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ON SETS THAT MEET EVERY HYPERPLANE IN n-SPACE IN AT MOST n POINTS

JAN J. DIJKSTRA AND JAN VAN MILL

Abstract

A simple proof that no subset of the plane that meets every line in precisely two points is an F_σ-set in the plane is presented. It was claimed that this result can be generalized for sets that meet every line in either one point or two points. No proof of this assertion is known, however. The main results in this paper form a partial answer to the question of whether the claim is valid. In fact, it is shown that a set that meets every line in the plane in at least one but at most two points must be zero-dimensional, and that if it is σ-compact then it must be a nowhere dense G_δ-set in the plane. Generalizations for similar sets in higher-dimensional Euclidean spaces are also presented.

Let n be a natural number greater than 1, and let V be an n-dimensional real vector space. Let Z be a subset of V. A k-dimensional affine subspace of V is called a *k-plane* in V, and a *hyperplane* is an $(n-1)$-plane in V. The set Z is called an *n-point set in V* if every hyperplane intersects Z in precisely n points, a *weak n-point set* if every hyperplane intersects Z in either $n-1$ or n points, and a *partial n-point set* if every hyperplane intersects Z in at most n points.

Mazurkiewicz [7] proved that $\mathbb{R}^2$ contains a two-point set. His construction can easily be adapted to prove that $\mathbb{R}^n$ contains an n-point set for every $n \geqslant 2$. The question of whether a two-point set can be a Borel set is a long-standing and difficult open problem; see Mauldin [**6**] for details. The descriptive complexity problem for weak two-point sets was raised by Sierpiński [**8**, p. 447].

The following two statements were presented as theorems by Larman [**5**].

CONJECTURE 1. No n-point set is an F_σ-set in $\mathbb{R}^n$.

CONJECTURE 2. No weak 2-point set is an F_σ-set in the plane.

Unfortunately, the proofs were incorrect, as was pointed out by Baston and Bostock [**1**]. In addition, the latter paper gives a rather elaborate proof of Conjecture 1 for the case $n = 2$. We first present a short, elegant proof for Conjecture 1 for general n. We were not able to decide whether Conjecture 2 is valid. However, we obtain some partial results that perhaps suggest that Conjecture 2 is true: weak n-point sets are zero-dimensional and the only candidates for σ-compact weak n-point sets are nowhere dense G_δ-sets. The first of these two results is a generalisation of, and is grounded on, Kulesza's theorem [**3**] that every two-point set is zero-dimensional.

Please note that the extension of Mazurkiewicz's original two-point set concept that we employ in this paper is different from the notion of a planar n-point set found in, for example, [**2**] and [**6**].

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An *arc* is a homeomorphic image of the closed unit interval $I = [0,1]$. It should be noted that [5] does contain a proof of the following useful result.

LEMMA 1. *No n-point set can contain arcs.*

We shall use the standard dot product in $\mathbb{R}^n$ and the standard norm $\|x\| = \sqrt{x \cdot x}$. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\hat{x} = (x_1, \ldots, x_n, 1) \in \mathbb{R}^{n+1}$. If $a \in \mathbb{R}^{n+1}$, then the function $h_a : \mathbb{R}^n \to \mathbb{R}$ is given by $h_a(x) = a \cdot \hat{x} = a_1x_1 + a_2x_2 + \ldots + a_nx_n + a_{n+1}$ for $x \in \mathbb{R}^n$.

Let $\mathbb{P}^n$ be the n-dimensional real projective space, whose topology is obtained by regarding it as a quotient space of $S^n = \{v \in \mathbb{R}^{n+1} : \|v\| = 1\}$ through identification of antipodal points. Let $\mathbb{H}^n$ be the set of all hyperplanes in $\mathbb{R}^n$, and note that every hyperplane equals $\{x \in \mathbb{R}^n : h_a(x) = 0\}$ for some point in $a \in S^n$. Observe that if $a, b \in S^n$, then the equations $h_a(x) = 0$ and $h_b(x) = 0$ are equivalent precisely if $a = \pm b$. So every hyperplane corresponds in a canonical way to a point of $\mathbb{P}^n$. There is one point in $\mathbb{P}^n$, namely $\pm(0, \ldots, 0, 1)$, that does not correspond to a hyperplane because it produces the equation $1 = 0$. So we can topologize $\mathbb{H}^n$ by identifying it with the punctured projective space $\mathbb{P}^n \setminus \{\pm(0, \ldots, 0, 1)\}$.

