Infinite-Dimensional Topology

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1. Introduction

The aim of this note is to present a survey of the main developments in infinite-dimensional manifold theory that have occurred since we wrote Dijkstra and van Mill [24]. Our focus is on topological vector spaces, function spaces, homotopy dense imbeddings, topological classification of semicontinuous functions and hyperspaces.

Infinite-dimensional topology is the creation of R. D. Anderson (see Anderson [3] for some remarks on the early development of infinite-dimensional topology). Several books were written on the subject, or deal with aspects of infinite-dimensional topology. The highlights of infinite-dimensional topology are the theorems of Anderson [2] on the homeomorphy of ℓ^2 and s, of Chapman [20] on the invariance of Whitehead torsion, of West [50] on the finiteness of homototopy types of compact ANR's and of Toruńczyk [48, 49] on the topological characterization of manifolds modeled on the Hilbert cube and Hilbert space. A large collection of open problems is West's paper [51]. The subjects that are being touched upon there range from absorbing sets and function spaces to ANR-theory.

2. Definitions and basic theory

We recall the basic ideas that play an important role in infinite-dimensional topology.

A subset A of a space X is called *homotopy dense in* X if there is a homotopy $H\colon X \times \mathbb{I} \to X$ such that H_0 is the identity and $H(X \times (0,1]) \subset A$. A closed subset F of a space X is called a *Z-set* if $X \setminus F$ is homotopy dense in X. A closed subset F of an ANR X is called a *strong Z-set* if for each open cover $\mathcal{U}$ of X there is a continuous function $f\colon X \to X$ that is $\mathcal{U}$-close to the identity such that $\mathrm{cl}_X(f(X])) \cap F = \emptyset$. A countable union of (strong) Z-sets is called a (*strong*) *σZ-set*. A space X that can be written $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is a (strong) Z-set in X, is called a (*strong*) *σZ-space*. An imbedding $f\colon X \to Y$ is called a *Z-imbedding* if $f[X]$ is a Z-set in Y.

It is clear that a Z-set is nowhere dense. It is tempting to think that a 'nice' space, e.g., a vector space, which is meager in itself is in fact a σZ-space. If this were true then some proofs in infinite-dimensional topology would be simpler. However, it is *not* true, as was shown by Banakh [5]. His example is the linear span in ℓ^2 of Erdős' space in [35]. It is even absolutely Borel. See also Banakh, Radul and Zarichnyi [7, Theorem 5.5.19] for details.

We will now recall the definition of an absorber after Bestvina and Mogilski [10]. Let $\mathcal{C}$ be a topological class that is closed hereditary. In addition, assume that $\mathcal{C}$ is additive: $A \in \mathcal{C}$ whenever A can be written as a union of two closed subsets that are in $\mathcal{C}$. Important examples of such classes are $\mathcal{M}_\alpha$ and $\mathcal{A}_\alpha$, the multiplicative respectively the additive Borel class of level α, $\alpha < \omega_1$. Let $\mathcal{C}_\sigma$ denote the class of spaces that have a countable closed covering consisting of spaces from $\mathcal{C}$. An AR X is called *$\mathcal{C}$-universal* if for every $A \in \mathcal{C}$ there exists a closed imbedding $g\colon A \to X$. An AR X is called *strongly $\mathcal{C}$-universal* if for every $A \in \mathcal{C}$ and every map $f\colon A \to X$ that restricts to a Z-imbedding on a closed set $K \subset A$ there exists a Z-imbedding $g\colon A \to X$ that can be chosen arbitrarily close to f with the property $g|K = f|K$. The AR X is called a $\mathcal{C}$-*absorber* if

1. X is a strong σZ-space,
2. $X \in \mathcal{C}_\sigma$,
3. X is strongly $\mathcal{C}$-universal.

Let us call a $\mathcal{C}$-absorber X a *standard* $\mathcal{C}$-*absorber* if X is a homotopy dense subspace of Hilbert space.

Bestvina and Mogilski proved the Uniqueness Theorem for absorbers:

2.1. THEOREM. *If there exists a standard $\mathcal{C}$-absorber and the spaces X and Y are both $\mathcal{C}$-absorbers then X and Y are homeomorphic.*

If we combine this theorem with Theorem 5.2 of BANAKH [4] then we get an improved Uniqueness Theorem:

2.2. THEOREM. *If the spaces X and Y are both $\mathcal{C}$-absorbers then X and Y are homeomorphic.*

Bestvina and Mogilski also show that there exists a standard absorber for every Borel class. Let us denote by Ω_α the standard $\mathcal{M}_\alpha$-absorber and by Λ_α the standard $\mathcal{A}_\alpha$-absorber.

