EMBEDDINGS INTO $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ AND EXTENSION OF AUTOMORPHISMS

A. BELLA, A. DOW, K. P. HART, M. HRUŠÁK[†], J. VAN MILL, AND P. URSINO

ABSTRACT. Given a Boolean algebra $\mathbb{B}$ and an embedding $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ we consider the possibility of extending each or some automorphism of $\mathbb{B}$ to the whole $\mathcal{P}(\mathbb{N})/\operatorname{fin}$. Among other things, we show, assuming CH, that for a wide class of Boolean algebras there are embeddings for which no non-trivial automorphism can be extended.

1. Introduction

A general problem in the theory of Boolean algebras concerns the possibility of extending all or some automorphisms with respect to a fixed embedding [12]. More precisely, given two Boolean algebras $\mathbb{A}$ and $\mathbb{B}$ one asks whether there is an embedding $e : \mathbb{B} \to \mathbb{A}$ for which each automorphism of $\mathbb{B}$ can be extended to an automorphism of $\mathbb{A}$. In the opposite direction, one may wonder if there is an embedding for which no non-trivial automorphism of $\mathbb{B}$ extends.

For instance, P. Štěpánek in [12] quotes as open the problem whether any Boolean algebra $\mathbb{B}$ can be embedded into a homogeneous Boolean algebra $\mathbb{A}$ in such a way that no non-trivial automorphism of $\mathbb{B}$ extends to an automorphism of $\mathbb{A}$. A result in this direction due to S. Koppelberg says that, assuming $\diamondsuit$, there exists an $\aleph_1$-Suslin tree T such that the corresponding complete Boolean algebra $\mathbb{A}(T)$ is homogeneous and has a regular complete subalgebra onto which no non-trivial automorphism of $\mathbb{A}(T)$ restricts.

This paper focuses on the case $\mathbb{A} = \mathcal{P}(\mathbb{N})/\operatorname{fin}$. The main reason for this is Parovičenko's theorem:

Theorem 1.1. *Every Boolean algebra of size $\leqslant \omega_1$ embeds into $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.*

So assuming the Continuum Hypothesis, for algebras of size at most continuum at least the minimal requirement of an existence of an embedding is satisfied. One should also mention the following result of van Douwen and Przymusiński (see [2] or [11])

Theorem 1.2 (MA). *Every Boolean algebra of size $< \mathfrak{c}$ embeds into $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.*

However, this result does not hold for all algebras of size $\mathfrak{c}$. In fact, assuming the Proper Forcing Axiom even algebras as nice as the measure algebra $\mathbb{M} = \operatorname{Borel}(\mathbb{R})/\operatorname{Null}$ do not embed into $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ (see [4]).

The investigation of how the measure algebra can be embedded into $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ was suggested by the last mentioned author as an attempt to compare, as composition groups, certain measure-preserving endomorphisms on the real line [6] with

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the permutations of the integers. Since a straightforward attempt presents unsurmountable obstacles, a possible way to approach this problem indirectly could be to view both of them as embedded in the larger group $\mathrm{Aut}(\mathcal{P}(\mathbb{N})/\operatorname{fin})$. Indeed, the latter give rise to automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$, and as we show, the same happens for the former.

The results presented here show that, under CH, for a wide class of Boolean algebras, including the measure algebra, there are embeddings such that all automorphisms can be extended, as well as embeddings such that no non-trivial automorphism can be extended.

All basic concepts of Boolean algebra and notation can be found in [8], set theoretic notation follows [9]. Henceforth, CH stands for the Continuum Hypothesis, MA, $\mathrm{MA}_{\sigma-\text{centered}}$ and $\mathrm{MA}_{\sigma-\text{linked}}$ indicate the use of Martin's Axiom or its variants, while PFA stands for the Proper Forcing Axiom. As usual, given a Tychonoff space X, βX denotes the Čech-Stone compactification of X, and X^*

the Čech-Stone remainder $\beta X \setminus X$. $\mathrm{CO}(X)$ and $\mathrm{RO}(X)$ are the Boolean algebras of clopen and regular open subsets of X, respectively. In particular, $\mathrm{CO}(2^\kappa)$ denotes free Boolean algebra on κ many generators. The Stone space of a Boolean algebra $\mathbb{B}$ is written $\mathrm{St}(\mathbb{B})$.