A *side* of a hyperplane $H \in \mathbb{H}^n$ is a component of $\mathbb{R}^n \setminus H$. If A and B are subsets of $\mathbb{R}^n$, then an $H \in \mathbb{H}^n$ is said to *separate* A and B if A and B are contained in different sides of H.

LEMMA 2. *If A and B are compacta in $\mathbb{R}^n$, then $\{H \in \mathbb{H}^n : H \text{ separates } A \text{ and } B\}$ is an open subset of $\mathbb{H}^n$.*

Proof. Let $H \in \mathbb{H}^n$ be given by $\{x \in \mathbb{R}^n : h_a(x) = 0\}$, where $a \in S^n$. If H separates the compacta A and B, then we may assume that there is some $\varepsilon > 0$ such that $h_a(A) \subset [\varepsilon, \infty)$ and $h_a(B) \subset (-\infty, -\varepsilon]$. If $b \in S^n$, then $|h_b(x) - h_a(x)| = |(b-a) \cdot \hat{x}| \leqslant \|b - a\| \|\hat{x}\|$. Since $A \cup B$ is bounded, it is obvious that there is a neighbourhood U of a such that $|h_b(x) - h_a(x)| < \varepsilon$ for any $b \in U$ and $x \in A \cup B$. So every hyperplane that corresponds to a point in U also separates A from B. □

The cardinality of a set A is denoted by $|A|$. A subset A of $\mathbb{R}^n$ with $|A| \leqslant n+1$ is called *independent* if every affine subspace of $\mathbb{R}^n$ that contains A has dimension at least $|A| - 1$.

LEMMA 3. *If $A \subset X \subset \mathbb{R}^n$ such that $|A| \leqslant n+1 \leqslant |X|$, and every hyperplane that contains A intersects X in precisely n points, then A is independent.*

Proof. Let B be such that $A \subset B \subset X$ and $|B| = n+1$. The set B cannot be contained in a hyperplane. So B is independent, and so is A. □

If a polar coordinate system has been chosen for $\mathbb{R}^2$, then we have a parametrization of $\mathbb{P}^1$ by identifying the space with the circle group $\mathbb{R}/\pi\mathbb{Z}$. Let $\theta : \mathbb{H}^2 \to \mathbb{P}^1$ be the continuous map that assigns to every line its angle of inclination. Let $L(u, v)$ stand for the line in the plane through u and v whenever $u, v \in \mathbb{R}^2$ and $u \neq v$.

The following result proves Conjecture 1 when combined with Lemma 1. After this paper was submitted, we were informed by James Foran that it was known to Frederick Bagemihl before 1970 that 2-point sets cannot be σ-compact. In addition, his proof used the same idea as we employ here.

LEMMA 4. *Every n-point set that is an F_σ-set in $\mathbb{R}^n$ contains arcs.*

and let U be the open subset of $\mathbb{H}^2$ that consists of all the lines that separate A from B (see Lemma 2). Since U is obviously nonempty, we can find an ℓ' in U that meets X. Select a $u_\varepsilon = (c,d) \in \ell' \cap X$. Since ℓ' separates o from both p_1 and q_1, we see that ℓ intersects the lines $L(p,o)$ and $L(q,o)$ between o and p_1, and between o and q_1, respectively. Put $\{p_2\} = \ell' \cap L(p,o)$ and $\{q_2\} = \ell' \cap L(q,o)$. Since X is a partial two-point set, we have $u_\varepsilon \neq p_2$ and $u_\varepsilon \neq q_2$.