The Hilbert cube Q is the product space $\mathbb{I}^\omega$, where $\mathbb{I} = [0,1]$. The pseudointerior and pseudoboundary of Q are the subspaces $s = (0,1)^\omega$ and $B = Q \setminus s$, respectively. According to ANDERSON [2] s is homeomorphic to the separable Hilbert space ℓ^2. The space B is an important example of an $\mathcal{A}_1$-absorber and B^ω is an example of an $\mathcal{M}_2$-absorber. So B and B^ω are homeomorphic to Λ_1 respectively Ω_2.

Absorbers are generalizations of so-called capsets, which were introduced independently by to ANDERSON [1] and BESSAGA and PEŁCZYŃSKI [9]. The notion of a capset was a fundamental tool in the early days of infinite-dimensional topology for recognizing topological Hilbert spaces.

3. Topological vector spaces

DUGUNDJI proved in [34] that every locally convex vector space is an AR. This raised the question whether the local convexity assumption is essential in this result. This was a formidable open problem for several decades. In [32] DOBROWOLSKI and TORUŃCZYK proved that every separable, infinite-dimensional, complete topological vector space that is an AR is homeomorphic to Hilbert space. This result gave additional importance to finding an answer to the above question. R. Cauty answered it in the negative:

3.1. THEOREM (CAUTY [15]). *There is a separable, topologically complete vector space that is not an* AR.

Cauty's construction proceeds as follows. The complete example is obtained as a completion of a σ-compact and metrizable vector space E. As basis for the construction of E Cauty considers an infinite-dimensional compact metric space

X with the property that it is the cell-like image of a finite-dimensional polyhedron. The existence of such a space follows from DRANISHNIKOV's celebrated construction [33] of an infinite-dimensional space with cohomological dimension three. Algebraically, E is the free vector space over X. The canonical topology on E is the strongest linear topology that induces on X the original topology. This topology, however, is not metrizable. Cauty constructs a weaker topology τ for E that is metrizable and linear and that has the property that it contains an open set U that does not have the homotopy type of a CW-complex, thereby showing that U is not an ANR and hence (E, τ) cannot be an AR.

An immediate corollary of Theorem 3.1 is:

3.2. COROLLARY. *There exists a separable, topologically complete vector space that is not homeomorphic to any convex subset of a locally convex vector space.*

Bessaga and Dobrowolski proved the following positive result in this direction.

3.3. THEOREM (BESSAGA and DOBROWOLSKI [8]). *Every locally convex σ-compact metric vector space is homeomorphic to a pre-Hilbert space.*

This result suggested the possibility of simplifying the (difficult) classification problem of incomplete locally convex vector spaces by considering only linear subspaces of Hilbert space. Recall that by the Anderson-Kadeč-Toruńczyk Theorem (see TORUŃCZYK [49]) complete locally convex metric spaces are characterized by their weight. For incomplete spaces, however, Marciszewski found the following obstructions.

3.4. THEOREM (MARCISZEWSKI [44]). *There exists a separable, normed vector space that is not homeomorphic to any convex subset of Hilbert space.*

3.5. THEOREM (MARCISZEWSKI [44]). *There exists a separable, locally convex metric vector space that is not homeomorphic to any convex subset of a normed vector space.*

Marciszewski's counterexamples are constructed by transfinite induction and the method of "killing homeomorphisms" that was invented by SIERPIŃSKI [47]. It is unknown whether there are such examples that are absolute Borel sets.

Even the classification problem for σ-compact pre-Hilbert spaces appears difficult as the following result shows.

3.6. THEOREM (CAUTY [14]). *There exist a continuum of σ-compact pre-Hilbert spaces such that no two of them have a continuous injection between them.*

Let X be an ANR. It is easy to prove that for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for every space Y, any two $\mathcal{V}$-close maps $f, g\colon Y \to X$ are $\mathcal{U}$-homotopic. It is a natural problem whether this property of ANR's in fact characterizes the class of all ANR's. This was also a difficult and fundamental problem which remained unanswered for decades. Cauty's example in Theorem 3.1 also solves this problem in the negative. This is because in every vector space close maps can be connected by small homotopies.

