfrak{c}$. Since $\bar{i}$ is an embedding, this implies that $\bar{f}$ has a fibre of cardinality $\mathfrak{c}$ as well. □

Remark 2. Both sets A from Theorem 2 and X from Theorem 3 are zero-dimensional. So they admit many embeddings into I. But none of those can be extended over $C \times I$ and I^2, respectively.

Let us remark that Lavrentieff's Theorem not only works for homeomorphisms but also for continuous maps. That is, any continuous map into a Polish space can be extended over some G_δ-set.

4. A strong version of the Eilenberg Theorem

In this section we will prove our announced strong version of the Eilenberg Theorem.

Theorem 4. *For every zero-dimensional map $f : X \to Y$ from a σ-compact space X into a space Y with $\dim B_2(f) \leqslant m < \infty$, there exists a family $\{h_1, h_2, \ldots\}$ of maps from X to I such that for all pairwise distinct $j_1, \ldots, j_{m+1}$ in $\mathbb{N}$ the map*

$$f \times h_{j_1} \times \cdots \times h_{j_{m+1}} : X \to Y \times I^{m+1}$$

is one-to-one.

The proof of this result is based on the following lemma.

Lemma 3. *Let $f : X \to Y$ be a map from a σ-compact space X into a space Y, and let $B \subseteq Y$ be a zero-dimensional F_σ-set with $\dim f^{-1}(y) \leqslant 0$ for any $y \in B$. Then there exist a map $h : X \to I$ and a G_δ-subset A of Y with $A \supseteq B$ such that the map $f \times h : X \to Y \times I$ is one-to-one on $f^{-1}(A)$.*

Proof. Let $X = \bigcup_{i=1}^{\infty} X_i$, with X_i compact for every i. The map $f|X_i : X_i \to Y$ is closed. Hence, the set

$$f^{-1}(B) \cap X_i = \bigl(f|X_i\bigr)^{-1}(B)$$

is σ-compact being an F_σ-subset of X_i, and zero-dimensional by Hurewicz's Theorem [4, Theorem 1.12.4] on dimension-lowering mappings. Then in view of the countable sum theorem [4, Theorem 1.3.1], the set $f^{-1}(B)$ is also zero-dimensional. By Corollary 2 there exists a map $h : X \to I$ which is one-to-one on $f^{-1}(B)$.

The set $B_2(f \times h) \subseteq Y \times I$ is an F_σ-set in $(f \times h)(X)$ [4, Lemma 4.3.7]. Consequently, $B_2(f \times h)$ is σ-compact because $(f \times h)(X)$ is. Since $h|f^{-1}(B)$ is a one-to-one map, we have

$$p_1\bigl(B_2(f \times h)\bigr) \cap B = \emptyset,$$

where $p_1 : Y \times I \to Y$ is the projection. Then the set $A = Y \setminus p_1(B_2(f \times h))$ is the required G_δ-set. $\square$

We are now in a position to present the proof of Theorem 4.

Proof of Theorem 4. In [1] it was shown that there exists a family $\mathcal{A} = \{A_1, A_2, \ldots\}$ of zero-dimensional G_δ-subsets of $B_2(f)$ such that for every $F \subset \mathbb{N}$ of cardinality $m + 1$ we have $B_2(f) = \bigcup_{j \in F} A_j$.

By induction on j we shall construct a map $h_j : X \to I$. For every subset $F \subset \mathbb{N} \setminus \{1\}$ of cardinality m put

$$H(F) = B_2(f) \setminus \bigcup_{j \in F} A_j.$$

Since $B_2(f)$ is an F_σ-subset of $f(X)$ [4, Lemma 4.3.7], it follows that $H(F)$ is σ-compact, and is clearly contained in A_1. The set

$$B = \bigcup \{H(F)\colon\ F \subset \mathbb{N}, |F| = m\}$$

consequently is a zero-dimensional σ-compact subset of A_1. So by Lemma 3 there exists a map $h_1 : X \to I$ and a G_δ-set $A \supseteq B$ such that the map $f \times h_1 : X \to Y \times I$ is one-to-one on the set $f^{-1}(A)$.