The set $\ell' \setminus \{p_2, q_2\}$ has three components: one bounded and two unbounded. We shall show that u_ε lies in the bounded component. Assume that u_ε is in one of the two unbounded components. Then the ray from u_ε towards p_2 is the same as the ray from u_ε towards q_2. Since p_2 and q_2 lie opposite to p and q with respect to o, we can draw a line ℓ'' through u_ε that separates o from p and q, Then ℓ' intersects $\alpha((0,r))$ and $\alpha((r,1))$, and since ℓ' separates A from B we have $u_\varepsilon \notin A$. So $|X \cap \ell''| \geqslant 3$, contradicting the properties of X. We have proved that u_ε lies between p_2 and q_2 on ℓ''. Consider the triangle Δ with vertices o, p_1, and q_1. Since ℓ' separates o from B, we know that u_ε lies in the interior of Δ. So $0 < d < \varepsilon$ and $-db < c < -da$. We may conclude that $\lim_{\varepsilon \to 0} u_\varepsilon = o$.

Consider now the arc $\alpha([0,r])$. Let $s \in (0,r)$, and note that $\alpha(s) \notin L(p,o)$. If $\alpha(s)$ lies on the same side of $L(p,o)$ as q, then we can find a line ℓ'', parallel to $L(p,o)$, that separates $\alpha(s)$ and q from $L(p,o)$. Then ℓ'' intersects $\alpha((0,s))$, $\alpha((s,r))$, and $\alpha((r,1))$, contradicting the assumption that X is a partial two-point set. If $\alpha(s)$ and q lie on opposite sides of $L(p,o)$, then choose a u_ε that is closer to o than $\alpha(s)$ is to the line $L(p,o)$. Let ℓ'' be the line through u_ε that is parallel to $L(p,o)$. Since u_ε and $\alpha(s)$ lie on the same side of $L(p,o)$, we know that ℓ'' separates $\alpha(s)$ from $L(p,o)$. So ℓ'' meets X in a point in $\alpha((0,s))$, a point in $\alpha((s,r))$, and in $u_\varepsilon \notin A$. The proof is complete. □

LEMMA 7. *If X is a one-dimensional finite-point set in $\mathbb{R}^n$ and L is a nontrivial linear subspace, then the orthogonal projection of X into L is a one-dimensional finite-point set in L.*

Proof. It suffices to prove the result for the case where L has codimension one; induction will do the rest. So let $n \geqslant 2$, and let L be an $(n-1)$-dimensional linear subspace. By a suitable choice of coordinates, we arrange that $L = \{0\} \times \mathbb{R}^{n-1}$. We shall identify L with $\mathbb{R}^{n-1}$. Assume that X is a one-dimensional finite-point set in $\mathbb{R}^n$. Let $\xi : \mathbb{R}^n \to L$ be the projection. If H is a hyperplane in L, then $\xi^{-1}(H)$ is a hyperplane in $\mathbb{R}^n$. Consequently, $\xi^{-1}(H) \cap X$ is finite, and so is $H \cap \xi(X)$. So $\xi(X)$ is a finite-point set, and hence $\operatorname{ind} \xi(X) \leqslant 1$.

Assume now that $\xi(X)$ is zero-dimensional. Let $w = (w_1, \ldots, w_n)$ be an arbitrary point in X, and put $v = \xi(w)$. Note that $\xi^{-1}(v) = \mathbb{R} \times \{v\}$ is a line in $\mathbb{R}^n$, and hence is contained in some hyperplane of $\mathbb{R}^n$. So $\xi^{-1}(v) \cap X$ is finite. Consider an arbitrary neighbourhood (a,b) of w_1 such that $([a,b] \times \{v\}) \cap X = \{w\}$. Note that $(\{a\} \times \mathbb{R}^{n-1}) \cap X$ and $(\{b\} \times \mathbb{R}^{n-1}) \cap X$ are finite, so $F = \xi((\{a,b\} \times \mathbb{R}^{n-1}) \cap X)$ is a finite set that does not contain v. Since $\xi(X)$ is zero-dimensional, we can find a closed neighbourhood U of v in L such that $U \cap F = \emptyset$ and the boundary ∂U of U is disjoint from $\xi(X)$. Then $V = [a,b] \times U$ is a neighbourhood of w whose boundary $(\{a,b\} \times U) \cup ([a,b] \times \partial U)$ is disjoint from X. Since V can be made arbitrarily small, we may conclude that $\operatorname{ind} X \leqslant 0$. □