To see this, let L be a topological vector space. In addition, let $\mathcal{U}$ be an open cover of L. The function $\lambda\colon L \times L \times \mathbb{I} \to L$ defined by

$$\lambda(x, y, t) = (1 - t) \cdot x + t \cdot y,$$

is defined in terms of the algebraic operations on L and is therefore continuous. For every $x \in L$ pick an element $U_x \in \mathcal{U}$ containing x. Since λ is continuous and $\lambda(\{x\} \times \{x\} \times \mathbb{I}) = \{x\}$, by compactness of $\mathbb{I}$ there exists for every $x \in X$ a neighborhood V_x of x such that $\lambda(V_x \times V_x \times \mathbb{I}) \subseteq U_x$. Put $\mathcal{V} = \{V_x : x \in L\}$. We claim that $\mathcal{V}$ is as required. To this end, let X be a space and let $f, g\colon X \to L$ be continuous $\mathcal{V}$-close functions. Define a homotopy $H\colon X \times \mathbb{I} \to L$ in the obvious way by the formula

$$H(x,t) = (1-t) \cdot f(x) + t \cdot g(x).$$

Then clearly $H_0 = f$ and $H_1 = g$. Fix an arbitrary $x \in X$. Since f and g are $\mathcal{V}$-close, there exists an element $p \in L$ such that $f(x), g(x) \in V_p$. But then

$$\{f(x)\} \times \{g(x)\} \times \mathbb{I} \subseteq V_p \times V_p \times \mathbb{I}$$

from which it follows that $H(x,t) = \lambda(f(x), g(x), t) \in U_p$ for every $t \in \mathbb{I}$. So this indeed proves that f and g are $\mathcal{U}$-homotopic.

The classical (Brouwer)-Schauder-Tychonoff Theorem states that every convex compactum in a locally convex vector space has the fixed point property. Schauder's unsupported claim that this theorem is valid in any metric vector space lead to the formulation of the Schauder Conjecture, which states that every convex compactum in a topological vector space should have the fixed point property. Theorem 3.1 shows that the Schauder Conjecture is a substantially stronger statement than the Schauder-Tychonoff Theorem. Recently, however, Cauty also proved the Schauder conjecture.

3.7. Theorem (Cauty [18]). *Every compact, convex subset of a topological vector space has the fixed point property.*

Cauty's proof is very interesting. For a compact space X, he first considers the space $P(X)$ of probability measures on X with finite support, and let $P_n(X)$ be the subspace of $P(X)$ consisting of those measures whose support has at most n elements. The spaces $P_n(X)$ have a natural compact topology, and the topology on $P(X)$ is just the inductive limit topology induced by the sequence

$$P_1(X) \subset P_2(X) \subset \cdots \subset P_n(X) \subset \cdots;$$

that is, $U \subset P(X)$ is open if and only if $U \cap P_n(X)$ is open in $P_n(X)$ for every n. It is clear that we may identify $P_1(X)$ and X. Cauty proves the following surprising result:

3.8. Theorem. *Let X be a compact space. Every continuous function $f\colon P(X) \to X$ has a fixed point, i.e., there is an element $x \in X$ such that $f(x) = x$.*

To see that this proves Theorem 3.7, consider a compact convex subset C of some vector space L, and let $f\colon C \to C$ be continuous. It is clear that f can be extended to a continuous function $\bar{f}\colon P(C) \to C$. Hence by Theorem 3.8, f has indeed a fixed point.

For a compact metrizable space X, let $E(X)$ be the free topological vector space over X, and let $\mathcal{T}(X)$ be the collection of all metrizable vector space topologies on $E(X)$ which are finer that the (nonmetrizable) free topology on $E(X)$. Observe that $P(X)$ is homeomorphic to a closed convex subspace of $E(X)$.

If X and Y are compact and $f: X \to Y$ is continuous then $\hat{f}: P(X) \to P(Y)$ is the natural continuous extension of f.

Observe that no metrizability is assumed in Theorem 3.7. So Cauty first reduces Theorem 3.7 to the metrizable case. Then he proceeds to prove the following result, which is the central element in his construction.

3.9. THEOREM. *Let X be a compact metrizable space. Then there are a compact metrizable space Z and a continuous function $\varphi: Z \to X$ such that*

(1) *Z is countable dimensional,*

(2) *If $\tau \in \mathcal{T}(X)$ and $\tau' \in \mathcal{T}(Z)$ are such that*

$$\hat{\varphi}: \bigl(P(Z), \tau'\bigr) \to \bigl(P(X), \tau\bigr)$$

is continuous, then for every τ-open cover $\mathcal{U}$ of $P(X)$ and every countable locally finite simplicial complex N and every continuous function $\xi: N \to X$ there is a continuous function $\eta: N \to \bigl(P(Z), \tau'\bigr)$ such that $\hat{\varphi} \circ \xi$ is $\mathcal{U}$-close to ξ and $\eta(N) \cup P_2(Z)$ is τ'-compact.