2. Nice embeddings

The following fact that, assuming CH, nice embeddings do exist appeared in [7] with a different proof.

Theorem 2.1 (CH). *For every Boolean algebra $\mathbb{B}$ of size at most continuum there is an embedding $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ such that every automorphism of $\mathbb{B}$ can be extended to an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.*

Proof. Given a Boolean algebra $\mathbb{B}$ of size at most $\mathfrak{c}$ note that the algebra $\mathbb{B}^\omega/\text{fin}$ is isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ ([11]). Define an embedding $e : \mathbb{B} \to \mathbb{B}^\omega/\text{fin}$ by $e(B) = [(B, B, B, \dots)]$. Clearly, if h is an automorphism of $\mathbb{B}$, putting $H([(A_n)_{n\in\omega}]) = [h(A_n)_{n\in\omega}]$ defines an extension of h, which is an automorphism of $\mathbb{B}^\omega/\text{fin}$. □

The above theorem can be strengthened by requiring that the embedding e be such that every automorphism h has the largest possible number, i.e., $2^\mathfrak{c}$ many, extensions. This is accomplished by identifying

$\mathcal{P}(\mathbb{N})/\operatorname{fin}$ with $\mathbb{A}^\omega/\text{fin}$, where $\mathbb{A} = \mathbb{B} \oplus \mathrm{CO}(2^\mathfrak{c})$.

A natural question as to whether the hypothesis of Theorem 2.1 can be weakened was also addressed in [7] where it is shown that

Theorem 2.2. *It is relatively consistent with $\mathrm{MA}_{\sigma-\text{linked}}$ that for every Boolean algebra $\mathbb{B}$ of size at most continuum there is an embedding $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ such that each automorphism of $\mathbb{B}$ can be extended to an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.*

It should be noted here that for certain Boolean algebras $\mathbb{B}$ such as any countable algebra all embeddings are good. Let $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ be an embedding. We say that e *lifts* if there is a Boolean algebra homomorphism $E : \mathbb{B} \to \mathcal{P}(\mathbb{N})$ such that $E(B) \in e(B)$ for every $B \in \mathbb{B}$.

Theorem 2.3 ($\mathrm{MA}_{\sigma-\text{centered}}(\kappa)$). *Let $\mathbb{B}$ be a Boolean algebra of size κ and let $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ be an embedding which lifts. Then every automorphism of $\mathbb{B}$ extends to an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.*

Proof. Let a Boolean algebra $\mathbb{B}$ and an embedding $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ be given. Let $E : \mathbb{B} \to \mathcal{P}(\mathbb{N})$ be a lifting of e. Let h be an automorphism of $\mathbb{B}$. Consider the following partial order

$$\mathbb{P} = \{\langle s, F\rangle : s \text{ is a finite partial one-to-one function } \mathbb{N} \to \mathbb{N},\ F \in [\mathbb{B}]^{<\aleph_0}\}$$

ordered by $\langle s,F\rangle \leqslant \langle t,G\rangle$ if $s \supseteq t$, $F \supseteq G$ and for every $n \in \operatorname{dom}(s)\setminus\operatorname{dom}(t)$ and every $B \in G$, $n \in E(B)$ if and only if $s(n) \in E(h(B))$.

The partial order $\mathbb{P}$ is σ-centered as conditions with the same working part are compatible. It is also easily seen that the following sets are dense in $\mathbb{P}$:

- $D_n = \{\langle s,F\rangle \in \mathbb{P} : n \in \operatorname{dom}(s)\}$,
- $R_n = \{\langle s,F\rangle \in \mathbb{P} : n \in \operatorname{rng}(s)\}$,
- $E_B = \{\langle s,F\rangle \in \mathbb{P} : B \in F\}$.

By $\mathrm{MA}_{\sigma-\text{centered}}$ there is a filter G on $\mathbb{P}$ which intersects all of these dense sets. Let $\pi = \bigcup\{s : (\exists F \in [\mathbb{B}]^{<\aleph_0})(\langle s,F\rangle \in G)\}$. It is straightforward to check that π is a permutation on $\mathbb{N}$ that defines an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ extending h.