THEOREM 8. *Every weak n-point set is zero-dimensional.*

Proof. Let X be an n-point set that is F_σ in $\mathbb{R}^n$. Select $n-1$ points in X and an $(n-2)$-plane P that contains these points. Choose a coordinate system for $\mathbb{R}^n$ such that P equals $\{(x_1,\dots,x_n) : x_1 = x_2 = 0\}$. We define the continuous map $\alpha : \mathbb{R}^n \setminus P \to \mathbb{P}^1$ by $\alpha(x_1,\dots,x_n) = \theta(L((0,0),(x_1,x_2)))$. We have by Lemma 3 that $|P \cap X| = n-1$ and hence for each hyperplane H that contains P, we have $|H \cap X \setminus P| = 1$. This means that $\alpha|X \setminus P : X \setminus P \to \mathbb{P}^1$ is a continuous bijection. Write $X \setminus P$ as a countable union of compacta $F_1, F_2, \dots$. We have $\mathbb{P}^1 = \bigcup_{i=1}^{\infty} \alpha(F_i)$, and so by the Baire category theorem, some $\alpha(F_i)$ has nonempty interior in the simple closed curve $\mathbb{P}^1$, and therefore $\alpha(F_i)$ contains an arc. Since F_i is compact, $\alpha|F_i$ is an embedding, and hence F_i contains an arc as well. □

Note that we have proved the slightly stronger statement that if X is an F_σ-set in $\mathbb{R}^n$ consisting of at least $n+1$ points, and $A \subset X$ is such that $|A| = n-1$ and every hyperplane that contains A meets X in exactly n points, then X contains arcs.

A space is called *rim-finite* if there is a basis for the topology consisting of sets with finite boundaries. Let 'ind' stand for the small inductive dimension. If X is a subset of a finite-dimensional vector space V, then X is called a *finite-point set in* V if every hyperplane in V meets X in only finitely many points. It is obvious that every finite-point set is rim-finite, and hence at most one-dimensional.

EXAMPLE 1. It is easy to construct a one-dimensional partial n-point set. Define $Z = \{(t, t^2, \dots, t^n) : t \in \mathbb{R}\}$. If H is a hyperplane in $\mathbb{R}^n$, then there exist a point $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ that is not the origin, and a value $b \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^n : \sum_{i=1}^n a_i x_i = b\}$. Intersecting Z with H, we obtain $\sum_{i=1}^n a_i t^i = b$, which is a nontrivial polynomial equation, and hence has at most n solutions. So the curve Z is a partial n-point set.

Kulesza [3] proved that every two-point set is zero-dimensional by establishing the following result.

THEOREM 5. *Every one-dimensional partial* 2*-point set contains arcs.*

Note that a weak two-point set is a partial two-point set that meets every line in the plane. We define an *ultra weak* 2*-point set* X as a partial two-point set such that $\{\ell \in \mathbb{H}^2 : \ell \cap X \neq \emptyset\}$ is dense in $\mathbb{H}^2$.

THEOREM 6. *Every ultra weak* 2*-point set is zero-dimensional.*

Proof. In view of Theorem 5, it suffices to show that such sets contain no arcs. Let X be an ultra weak 2-point set, and let $\alpha : I \to X$ be an embedding. Put $A = \alpha([0,1])$, $p = \alpha(0)$, and $q = \alpha(1)$. Let ℓ be a line that intersects the arc A, and is parallel to $L(p,q)$, with the maximum distance towards $L(p,q)$. Let $o \in \ell \cap A$, and let $r \in (0,1)$ be such that $\alpha(r) = o$. Select an affine coordinate system for the plane such that o is the origin, ℓ is the x-axis, and $L(p,q)$ corresponds to the equation $y = -1$. Note that $A \subset (-\infty, 0] \times \mathbb{R}$, and that we may assume that $p = (a, -1)$ and $q = (b, -1)$ with $a < b$.