To see that this result implies Theorem 3.8, striving for a contradiction, assume that there are a compact metrizable space X and a continuous function $f: P(X) \to X$ without fixed point. Let Z and φ be as in Theorem 3.9 for X. It is not difficult to see that there are topologies $\tau \in \mathcal{T}(X)$ and $\tau' \in \mathcal{T}(Z)$ such that the functions

$$f: \bigl(P(X), \tau\bigr) \to X \quad \text{and} \quad \hat{\varphi}: \bigl(P(Z), \tau'\bigr) \to \bigl(P(X), \tau\bigr)$$

are continuous. There is a τ-open cover $\mathcal{U}$ of $P(X)$ such that

$$U \cap f(U) = \emptyset \tag{1}$$

for every $U \in \mathcal{U}$. Let $\mathcal{V}$ be a τ-open cover of $P(X)$ which is a star-refinement of $\mathcal{U}$. It is not difficult to see that $\bigl(P(Z), \tau'\bigr)$ is countable dimensional, hence it is an AR by a result of Gresham [40]. Since $\bigl(P(Z), \tau'\bigr)$ is separable, there consequently are a countable locally finite simplicial complex N and continuous functions

$$\mu: \bigl(P(Z), \tau'\bigr) \to N, \quad \xi: N \to \bigl(P(Z), \tau'\bigr)$$

such that

$$\xi \circ \mu: \bigl(P(Z), \tau'\bigr) \to \bigl(P(Z), \tau'\bigr)$$

is $\hat{\varphi}^{-1}[\mathcal{V}]$-close to the identity on $P(Z)$. The function $f \circ \hat{\varphi} \circ \xi: N \to X$ is continuous. By (ii) of Theorem 3.9 there is a continuous function $\eta: N \to \bigl(P(Z), \tau'\bigr)$ such that $\hat{\varphi} \circ \eta$ and $f \circ \hat{\varphi} \circ \xi$ are $\mathcal{V}$-close, while moreover $\eta(N) \cup P_2(Z)$ is τ'-compact. Put $h = \eta \circ \mu$. Then h is a continuous function from $\bigl(P(Z), \tau'\bigr)$ into itself, the range of which has compact closure. Since $\bigl(P(Z), \tau'\bigr)$ is an AR, the function h has a fixed point, say x_0. There is an element V_1 of $\mathcal{V}$ containing the points

$$\hat{\varphi}(x_0) = \hat{\varphi} \circ \eta \circ \mu(x_0), \quad f \circ \hat{\varphi} \circ \xi \circ \mu(x_0).$$

There is also an element V_2 of $\mathcal{V}$ containing the points

$$\hat{\varphi}(x_0), \quad \hat{\varphi} \circ \xi \circ \mu(x_0).$$

Since $\hat{\varphi}(x_0) \in V_1 \cap V_2$ there consequently is an element U_0 of $\mathcal{U}$ which contains the points
$$\hat{\varphi} \circ \xi \circ \mu(x_0), \quad f \circ \hat{\varphi} \circ \xi \circ \mu(x_0).$$
But this contradicts (1).

This most fundamental open problem in this area now seems to be the question whether every *compact* convex subset of a metrizable vector space is an AR.

4. Function spaces

We first consider the function spaces $C_{\mathrm{p}}(X)$, that is the space of all real-valued continuous function on a Tychonoff space X and equipped with the topology of point-wise convergence. Since we are interested in metric spaces we will restrict our attention to spaces X that are countable. The main problem in this field is the topological classification of all such spaces $C_{\mathrm{p}}(X)$ that are Borel. There are many examples of spaces X such that $C_{\mathrm{p}}(X) \in \mathcal{M}_2$, for instance all metric spaces X have this property. In [23] Dijkstra, Grilliot, Lutzer, and van Mill showed that $C_{\mathrm{p}}(X) \in \mathcal{A}_2$ implies that X is discrete. The following result was a major step forward.

4.1. Theorem (Dobrowolski, Marciszewski, and Mogilski [29]). *If X is a non-discrete countable space with $C_{\mathrm{p}}(X) \in \mathcal{M}_2$ then $C_{\mathrm{p}}(X)$ is an $\mathcal{M}_2$-absorber and hence homeomorphic to Ω_2.*

This result prompted Dobrowolski et al. to conjecture that every $C_{\mathrm{p}}(X)$ that is Borel should be the absorber of the exact Borel class to which it belongs, which would imply by the Uniqueness Theorem that $C_{\mathrm{p}}(X)$ is topologically characterized by its Borel complexity. Further supporting evidence for this conjecture was supplied by the following results.