□

Corollary 2.4 ($\mathrm{MA}_{\sigma-\text{centered}}(\kappa)$)**.** *Every embedding* $e : \mathrm{CO}(2^\kappa) \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ *is such that every automorphism of* $\mathrm{CO}(2^\kappa)$ *extends to an automorphism of* $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.

Proof. It is easy to see that every $e : \mathrm{CO}(2^\kappa) \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ lifts. □

Note the fundamental difference between the two proofs presented so far;

whereas in Theorem 2.1 it was in effect shown that the map that sends h to H is actually an injective group homomorphism from $\operatorname{Aut}(\mathbb{B})$ into $\operatorname{Aut}(\mathcal{P}(\mathbb{N})/\operatorname{fin})$, the method of the proof of Theorem 2.3 does not seems to produce such an embedding.

3. Ugly Embeddings

This section is devoted to showing that, assuming CH, many Boolean algebras can be embedded into $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ in such a way that only the

trivial automorphism extends to an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$. Recall that if A and B are elements of a Boolean algebra $\mathbb{B}$ then A *splits* B if both $A \wedge B$ and $A^c \wedge B$ are non-zero. Denote by $\mathfrak{u}(\mathbb{B})$ the minimal character of an ultrafilter on $\mathbb{B}$.

Lemma 3.1 (CH)**.** *Let* $\mathbb{B}$ *be a Boolean algebra of size* $\mathfrak{c}$ *such that* $\mathfrak{u}(\mathbb{B}) > \omega$. *Then for every family of ultrafilters* $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathrm{St}(\mathbb{B})$ *there exists an embedding* $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ *and a set* $\{C_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\mathbb{N})/\operatorname{fin}^+$ *such that*

(1) $C_\alpha < e(B)$ *for each* $B \in p_\alpha$ *and* $\alpha < \mathfrak{c}$,
(2) *If* $A \in \mathcal{P}(\mathbb{N})/\operatorname{fin}^+$ *and* $A \wedge C_\alpha = \mathbf{0}$ *for every* $\alpha < \mathfrak{c}$ *then there is a* $B \in \mathbb{B}$ *such that* $e(B)$ *splits* A.

Proof. Enumerate $\mathbb{B}$ as $\{B_\alpha : \alpha \in \omega_1\}$ and $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ as $\{A_\alpha : \alpha \in \omega_1\}$. Recursively construct

an increasing chain $\{\mathbb{B}_\beta : \beta \in \omega_1\}$ of countable subalgebras of $\mathbb{B}$ and compatible embeddings $e_\beta : \mathbb{B}_\beta \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ together with a set of non-zero pairwise disjoint elements $\{C_\beta : \beta < \omega_1\} \subseteq \mathcal{P}(\mathbb{N})/\operatorname{fin}$ so that for $\alpha < \mathfrak{c}$:

(1) $B_\alpha \in \mathbb{B}_\alpha$,
(2) $C_\gamma < e_\alpha(B)$ for every $B \in p_\gamma \cap \mathbb{B}_\alpha$ and $\gamma \leqslant \alpha$,
(3) if $(\forall\gamma \leqslant \alpha)(A_\alpha \wedge C_\gamma = \mathbf{0})$ then $(\exists B \in \mathbb{B}_\alpha)(e_\alpha(B)$ splits $A_\alpha)$.

At stage $\alpha < \mathfrak{c}$ we have to define the subalgebra $\mathbb{B}_\alpha$, the embedding e_α and the element $C_\alpha \in \mathcal{P}(\mathbb{N})/\operatorname{fin}$. Let $\mathbb{B}' = \bigcup\{\mathbb{B}_\beta : \beta < \alpha\}$ and let $e' = \bigcup\{e_\beta : \beta < \alpha\}$ be the corresponding embedding. Let $\mathbb{C}$ be the subalgebra of $\mathbb{B}$ generated by $\mathbb{B}' \cup \{B_\alpha\}$. Let $c : \mathbb{C} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ be an embedding which extends e' and such that $C_\beta < c(B_\alpha)$ whenever $B_\alpha \in p_\beta$, and $C_\beta \wedge c(B_\alpha) = \mathbf{0}$ if $B_\alpha \notin p_\beta$. Then pick a $C_\alpha \in \mathcal{P}(\mathbb{N})/\operatorname{fin}^+$ so that $C_\alpha < c(B)$ for all $B \in p_\alpha \cap \mathbb{C}$ and $C_\alpha \wedge C_\beta = \mathbf{0}$ for each $\beta < \alpha$.