Let $\varepsilon > 0$, and consider the points $p_1 = (-\varepsilon a, \varepsilon)$ and $q_1 = (-\varepsilon b, \varepsilon)$, which are the points of intersection of $L(p,o)$ and $L(q,o)$ respectively with the horizontal line $y = \varepsilon$. Let B be the line segment $[-\varepsilon b, -\varepsilon a] \times \{\varepsilon\}$ that has q_1 and p_1 as endpoints, Note that if we substitute $n = 2$ into this proposition, we get Theorem 6 back.

LEMMA 10. *Let X be an ultra weak two-point set such that for each $u \in \mathbb{R}^2$, $\{\theta(\ell) : u \in \ell \in \mathbb{H}^2$ and $\ell \cap X \neq \emptyset\}$ is one-dimensional. If $F_1 \subset F_2 \subset \ldots \subset X$ is a countable covering of X by compacta, then there is a countable closed covering $G_1 \subset G_2 \subset \ldots$ of the plane such that $G_i \cap X = F_i$ for each $i \in \mathbb{N}$.*

Proof. Let D denote the open subset $\{(u, v) : u, v \in \mathbb{R}^2$ and $u \neq v\}$ of $\mathbb{R}^4$. The continuous map $\alpha : D \to \mathbb{P}^1$ is given by the rule $\alpha(u, v) = \theta(L(u, v))$. If M is a metric space, then $\mathscr{K}(M)$ denotes the space of compacta in M equipped with the Hausdorff metric.

Let X be a σ-compact ultra weak two-point set such that for each $u \in \mathbb{R}^2$, $\Theta(u) = \{\theta(\ell) : u \in \ell \in \mathbb{H}^2$ and $\ell \cap X \neq \emptyset\}$ is one-dimensional. Write X as the union of an increasing sequence $F_1 \subset F_2 \subset \ldots$ of compacta. Define, for each $i \in \mathbb{N}$, the map $\beta_i : \mathbb{R}^2 \setminus F_i \to \mathscr{K}(\mathbb{P}^1)$ by $\beta_i(u) = \alpha(\{u\} \times F_i)$. Since the induced map $\alpha : \mathscr{K}(D) \to \mathscr{K}(\mathbb{P}^1)$ is also continuous, we find that every β_i is continuous as well.

Define, for each $i \in \mathbb{N}$, the following subset of $\mathbb{R}^2 \setminus F_i$:

$$G'_i = \{u \in \mathbb{R}^2 \setminus F_i : \beta_i(u) \text{ contains a continuum with diam} \geqslant 1/i\}.$$

Since β_i is continuous, and the limit (in the Hausdorff metric) of a sequence of continua with diameter at least ε is a continuum with diam $\geqslant \varepsilon$, every G'_1 is closed in $\mathbb{R}^2 \setminus F_i$.

Next, we show that G'_i is disjoint from X. Assume that $u \in X \setminus F_i$. Define the continuous surjection $\varphi : F_i \to \beta_i(u)$ by $\varphi(v) = \alpha(u, v)$. If $v_1, v_2 \in F_i$ and $\varphi(v_1) = \varphi(v_2)$, then $L(u, v_1) = L(u, v_2)$, and this line can intersect X in only two points, one of which is $u \notin F_i$. So $v_1 = v_2$, and we may conclude that φ is a bijection. Since F_i is compact, F_i and $\beta_i(u)$ are homeomorphic. So, using Theorem 6, we see that $\beta_i(u)$ is zero-dimensional and cannot contain a nontrivial continuum. We may conclude that $u \notin G'_i$.

We now prove that $\bigcup_{i=1}^{\infty} G'_i = \mathbb{R}^2 \setminus X$. Let $u \in \mathbb{R}^2 \setminus X$. We obviously have $\Theta(u) = \bigcup_{i=1}^{\infty} \beta_i(u)$. Since $\operatorname{ind} \Theta(u) = 1$ and $\Theta(u)$ is a subset of the circle $\mathbb{P}^1$, we know that $\Theta(u)$ contains an arc. So, according to the Baire category theorem, some $\beta_i(u)$ contains an arc A as well. Let $k \in \mathbb{N}$ be such that $k \geqslant i$ and $\operatorname{diam}(A) \geqslant 1/k$. Then $A \subset \beta_i(u) \subset \beta_k(u)$, and hence $u \in G'_k$.