4.2. Theorem (Cauty, Dobrowolski, and Marciszewski [19]). *If $C_{\mathrm{p}}(X)$ is Borel then it belongs to $\mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$ for some $\alpha \geq 2$, provided that X is not discrete.*

4.3. Theorem (Cauty, Dobrowolski, and Marciszewski [19]). *For each $\alpha \geq 2$ there exists a countable space X_α such that $C_{\mathrm{p}}(X_\alpha)$ is an $\mathcal{M}_\alpha$-absorber and hence homeomorphic to Ω_α.*

A major and surprising break through was Cauty's proof that the conjecture is false.

4.4. Theorem (Cauty [16]). *For each $\alpha > 2$ there exists a countable space Y_α such that $C_{\mathrm{p}}(Y_\alpha) \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$ and yet $C_{\mathrm{p}}(Y_\alpha)$ is not an $\mathcal{M}_\alpha$-absorber.*

In fact, the construction is such that the space $C_{\mathrm{p}}(Y_\alpha)$ does not even contain a closed copy of Λ_2, the $\mathcal{A}_2$-absorber. The spaces Y_α were actually constructed by Lutzer, van Mill, and Pol [42] to show that there exist spaces $C_{\mathrm{p}}(X)$ of arbitrarily high Borel complexity. We let T_n be the set of functions from $\{0, 1, \ldots, n-1\}$ to $\{0, 1\}$ and define the countable set $T = \bigcup_{n=1}^{\infty} T_n$. If x is an element of the Cantor set 2^ω then $x|n \in T_n$ denotes the restriction of x to the domain $\{0, 1, \ldots, n-1\}$. Let $A_\alpha \subset 2^\omega$ be an element of $\mathcal{M}_\alpha \setminus A_\alpha$ and consider the filter $\mathcal{F}_\alpha$ on T that is generated by the co-finite sets and all sets of the form

$\bigcup_{n=1}^{\infty} T_n \setminus \{x|n\}$, where $x \in A_\alpha$. The space Y_α is $T \cap \{\infty\}$ where all points of T are isolated and the neighbourhoods of ∞ are the sets $F \cap \{\infty\}$ for $F \in \mathcal{F}_\alpha$. According to CALBRIX [12] $C_{\mathrm{p}}(Y_\alpha)$ is also in $\mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$.

Let us define

$$s_\alpha = \{f \in \mathbb{R}^T : f|F = 0 \text{ for some } F \in \mathcal{F}_\alpha.\}$$

It is not hard to see that $C_{\mathrm{p}}(Y_\alpha)$ is homeomorphic to a closed subset of $(s_\alpha)^\omega$ so if $C_{\mathrm{p}}(Y_\alpha)$ is $\mathcal{A}_2$-universal then so is $(s_\alpha)^\omega$. But then, according to BANAKH and CAUTY [6], the pair $((\mathbb{R}^T)^\omega, (s_\alpha)^\omega)$ is $(\mathcal{M}_0, \mathcal{A}_2)$-universal, that is for each compactum K and subset C of K with $C \in \mathcal{A}_2$ there is an imbedding φ of K into $(\mathbb{R}^T)^\omega$ such that $\varphi^{-1}[(s_\alpha)^\omega] = C$.

Let $W(Q, s)$ be the subset of Q^ω consisting of all sequences chosen from Q such that all but finitely many elements are in the pseudo-interior s. It is obvious that $W(Q, s)$ is in $\mathcal{A}_2$. The desired contradiction is obtained by Cauty via the following lemma, the proof of which occupies essentially the entire paper [16] and makes extensive use of Homology Theory.

4.5. LEMMA. *For each α there is no continuous function $\varphi: Q^\omega \to (\mathbb{R}^T)^\omega$ such that $\varphi^{-1}[(s_\alpha)^\omega] = W(Q, s)$.*

We now turn beyond Borel to the classes of analytic and co-analytic spaces. The following results also exclude a simple answer to the classification problem in these classes.

4.6. THEOREM (MARCISZEWSKI [43]). *Under $V = L$ there exist countable spaces X and Y such that $C_{\mathrm{p}}(X)$ and $C_{\mathrm{p}}(Y)$ are non-homeomorphic spaces that are both analytic but not co-analytic.*

4.7. THEOREM (MARCISZEWSKI [43]). *Under $V = L$ there exist countable spaces X and Y such that $C_{\mathrm{p}}(X)$ and $C_{\mathrm{p}}(Y)$ are non-homeomorphic spaces that are both co-analytic but not analytic.*

We now consider the space $\mathcal{C}$ of continuous real-valued functions on the interval $\mathbb{I}$ with the topology of uniform convergence. Let $\mathcal{D}$ and $\mathcal{D}^*$ stand for the subspaces of $\mathcal{C}$ consisting of all differentiable functions respectively all function that are differentiable in at least one point.