If there is an element $B \in \mathbb{C}$ such that $c(B)$ splits A_α or if $A_\alpha \wedge C_\beta > \mathbf{0}$ for some $\beta \leqslant \alpha$, then set $\mathbb{B}_\alpha = \mathbb{C}$ and $e_\alpha = c$. If not, let $\mathcal{C} = \{B \in \mathbb{B}'_\alpha : A_\alpha < c(B)\}$. The set $\mathcal{C}$ generates a filter in $\mathbb{B}$. Being countably generated, this filter cannot be an ultrafilter so there is a $G \in \mathbb{B}$ which splits every element of $\mathcal{C}$. Let $\mathbb{B}_\alpha$ be the subalgebra of $\mathbb{B}$ generated by $\mathbb{C} \cup \{G\}$. All that has to be done is to extend c to an embedding $e_\alpha : \mathbb{B}_\alpha \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ so that $e_\alpha(G)$ splits A_α. To that end let

$$F = \{c(B) : B < G\} \cup \{C_\beta : \beta \leqslant \alpha \ \&\ G \in x_\beta\},$$

$$G = \{c(B) : G < B\} \cup \{{C_\beta}^c : \beta \leqslant \alpha \ \&\ G \notin x_\beta\},$$

$$H = \{c(B) : B \not< G \ \&\ B \not> G\} \cup \{A_\alpha\}.$$

As $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ satisfies condition $\mathcal{R}_\omega$ (see [11]) there is an element $Y \in \mathcal{P}(\mathbb{N})/\operatorname{fin}$ which separates F from G and splits every element of H. Setting $e_\alpha(G) = Y$ defines the desired embedding $e_\alpha : \mathbb{B}_\alpha \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$

Finally, $e = \bigcup\{e_\alpha : \alpha < \omega_1\}$ is then the required embedding. □

Theorem 3.2 (CH)**.** *Let $\mathbb{B}$ be a Boolean algebra of size $\mathfrak{c}$ and $\mathfrak{u}(\mathbb{B}) > \omega$. Let $S \subseteq \operatorname{Aut}(\mathbb{B}) \setminus \{\mathrm{id}\}$, $|S| \leqslant \mathfrak{c}$ be given. There is an embedding $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ such that no element of S can be extended to an element of* $\operatorname{Aut}(\mathcal{P}(\mathbb{N})/\operatorname{fin})$*.*

Proof. Let $\{h_\alpha : \alpha < \omega_1\}$ be an enumeration of S and let H_α be the homeomorphism of $\operatorname{St}(\mathbb{B})$ corresponding to h_α. Recursively choose ultrafilters $p_\alpha \in \operatorname{St}(\mathbb{B})$ so that $\{p_\alpha : \alpha < \omega_1\} \cap \{H_\alpha(p_\alpha) : \alpha < \omega_1\} = \varnothing$. Then let $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ be the embedding described in Lemma 3.1 with respect to the set $\{p_\alpha : \alpha < \omega_1\}$.

Suppose that, for some $\alpha < \omega_1$, $h_\alpha \in S$ has an extension $\tilde{h}_\alpha \in \operatorname{Aut}(\mathcal{P}(\mathbb{N})/\operatorname{fin})$. As $C_\alpha < e(B)$ for every $B \in p_\alpha$, $\tilde{h}_\alpha(C_\alpha) < \tilde{h}_\alpha(e(B)) = e(h_\alpha(B))$ for every $B \in p_\alpha$. As $H_\alpha(p_\alpha) \neq p_\beta$ and $C_\alpha \wedge C_\beta = \mathbf{0}$ we have $\tilde{h}_\alpha(C_\alpha) \wedge C_\beta = \mathbf{0}$ for all $\beta \in \omega_1$ different from α. Choose a $B \in \mathbb{B}$ which splits $\tilde{h}_\alpha(C_\alpha)$ and set $D = h_\alpha^{-1}(B)$. Note that $C_\alpha < e(D)$ or $C_\alpha < e(D)^c$ depending on whether $D \in p_\alpha$ or $D^c \in p_\alpha$. In both cases this contradicts the fact that $e(B)$ splits $\tilde{h}_\alpha(C_\alpha)$. □

In particular, assuming CH, every Boolean algebra $\mathbb{B}$ of size $\mathfrak{c}$ with no ultrafilters of countable character having only $\mathfrak{c}$ many automorphisms can be embedded into $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ so that no non-trivial automorphism of $\mathbb{B}$ extends. Examples of such algebras include the measure algebra or the algebra $\operatorname{RO}(2^\omega)$.