If we define $G_i = G'_i \cup F_i$, with $i \in \mathbb{N}$, then this sequence has the required properties. □

THEOREM 11. *Every weak n-point set that is an F_σ-set is also a G_δ-set in $\mathbb{R}^n$.*

Proof. If X is a σ-compact weak two-point set, then Lemma 10 applies, yielding a closed cover $\{G_i : i \in \mathbb{N}\}$ of the plane such that every $G_i \cap X$ is compact. Then every $G_i \setminus X$ is a closed subset of the open subset $\mathbb{R}^2 \setminus (G_i \cap X)$ of the plane. So every $G_i \setminus X$ is σ-compact, and since the union of these sets is the complement of X, we find that X is a G_δ-set.

Assume that $n \geqslant 3$, and that X is a σ-compact weak n-point set in $\mathbb{R}^n$. Precisely as in the proof of Theorem 8, we can find a hyperplane H, a continuous map $q : \mathbb{R}^n \setminus H \to \mathbb{R}^2$, and a finite subset Θ of $\mathbb{P}^1$ such that $Z = q(X \setminus H)$ is an ultra weak two-point set, $q|X \setminus H$ is one-to-one, and only lines that have their angle of inclination in Θ can miss Z.

Proof. Let X be a one-dimensional weak n-point set in $\mathbb{R}^n$ for $n \geqslant 3$. Select a subset A of X consisting of $n-2$ points. Let L be the $(n-3)$-plane in $\mathbb{R}^n$ with the property $L \cap X = A$ (Lemma 3), and select a coordinate system for $\mathbb{R}^n$ such that L corresponds to $x_1 = x_2 = x_3 = 0$. Let $\xi : \mathbb{R}^n \to \mathbb{R}^3$ be the projection onto the first three coordinates, and let $\mathbf{0}$ be the origin in $\mathbb{R}^3$. Put $Y = \xi(X)$, and note that according to Lemma 7, Y is a one-dimensional finite-point set in $\mathbb{R}^3$.

Consider the plane $P = \{0\} \times \mathbb{R}^2$ in $\mathbb{R}^3$, and note that $P \cap Y$ is finite. Let Θ stand for the finite set $\{\theta(\ell) : \ell \in \mathbb{H}^2 \text{ and } |(\{0\} \times \ell) \cap Y| \geqslant 2\}$. We define the radial projection $p : \mathbb{R}^3 \setminus P \to \mathbb{R}^2$ by $p(x_1, x_2, x_3) = (x_2/x_1, x_3/x_1)$. Note that $(1, p(x))$ is the point of intersection of the line through $\mathbf{0}$ and x and the plane $\{1\} \times \mathbb{R}^2$. Put $H = \xi^{-1}(P)$, $q = p \circ \xi|\mathbb{R}^n \setminus H$, and $Z = p(Y \setminus P) = q(X \setminus H)$.

Note that $q|X \setminus H$ (and hence also $p|Y \setminus P$) is a one-to-one map. If we have two distinct points $u, v \in X \setminus H$ that are mapped by q onto the the same point $z \in Z$, then $\xi(u)$ and $\xi(v)$ both belong to the line $p^{-1}(z) \cup \{\mathbf{0}\}$ in $\mathbb{R}^3$. So $q^{-1}(z) \cup L = \xi^{-1}(p^{-1}(z) \cup \{\mathbf{0}\})$ is an $(n-2)$-plane that contains the n points $A \cup \{u, v\} \subset X$, in violation of Lemma 3.