4.8. THEOREM (CAUTY [13]). *$\mathcal{D}$ and $\mathcal{D}^*$ are absorbers for the co-analytic respectively analytic classes.*

5. Homotopy dense imbeddings

The basic theorem concerning dense imbeddings reads as follows.

5.1. THEOREM (BOWERS [11]). *A separable metric space admits a dense imbedding in Hilbert space if and only if it is nowhere locally compact.*

A space X is said to have the *strong discrete approximation property* (SDAP) if for every sequence of continuous maps $f_1, f_2, \ldots : Q \to X$ and every open cover $\mathcal{U}$ of X there exists another sequence of continuous maps $g_1, g_2, \ldots : Q \to X$ such that each g_i is $\mathcal{U}$-close to f_i and the images of the g_i's form a discrete collection in X.

This concept was introduced by TORUŃCZYK [49] for the purpose of characterizing Hilbert space as the only separable complete metric AR with the SDAP. An imbedding $f\colon X \to Y$ is called *homotopy dense* if $f[X]$ is homotopy dense in Y. The following theorem gives an internal characterization of the homotopy dense subspaces of Hilbert space.

5.2. THEOREM (BANAKH [7, 4]). *A separable metric space admits a homotopy dense imbedding in Hilbert space if and only if it is an* AR *with the* SDAP*.*

A short proof for this theorem can be found in DOBROWOLSKI [28]. Since BESTVINA and MOGILSKI [10] have shown that every strong σZ-space has the SDAP an interesting consequence of Theorem 5.2 is that every absorber is imbeddable as a standard absorber and hence Theorem 2.1 improves to Theorem 2.2.

A different approach to homotopy dense imbeddings was taken by CHAPMAN and SIEBENMANN [21] who introduced the concept of a Z-compactification as the natural infinite-dimensional extension of adding a boundary to a finite-dimensional open manifold, which was the subject of SIEBENMANN's famous thesis [46]. A *Z-compactification* Y of a (locally compact) space X is a compact metric space containing X such that $Y \setminus X$ is a Z-set in Y. So a locally compact space is Z-compactifiable if and only if it admits a homotopy dense imbedding into some compact space. Model examples are for instance the case that X is the interior of a topological manifold Y or that X is the complement of an endface in the Hilbert cube.

In [21] CHAPMAN and SIEBENMANN presented criteria for a Hilbert cube manifold X to be Z-compactifiable. Formulated in geometric terms, their result is that X admits a Z-compactification if and only if X is homeomorphic to the product of an inverse mapping telescope with the Hilbert cube. Chapman and Siebenmann were not able to decide whether their characterization can be extended beyond Hilbert cube manifolds to all locally compact ANR's. The existence of that extension depended on an answer to the following question, which was posed in the paper: if $X \times Q$ is Z-compactifiable is X itself Z-compactifiable?

Guilbault answered this question in the negative:

5.3. THEOREM (GUILBAULT [41]). *There exists a locally compact* 2*-dimensional polyhedron* X *that is not Z-compactifiable but such that* $X \times Q$ *has a Z-compactification.*

Surprisingly, the construction of the example X is not complicated. X is the infinite mapping telescope of a direct sequence $S^1 \xrightarrow{\theta} S^1 \xrightarrow{\theta} S^1 \xrightarrow{\theta} \dots$ where θ is a degree 1 map which wraps the circle around itself twice counterclockwise, then once back in the clockwise direction. The fact that $X \times Q$ is Z-compactifiable follows easily from the characterization of Chapman and Siebenmann or by observing that CHAPMAN's characterization of simple homotopy equivalence [20] implies that $X \times Q$ is homeomorphic to $(S^1 \times Q) \times [0, \infty)$. The proof that X does not admit Z-compactifications, however, is very lengthy and involved.