Aiming for a similar result for algebras with many automorphisms we first need the following lemma of independent interest. Recall that the *reaping number* $\mathfrak{r}(\mathbb{B})$ of a Boolean algebra $\mathbb{B}$ denotes the minimal size of a family $\mathcal{R} \subseteq \mathbb{B}^+$ such that for every $A \in \mathbb{B}^+$ there is an $R \in \mathcal{R}$ such that $R \leqslant B$ or $R \wedge B = \mathbf{0}$ (in other words, no $A \in \mathbb{B}^+$ splits all elements of $\mathcal{R}$). A subset D of a topological space X is G_δ*-dense* if $D \cap G \neq \varnothing$ for every G_δ set $G \subseteq X$.

Lemma 3.3 (CH)**.** *Let $\mathbb{B}$ be a Boolean algebra of cardinality $\mathfrak{c}$ and with $\mathfrak{r}(\mathbb{B}) > \omega$. Then there is a G_δ-dense $D \subseteq \operatorname{St}(\mathbb{B})$ of size $\aleph_1$ such that every countable subset of D is relatively discrete.*

Proof. Enumerate $\mathbb{B}$ as $\{B_\alpha : \alpha < \mathfrak{c}\}$ and list all decreasing sequences in $\mathbb{B}^+$ as $\{\langle B_{\alpha,n}\rangle_{n\in\omega} : \alpha < \mathfrak{c}\}$ in such a way, so that $\{B_{\alpha,n} : n \in \omega\} \subseteq \{B_\beta : \beta < \alpha\}$ for every $\alpha < \mathfrak{c}$. We will recursively construct countable families $\{\mathcal{F}_{\alpha,\beta} : \beta < \alpha\}$ of countably generated filters on $\mathbb{B}$ so that for every $\alpha < \omega_1$

(1) $(\forall \beta \leqslant \alpha)(B_\alpha \in \mathcal{F}_{\alpha+1,\beta}$ or ${B_\alpha}^c \in \mathcal{F}_{\alpha+1,\beta})$,
(2) $(\forall \gamma < \beta \leqslant \alpha)(\mathcal{F}_{\beta,\gamma} \subseteq \mathcal{F}_{\alpha,\gamma})$,
(3) $(\exists \beta \leqslant \alpha)(\{B_{\alpha,n} : n \in \omega\} \subseteq \mathcal{F}_{\alpha+1,\beta})$,
(4) There is a pairwise disjoint collection$\{F_\beta : \beta < \alpha\} \subseteq \mathbb{B}^+$ such that $F_\beta \in \mathcal{F}_{\alpha+1,\beta}$ for every $\beta < \alpha$.

Assume that $B_0 = \mathbf{1}$ and $B_1 = \mathbf{0}$ and set $\mathcal{F}_{1,0} = \{\mathbf{1}\}$. For limit $\alpha < \omega_1$ let $\mathcal{F}_{\alpha,\beta} = \bigcup\{\mathcal{F}_{\gamma,\beta} : \beta < \gamma < \alpha\}$. Given $\{\mathcal{F}_{\alpha,\beta} : \beta < \alpha\}$, fix a bijection $\varphi : \alpha \to \omega$ and construct $\{\mathcal{F}_{\alpha+1,\beta} : \beta < \alpha\}$ as follows:

Recursively pick R_n ($n \in \omega$) so that R_n splits all elements of the family of all non-zero finite intersections from the family $\bigcup\{\mathcal{F}_{\alpha,\beta} : \beta < \alpha\} \cup \{R_i : i < n\}$ and let

$$F^\alpha_\beta = (\bigwedge_{m<\varphi(\beta)} R_m) \wedge R_{\varphi(\beta)}{}^c.$$