We claim that Z is one-dimensional. The proof is analogous to the proof of Lemma 7. Note that since $Y \cap P$ is finite, we have $\operatorname{ind}(Y \setminus P) = 1$. Let $y = (y_1, y_2, y_3) \in Y \setminus P$, and assume that Z is zero-dimensional. Put $z = p(y)$. Since $p|Y \setminus P$ is one-to-one, we have $p^{-1}(z) \cap Y = \{y\}$. Consider an arbitrary neighbourhood (a, b) of y_1 such that $0 \notin [a, b]$. Note that $(\{a\} \times \mathbb{R}^2) \cap Y$ and $(\{b\} \times \mathbb{R}^2) \cap Y$ are finite, so $F = p((\{a, b\} \times \mathbb{R}^2) \cap Y)$ is a finite set that does not contain z. Since Z is zero-dimensional, we can find a closed neighbourhood U of z in $\mathbb{R}^2$ such that $U \cap F = \emptyset$ and the boundary of U is disjoint from Z. Then $V = ([a, b] \times \mathbb{R}^2) \cap p^{-1}(U)$ is a neighbourhood of y whose boundary is disjoint from Y. Since V can be made arbitrarily small, we may conclude that $\operatorname{ind}(Y \setminus P) = 0$, a contradiction.

Next we prove that Z is a partial two-point set. Let $\ell \in \mathbb{H}^2$ be such that $\ell \cap Z$ has at least three points. Consider the 2-plane Q in $\mathbb{R}^3$ that contains $\mathbf{0}$ and the line $\{1\} \times \ell$. Note that since $p^{-1}(\ell) = Q \setminus P$, the sets $Q \setminus P$ and Y meet in at least three points. So the hyperplane $\xi^{-1}(Q)$ contains at least three points of X in addition to the $n-2$ points of A. This contradicts the assumption that X is a partial n-point set.

We now prove that Z is an ultra weak two-point set. Let ℓ be a line in the plane such that $\theta(\ell) \notin \Theta$. Since Θ is finite, such lines form a dense subset of $\mathbb{H}^2$. Consider the plane $Q \subset \mathbb{R}^3$ that contains $\mathbf{0}$ and the line $\{1\} \times \ell$. Since the line $P \cap Q$ is parallel to $\{1\} \times \ell$ and $\mathbf{0} \in Y \cap P \cap Q$, we see that $P \cap Q$ contains no points of Y other than $\mathbf{0}$. Since $\xi^{-1}(Q)$ is a hyperplane in $\mathbb{R}^n$, it intersects the weak n-point set X in at least $n-1$ points. Precisely $n-2$ of these points lie in $L = \xi^{-1}(\mathbf{0})$. So there is a point $y \in Y \cap Q \setminus P$. Then $p(y) \in \ell \cap Z$.

We found that Z is a one-dimensional ultra weak two-point set, in contradiction to Theorem 6. □

Inspection of this proof shows that we have actually proved the following stronger—albeit less attractive—statement.

PROPOSITION 9. *Let X be a finite-point set in $\mathbb{R}^n$. Let A be a subset of X with $|A| = n-2$, such that for every $H \in \mathbb{H}^n$ with $A \subset H$, we have $|H \cap X| \leqslant n$, and H can be approximated by hyperplanes H' that satisfy the conditions $A \subset H'$ and $|H' \cap X| \geqslant n-1$. Then X is zero-dimensional.*

Write $X \setminus H$ as the union of an increasing sequence $F_1, F_2, \dots$ of compacta. Lemma 10 applies to Z and $q(F_1), q(F_2), \dots,$ and we have a closed covering $G_1 \subset G_2 \subset \dots$ of the plane such that $G_i \cap Z = q(F_i)$ for each $i \in \mathbb{N}$. So $\{q^{-1}(G_i) : i \in \mathbb{N}\}$ is a closed covering of the open set $\mathbb{R}^n \setminus H$ and $q^{-1}(G_i) \cap X = F_i$ for each i because $q|X \setminus H$ is one-to-one. So $\{q^{-1}(G_i) \setminus F_i : i \in \mathbb{N}\}$ is a collection of σ-compact subsets of $\mathbb{R}^n \setminus (X \cup H)$ that covers the set. Consequently, $X \setminus H$ is a G_δ-set, and since $X \cap H$ is finite, we also see that X is G_δ in $\mathbb{R}^n$. □

Again, in Theorem 11 we may replace the assumption that the set X is a weak n-point set by the weaker condition that X is a set in $\mathbb{R}^n$ that contains a subset A with precisely $n-2$ points such that every hyperplane that contains A meets X in n or $n-1$ points.