Although Chapman and Siebenmann's question about Z-compactifications has its origin firmly in Hilbert cube manifold theory, Ferry showed recently that this problem is finite-dimensional rather than infinite-dimensional in nature:

5.4. Theorem (Ferry [36]). *If an n-dimensional polyhedron X is such that $X \times Q$ is Z-compactifiable then $X \times \mathbb{I}^{2n+5}$ is also Z-compactifiable.*

6. Topological classification of semicontinuous functions

The primary focus of this research concerns the question whether certain semicontinuous functions of analytic origin that are defined on Hilbert space are topologically indistinguishable. Interesting examples of such functions are the p-norm on the topological Hilbert space $s = \mathbb{R}^{\mathbb{N}}$:

$$|x|_p = \begin{cases} \sqrt[p]{\sum_{n=1}^{\infty} |x_n|^p} & \text{if } p < \infty \\ \sup\{|x_n| : n \in \mathbb{N}\} & \text{if } p = \infty. \end{cases}$$

Because s carries the topology of point-wise convergence these functions are lower semicontinuous but not continuous. In fact, according to van Mill and Pol [45] these functions are in a sense universal for all lower semicontinuous functions and they are not even countably continuous, that is their domain cannot be partitioned into countably many sets such that the restrictions are continuous.

If X is a (real) topological vector space endowed with the continuous norms $\|\cdot\|$ and $|\cdot|$, respectively, then there is a norm preserving homeomorphism $f\colon (X, \|\cdot\|) \to (X, |\cdot|)$ defined by $f(\mathbf{0}) = \mathbf{0}$ and

$$f(x) = \frac{\|x\|}{|x|} x$$

if $x \neq \mathbf{0}$. Observe that such a homeomorphism is in general not linear. Consider for example $\mathbb{R}^2$ endowed with the Euclidean norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ and the max norm $|x| = \max\{x_1, x_2\}$. So a norm preserving homeomorphism sends the unit ball $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ onto the unit brick $[-1, 1]^2$ and consequently changes the shape of a geometric object considerably.

These considerations for continuous norms are not very interesting and the question naturally arises whether something can be said in the case of discontinuous norms. All norms on finite-dimensional vector spaces are continuous, so the question only makes sense within the framework of infinite-dimensional spaces. If X is an infinite-dimensional vector space then it can be endowed with several discontinuous norms. This leads us to consideration of the well-known p-norms from the Banach spaces ℓ^p in combination with the topology of point-wise convergence.

By means of the Bing Shrinking Criterion the authors proved that all the p-norms are topologically indistinguishable:

6.1. Theorem (Dijkstra and van Mill [25]). *For every $p \in (0, \infty)$ there exists a homeomorphism $h\colon s \to s$ such that $|h(x)|_p = |x|_\infty$ for every $x \in s$.*

Sketch of Proof: For $p, q \in (0, \infty)$ it is easy to construct homeomorphisms $H_q^p\colon s \to s$ that are norm preserving, that is $|H_q^p(x)|_p = |x|_q$ for all $x \in s$. Let for $p \in (0, \infty)$ and $q \in (0, \infty]$ the map $H_q^p\colon s \to s$ be defined by the property that for each $x, y \in s$ and $n \in \mathbb{N}$ with $H_q^p(x) = y$ we have

$$y_n = \operatorname{sgn}(x_n) \sqrt[p]{|\xi_n(x)|_q^p - |\xi_{n-1}(x)|_q^p},$$

where $\xi_n(x) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ and $\mathrm{sgn}(x_n)$ is the sign of the number. Note that this definition also works in the case that $q = \infty$. However, H^p_∞ is never a homeomorphism but it is a norm-preserving cell-like surjection. The idea of the proof is to take a representative H^1_∞ and to show that this map is shrinkable by homeomorphisms of s that preserve the $|\cdot|_\infty$ norm. Then, according to Bing, H^1_∞ can be approximated by norm-preserving homeomorphisms.

The three figures show the shrinking process in (considerably) simplified form. The first figure shows the unit sphere with respect to the sup norm in the first octant of the first three dimensions with the fibres of the map H^1_∞ indicated by solid lines and a shaded region.

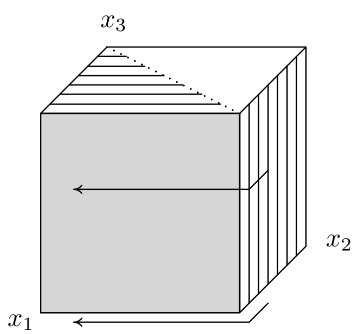

Figure 1

Assuming that the fibres need to be shrunk to a constant size ε the transition from Figure 1 to Figure 2 indicates how the projections of the fibres onto the x_1x_2-plane are shrunk by a rotational move in the x_1x_2-plane that does not involve x_3.