Let $A'_1 = A_1 \cap A$. It is clear that the collection $\mathcal{A}' = \{A'_1, A_2, A_3, \ldots\}$ has the same property as the original collection $\mathcal{A}$. So now we replace $\mathcal{A}$ by $\mathcal{A}'$ and consider the set A_2 in the second step of the construction. By a similar argument we find a map $h_2 : X \to I$ which is one-to-one on $f^{-1}(A'_2)$, where $A'_2 \subset A_2$ is a G_δ-set such that $\mathcal{A}'' = \{A'_1, A'_2, A_3, A_4, \ldots\}$ still has the same property as the original collection $\mathcal{A}$. Etc.

We claim that the maps h_j are as required. For let $j_1, \ldots, j_{m+1}$ be pairwise distinct elements of $\mathbb{N}$. Take arbitrary distinct elements $x_0, x_1 \in X$. If $f(x_0) \neq f(x_1)$ then there is nothing to prove. So assume that $f(x_0) = f(x_1)$. Since $B_2(f) = \bigcup_{i=1}^{m+1} A'_{j_i}$, for some $i \leqslant m+1$ we have that $f(x_0) = f(x_1) \in A'_{j_i}$. But then since h_{j_i} is one-to-one on $f^{-1}(A'_{j_i})$, it follows that $h_{j_i}(x_0) \neq h_{j_i}(x_1)$, which is clearly as required. $\square$

We let Δ^N denote an arbitrary N-dimensional simplex. Its n-dimensional skeleton will be denoted by Δ^N_n.

Corollary 5. *Let $n \leqslant m \leqslant 2n+1$ and let $f : \Delta^{2n+2}_n \to I^m$ be a zero-dimensional map. Then*

$$\dim B_2(f) \geqslant 2n - m.$$

Proof. Assume $\dim B_2(f) \leqslant 2n - m - 1$ for some zero-dimensional map $f : \Delta^{2n+2}_n \to I^m$. Then by Theorem 4, there exists an embedding $f \times h : \Delta^{2n+2}_n \to I^{m+2n-m-1+1} = I^{2n}$. But in view of the classical theorem by van Kampen–Flores [7,5], the polyhedron Δ^{2n+2}_n is not embeddable in $\mathbb{R}^{2n}$. $\square$

Let us note that for $n \leqslant m \leqslant 2n+1$ any regularly branched map $f : \Delta^{2n+2}_n \to I^m$ is zero-dimensional and so the equality $\dim B_2(f) = 2n - m$ holds.

5. Further remarks

Maps f with $B_{k+1}(f) = \emptyset$, i.e., maps of multiplicity $\leqslant k$, are of special interest. Suppose that X is an n-dimensional compactum and that we are interested in maps from X into $\mathbb{R}^m$. If there is an integer k such that $n+1 \leqslant (m-n)k$ then we are in an especially nice situation. First observe that $n \leqslant m$. So if $f : X \to \mathbb{R}^m$ is regularly branched then

$$\dim B_{k+1}(f) \leqslant (k+1)n - km \leqslant -1,$$

i.e., f is $\leqslant k$-to-one. So Hurewicz's Theorem implies the following

Hurewicz's Corollary. *If n, m, and k are integers such that*

$$n+1 \leqslant (m-n)k$$

then for any n-dimensional compactum X the set

$$\{f \in \mathcal{C}(X,\mathbb{R}^m)\colon\ f \text{ has multiplicity at most } k\}$$

is a dense G_δ-subset of $\mathcal{C}(X,\mathbb{R}^m)$.

This corollary suggests the rather vague question of whether it is possible to "transform" a given map into a map with small fibres. What we mean is described in the hypotheses below of which we remark that so far we were unable to prove nor to disprove them.

Hypothesis 1. For every q-dimensional map $f:X\to Y$ from a compact space X into m-dimensional compact space Y there exists a map $h:X\to I^{m+2q}$ such that $f\times h:X\to Y\times I^{m+2q}$ is a 2–1 map.

Hypothesis 2. For every q-dimensional map $f:X\to Y$ from a compact space X into m-dimensional compact space Y there exists a map $h:X\to I^{m+q+1}$ such that $f\times h:X\to Y\times I^{m+q+1}$ is a $(q+1)$–1 map.

In the remaining part of this section we will describe a natural approach to a possible proof of both hypotheses, and conclude that it leads nowhere. To begin with, let us first prove the following result.