EXAMPLE 2. There exist countable ultra weak two-point sets that are dense in $\mathbb{R}^2$. According to the Baire category theorem, these sets are not G_δ-sets, so Theorem 11 is not valid for ultra weak two-point sets. We use Mazurkiewicz's [7] inductive method to construct a counterexample X.

Let $\{\ell_i : i \in \mathbb{N}\}$ and $\{u_i : i \in \mathbb{N}\}$ be countable dense subsets of $\mathbb{H}^2$ and $\mathbb{R}^2$, respectively. We shall construct a monotone sequence $X_1 \subset X_2 \subset \dots$, consisting of finite partial two-point sets. Put $X_1 = \emptyset$, and assume that X_i has been constructed for some $i \in \mathbb{N}$. Let $\mathscr{L}$ be the collection of all lines that meet X_i in two points. Since $\mathscr{L}$ is a finite collection of nowhere dense sets, $\bigcup \mathscr{L}$ is nowhere dense, and we can find a $v_i \in \mathbb{R}^2 \setminus \bigcup \mathscr{L}$ such that $\|u_i - v_i\| < 1/i$. Put $F = X_i \cup \{v_i\}$, and consider the collection $\mathscr{L}'$ of all lines that meet F in two points. If $\ell_i \in \mathscr{L}'$, then put $X_{i+1} = F$. If $\ell_i \notin \mathscr{L}'$, then $\ell_i \cap \bigcup \mathscr{L}'$ is finite, and hence we can find a point $w \in \ell_i \setminus \bigcup \mathscr{L}'$ and define $X_{i+1} = F \cup \{w\}$. This completes the induction.

If we define $X = \bigcup_{i=1}^{\infty} X_i$, then X is obviously a countable partial two-point set that meets every ℓ_i, and that contains the dense set $\{v_i : i \in \mathbb{N}\}$.

Let X and Y be separable metric spaces. If $A \subset X \times Y$ and $x \in X$, then we let $A(x)$ stand for the trace $\{y \in Y : (x, y) \in A\}$. The Kuratowski–Ulam theorem [4] states that if A is a first category subset of $X \times Y$, then there is a first category set C in X such that for each $x \in X \setminus C$, the set $A(x)$ is first category in Y.

PROPOSITION 12. *Any finite-point set that is a G_δ-set in $\mathbb{R}^n$ is nowhere dense in $\mathbb{R}^n$.*

Proof. Let Z be a finite-point set that is G_δ in $\mathbb{R}^n$. Let $U \subset \mathbb{R}$ and $V \subset \mathbb{R}^{n-1}$ be arbitrary nonempty open sets. We put $A = (U \times V) \setminus Z$, and we note that for each $x \in \mathbb{R}$ the set $Z(x)$ is finite. Consequently, $A(x)$ is open and dense, and hence second category in V for each $x \in U$. So, by the Kuratowski–Ulam theorem, A is second category in $U \times V$. Since A is an F_σ set, it has nonempty interior, according to Baire. We may conclude that the closure of Z cannot contain $U \times V$, and hence that Z is nowhere dense in $\mathbb{R}^n$. □

It is an open problem as to whether there exist (weak) n-point sets that are G_δ-subsets of $\mathbb{R}^n$.

Theorem 8 generalizes Kulesza's result that every 2-point set is zero-dimensional. We do not know, however, whether Theorem 5 generalizes to partial n-point sets.

Let us take a stand, as follows.

CONJECTURE 3. Every one-dimensional partial n-point set contains arcs.

CONJECTURE 4. No weak n-point set is F_σ in $\mathbb{R}^n$.

CONJECTURE 5. No weak n-point set is G_δ in $\mathbb{R}^n$.

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Department of Mathematics
The University of Alabama
Box 870350
Tuscaloosa
Alabama 35487-0350
USA

dijkstra@cs.vu.nl

Vrije Universiteit
Faculty of Sciences
Division of Mathematics and Computer Science
De Boelelaan 1081a
1081 HV Amsterdam
The Netherlands

vanmill@cs.vu.nl

Current address:

Vrije Universiteit
Faculty of Sciences
Division of Mathematics and Computer Science
De Boelelaan 1081a
1081 HV Amsterdam
The Netherlands