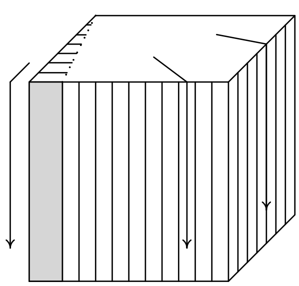

Figure 2

This operation is then followed by a similar move in the planes that contain the x_3-axis, as illustrated by the transition from Figure 2 to Figure 3. The result is that the projections onto the first three dimensions of all fibres have been shrunk to size ε. This process can be continued. If we would be working in the Hilbert cube Q then the process could stop once the length of the n-th coordinate dips below ε. It is not desirable to let the process run through infinitely many coordinates since the norm is not continuous and so limits in general do not preserve norm.

We are, however, working in a highly noncompact space, Hilbert space. This means that we have to work with ε-functions rather than constant ε's. Most of the effort in the paper [25] goes towards dealing with the tension between this requirement and the rigidity that is caused by the need to preserve the norms of vectors. Also, the use of ε-functions means that the shrinking homeomorphisms are obtained as limits of infinite sequences of homeomorphisms which are kept norm-preserving by making sure that every individual vector is moved only a finite number of times.

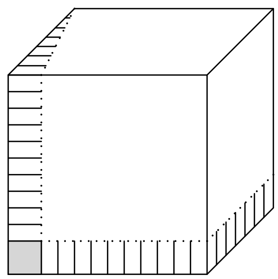

Figure 3

7. Hyperspaces of Peano continua

If X is a compact metric space, then 2^X denotes the hyperspace consisting of all non-empty closed subsets of X, endowed with the Hausdorff metric. $C(X)$ denotes the compact subspace of 2^X consisting of all subcontinua of X. The fundamental theorems are by Curtis and Schori [22]: 2^X is homeomorphic to Q if and only if X is a non-degenerate Peano continuum and $C(X)$ is homeomorphic to Q if and only if X is a non-degenerate Peano continuum without free arcs. For $k \in \{0,1,2,\ldots\}$ we let $\mathrm{Dim}_{\geq k}(X)$ denote the subspace consisting of all $\geq k$-dimensional elements of 2^X and we put $\mathrm{Dim}_\infty(X) = \bigcap_{k=0}^{\infty} \mathrm{Dim}_{\geq k}(X)$.

7.1. Theorem (Dijkstra, van Mill, and Mogilski, [26]). *There exists a homeomorphism α from 2^Q onto Q^ω such that for every $k \in \{0,1,2,\ldots\}$,*

$$\alpha(\mathrm{Dim}_{\geq k}(Q)) = \underbrace{B \times \ldots \times B}_{k \text{ times}} \times Q \times Q \times \cdots$$

and hence $\mathrm{Dim}_\infty(Q)$ is an $\mathcal{M}_2$-aborber and homeomorphic to B^ω and Ω_2.

The proof of this theorem is based on the technique of absorbing systems, which was introduced in the papers [26, 27]. Subsequently, several authors generalized Theorem 7.1 in different directions. Gladdines [37] proved that the theorem remains valid when we consider the sequences $\mathrm{Dim}_{\geq k}(X)$ and $\mathrm{Dim}_{\geq k+1}(X) \cap C(X)$ for X an countable infinite product of Peano continua instead of Q. Dobrowolski and Rubin [30] show that in Theorem 7.1 the covering dimension may be replaced by cohomological dimension. In addition, Gladdines and van Mill [38] give an example that shows that the theorem is not valid for all everywhere infinite dimensional Peano continua. The final word on this subject was spoken by Cauty:

7.2. Theorem (Cauty [17]). *A Peano continuum X has the property that every nonempty open subset contains compacta of arbitrarily high finite dimension if and only if here exists a homeomorphism α from 2^X onto Q^ω such that for every*

$k \in \{0, 1, 2, \ldots\}$,

$$\alpha(\mathrm{Dim}_{\geq k}(X)) = \underbrace{B \times \ldots \times B}_{k \text{ times}} \times Q \times Q \times \cdots .$$

This result remains valid if we consider $C(X)$ instead of 2^X and also if we replace covering dimension by cohomological dimension.

Gladdines and van Mill have also considered the space $L(X) \subset C(X)$ consisting of all Peano continua in X:

7.3. Theorem (Gladdines and van Mill [39]). *If $n \geq 3$ then $L(\mathbb{I}^n)$ is an $\mathcal{M}_2$-absorber and hence homeomorphic to Ω_2.*

Continuing in this direction Dobrowolski and Rubin found:

7.4. Theorem (Dobrowolski and Rubin [31]). *If $n \geq 3$ then both $\mathsf{AR}(\mathbb{I}^n)$ and $\mathsf{ANR}(\mathbb{I}^n)$ are $\mathcal{M}_3$-absorbers and hence homeomorphic to Ω_3.*

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