Theorem 5. *Let B be a σ-compact subset of a space X, and let $f:X\to Y$ be a map into a space Y such that* $\dim f(B)\leqslant m<\infty$ *and* $\dim f^{-1}(y)\cap B\leqslant 0$ *for all points $y\in Y$. Then there exists a map $h:X\to I^{m+1}$ such that the map $f\times h:X\to Y\times I^{m+1}$ is one-to-one on B.*

Proof. Let $B=\bigcup_{i=1}^\infty B_i$, where the B_i are compact. Then $f|B_i$ is zero-dimensional and closed. Hence, by Hurewicz's Theorem [4, Theorem 1.12.4],

$$\dim B_i \leqslant \dim f(B_i) + \dim f|B_i \leqslant m.$$

So $\dim B\leqslant m$ in view of the countable sum theorem. We will now prove our assertion by induction on $\dim f(B)$.

If $\dim f(B)=0$, then it suffices to apply Corollary 2 (for $n=0$).

So let $\dim f(B)=m\geqslant 1$ and assume that we have what we want for $m-1$. By Urysohn's Decomposition Theorem (see the proof of [4, Theorem 1.5.7]) there is an F_σ-subset $Y_0\subset f(B)\subset Y$ such that $\dim Y_0=m-1$ and $\dim f(B)\setminus Y_0=0$. The set $B_0=B\cap f^{-1}(Y_0)$ is an F_σ-set. According to our inductive hypothesis there is a map $h_0:X\to I^{\dim f(B_0)+1}\hookrightarrow I^m$ such that $f\times h_0:X\to Y\times I^m$ is one-to-one on B_0.

The set $B_2((f \times h_0)|B)$ is an F_σ-subset of a σ-compact space $(f \times h_0)(B)$. Hence, $B_1 = (f \times h_0)^{-1}(B_2((f \times h_0)|B))$ is a σ-compact subset of B. By the choice of h_0,

$$p_1\big(B_2((f \times h_0)|B)\big) \cap Y_0 = \emptyset,$$

here $p_1 : Y \times I^m \to Y$ is the projection, i.e.,

$$p_1\big(B_2((f \times h_0)|B)\big) = f(B_1) \subseteq f(B) \setminus Y_0.$$

The last inclusion means that $\dim f(B_1) \leqslant 0$. Hence by Corollary 2 there is a map $h_1 : X \to I$ such that the map $f \times h_1$ is one-to-one on B_1. Thus, $h = h_0 \times h_1 : X \to I^m \times I$ is the required map. □

Remark 3. The formulation and the proof of Theorem 5 are similar to those of Theorem 4 and Lemma 3. But it is rather difficult to find a general assertion of which these results are all special cases. So we have preferred to present them separately.

We now quote an interesting result which suggests an approach to a proof of our hypotheses.

Toruńczyk's Theorem [12]. *Let $f : X \to Y$ be a q-dimensional map from a compact space X into a finite-dimensional compact space Y, and let $0 \leqslant l \leqslant q - 1$. Then there is a σ-compact set $C_l \subset X$ such that* $\dim C_l \leqslant l$ *and* $\dim f|X \setminus C_l \leqslant q - l - 1$.

So let us now try to prove our hypotheses and see where we get into troubles. Indeed, let X and Y be compact spaces with $\dim Y = m$ and let $f : X \to Y$ be q-dimensional. From Toruńczyk's Theorem we get a σ-compact set $C_{q-1} \subset X$ such that $\dim C_{q-1} \leqslant q - 1$ and $\dim f|X \setminus C_{q-1} \leqslant 0$. From Tumarkin's Theorem [4, Theorem 1.5.11] we get a G_δ-set $D_{q-1} \supseteq C_{q-1}$ such that $\dim D_{q-1} = \dim C_{q-1} \leqslant q - 1$. We could apply Theorem 5 to the set $B = X \setminus D_{q-1}$. But then in order to complete the proof we need a version of Hurewicz's Theorem (Corollary 2) for the set D_{q-1} which is not σ-compact. So we run into troubles here. We could apply Corollary 2 to the set $B = C_{q-1}$. But then in order to complete the proof we need a version of Theorem 5 for the set $X \setminus C_{q-1}$ which is not σ-compact. So we run into troubles here too.

It seems that there are only two possibilities. Either to enlarge C_{q-1} to an appropriate G_δ or to enlarge $X \setminus D_{q-1}$ to an appropriate G_δ. But the Examples 1 and 2 show that enlarging F_σ-sets to G_δ-sets may increase the sizes of fibres from 1 to infinite, or from 1 to $\mathfrak{c}$. So our approach indeed leads nowhere.

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