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On Dow's solution of Bell's problem

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Abstract

We prove under CH that if B is a Boolean algebra of size at most $\mathfrak{c}$ then there is a subalgebra C of $\mathcal{P}(\omega)$ containing fin such that C/fin is isomorphic to B and C contains no infinite completely separated set. This is a generalization of a result of Dow. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Murray Bell has raised the following problem: if a compact zero-dimensional space has a zero-set which maps onto $\beta\omega$, must the space map onto $\beta\omega$? It will be convenient to translate this problem into Boolean algebraic language. Let B be a Boolean algebra. A subset $A \subseteq B$ is *completely separated* if for each $C \subseteq A$ there exists $d \in B$ such that $c \leqslant d$ for all $c \in C$ and $d \wedge a = 0$ for all $a \in A \setminus C$. If $\mathcal{P}(\omega)$ can be embedded in B then obviously B contains an infinite completely separated subset. Simple examples show that the converse need not be true. Observe that Bell's question in Boolean algebraic language is the following one: if I is a countably generated ideal of a Boolean algebra B, and if B/I contains $\mathcal{P}(\omega)$, must B itself contain $\mathcal{P}(\omega)$? Dow [1] proved that the answer to this question is in the negative under CH. His example is quite complicated and the aim of this note is to present a stronger theorem with a much simpler proof.

Dow [1] also considered the following natural generalization of Bell's question: if I is a countably generated ideal of a Boolean algebra B, and if B/I contains $\mathcal{P}(\omega)$, must B itself contain an infinite completely separated subset? He answered this question in

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the affirmative under OCA, and raised the question whether in this result it suffices to just assume that B/I has an infinite completely separated subset. We leave this question unanswered.

2. The main result

All spaces under discussion are Tychonoff. If X is a space then X^* denotes $\beta X \setminus X$. If X is normal and Y is closed then we identify βY and the closure of Y in βX.

Theorem 2.1. (CH) *There is a continuous surjection $f : \omega^* \to \omega^*$ such that for every non-empty clopen subset $C \subseteq \omega^*$, $f \restriction C$ is not one-to-one.*

Proof. Let X denote $\omega \times 2^\omega$ and let $\pi : X \to \omega$ be the projection. Since π is perfect, its Čech–Stone extension $\beta\pi$ maps X^* onto ω^*. Now let C be an arbitrary non-empty clopen subset of X^*. There is a clopen subset $C' \subseteq \beta X$ such that $C' \cap X^* = C$. There is an infinite subset $E \subseteq \omega$ such that $C' \cap (\{n\} \times 2^\omega) \neq \emptyset$ for every $n \in E$. Since 2^ω has no isolated points, we may pick for every $n \in E$ distinct points $x_n, y_n \in C' \cap (\{n\} \times 2^\omega)$. Put $A = \{x_n \colon n \in E\}$ and $B = \{y_n \colon n \in E\}$, respectively.

Now pick an arbitrary point $p \in E^* \subseteq \omega^*$. Since $\pi[A] = E$ and $\pi[B] = E$ and $\beta\pi$ is closed, there exist points $u \in A^*$ and $v \in B^*$ with $\beta\pi(u) = \beta\pi(v) = p$. Since A and B have disjoint closures in βX and clearly $u, v \in C$ this shows that $\beta\pi \restriction C$ is not one-to-one.

Now since by Parovičenko's Theorem [3] X^* and ω^* are homeomorphic under CH, we are done. □

We now show that this result easily implies our main result.

Theorem 2.2. (CH) *If X is a compact space of weight at most $\mathfrak{c}$ then there is a compactification $\gamma\omega$ of ω such that*

(1) *$\gamma\omega \setminus \omega$ is homeomorphic to X.*

(2) *For every $E \in [\omega]^\omega$ there exist disjoint $E', E'' \in [E]^\omega$ such that $\overline{E'} \cap \overline{E''} \neq \emptyset$.*

Proof. It is well known that there exists a continuous surjection $\xi : \omega^* \to X$ under CH (Parovičenko [3]). Let $\eta : \omega^* \to X$ be the composition $\xi \circ f$, where f is the map in Theorem 2.1. The map η defines a compactification $\gamma\omega$ of ω for which the natural map $g : \beta\omega \to \gamma\omega$ has the property that $g \restriction \omega^* = \eta$. (Formally, $\gamma\omega$ is the adjunction space $\beta\omega \cup_\eta X$, see Dugundji [2, 6.6.1].) We will prove that $\gamma\omega$ is as required. To this end, pick an arbitrary $E \in [\omega]^\omega$. Since $f \restriction E^*$ is not one-to-one, there exist distinct points $x, y \in E^*$ such that $f(x) = f(y)$. Pick disjoint sets $E', E'' \subseteq E$ with $x \in \overline{E'}$ and $y \in \overline{E''}$. Then in $\gamma\omega$, the point $\eta(x) = \eta(y)$ is in the closure of both E' and E''. □

Now let X be a zero-dimensional compact space of weight $\mathfrak{c}$. Then the compactification $\gamma\omega$ we get from Theorem 2.2 is obviously zero-dimensional and its clopen algebra has no infinite completely separated subset. For from such a subset one would get an infinite subset $E \subseteq \omega$ whose closure is βE and that is impossible by Theorem 2.2(2). So if we let X be $\beta\omega$ or $\beta\omega \setminus \omega$, or any compact zero-dimensional F-space of weight $\mathfrak{c}$, then the clopen algebra of $\gamma\omega \setminus \omega$ has an infinite completely separated subset but the clopen algebra of $\gamma\omega$ does not have such a subset.

Let $X = \omega^*$. The compactification $\gamma\omega$ from Theorem 2.2 with X as input has the amusing property that $\gamma\omega \setminus \omega$ is homeomorphic to ω^* yet for every $E \in [\omega]^\omega$ there exist disjoint $E', E'' \in [E]^\omega$ such that $\overline{E'} \cap \overline{E''} \neq \emptyset$.

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