Let, for all $\beta < \alpha$, $\mathcal{F}_{\alpha+1,\beta}$ be the filter generated by $\mathcal{F}_{\alpha,\beta} \cup \{F^\alpha_\beta, B_\alpha\}$ if possible, otherwise let $\mathcal{F}_{\alpha+1,\beta}$ be the filter generated by $\mathcal{F}_{\alpha,\beta} \cup \{F^\alpha_\beta, B_\alpha{}^c\}$ Finally, choose $\mathcal{F}_{\alpha+1,\alpha}$ so that $\{B_{\alpha,n} : n \in \omega\} \subseteq \mathcal{F}_{\alpha+1,\alpha}$ and so that $B_\beta \in \mathcal{F}_{\alpha+1,\alpha}$ or $B_\beta{}^c \in \mathcal{F}_{\alpha+1,\alpha}$ for all $\beta \leqslant \alpha$. This concludes the recursive construction.

In the end set $x_\alpha = \bigcup_{\alpha<\beta} \mathcal{F}_{\beta,\alpha}$ and note that:

- x_α is an ultrafilter,
- the family $\{F^\alpha_\beta : \beta < \alpha\}$ witnesses that $\{x_\beta : \beta < \alpha\}$ is relatively discrete, and
- $\{B_{\alpha,n} : n \in \omega\} \subseteq x_\alpha$ so $D = \{x_\alpha : \alpha < \omega_1\}$ is $G_\delta-$dense in $\mathrm{St}(\mathbb{B})$.

□

Lemma 3.4 (CH)**.** *There are $\mathfrak{c}$ many non-homeomorphic separable rigid P-sets in $\mathbb{N}^*$.*

Proof. Follows immediately from results of Dow, Gubbi and Szymański [3] and Balcar, Frankiewicz and Mills [1]. In [3] the authors prove that there are $2^{\mathfrak{c}}$ many pairwise non-homeomorphic rigid separable extremally disconnected compact spaces. In [1] it is shown that, assuming CH, any compact zero-dimensional F-space of weight at most $\mathfrak{c}$ is homeomorphic to a nowhere dense P-set in $\mathbb{N}^*$ (see Theorem 1.4.4 of [11]). As every extremally disconnected space is an F-space and compact separable spaces are of weight less or equal to $\mathfrak{c}$ the lemma easily follows. □

The proof of the next theorem is an advanced bookkeeping argument, moreover obscured by the fact that it uses both the language of Boolean algebra and the dual language of topology. In order not to cloud the argument any further by unnecessary notation we identify (denote by the same symbol) an element of a Boolean algebra and the corresponding clopen subset of the Stone space of the algebra.

Theorem 3.5 (CH)**.** *Let $\mathbb{B}$ be a Boolean algebra of size $\mathfrak{c}$ such that $\mathfrak{r}(\mathbb{B}) > \omega$. There is an embedding $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ such that no element of $\mathrm{Aut}(\mathbb{B})$ other than the identity can be extended to an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.*

Proof. Enumerate $\mathbb{B}$ as $\{B_\alpha : \alpha < \omega_1\}$ and let D be a $G_\delta-$dense subset of $\mathrm{St}(\mathbb{B})$ of size $\aleph_1$ such that every countable subset of D is relatively discrete. We will construct the embedding $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ as an increasing union of a chain of embeddings $\{e_\alpha : \alpha < \omega_1\}$, $e_\alpha : \mathbb{B}_\alpha \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ where $\mathbb{B}_\alpha$ is a countable subalgebra of $\mathbb{B}$ containing $\{B_\beta : \beta < \alpha\}$. During the construction we will recursively choose points $x_\beta \in D$ together with an $A_\beta \in \mathcal{P}(\mathbb{N})/\operatorname{fin}$, split into A^r_β and A^s_β with the intention that

(1) $E^{-1}(x_\beta) \subseteq A_\beta$,
(2) $E^{-1}(x_\beta) \cap A^r_\beta$ is regular closed subset of ω^*,
(3) $E^{-1}(x_\beta) \cap A^s_\beta$ is a separable P-set,
(4) $E^{-1}(x_\beta) \cap A^s_\beta$ and $E^{-1}(x_\gamma) \cap A^s_\gamma$ are not homeomorphic as long as $\beta \neq \gamma$,
(5) $\bigcup_{\beta<\omega_1} E^{-1}(x_\beta)$ is dense in $\mathbb{N}^*$.

where E denotes the continuous surjection $\omega^* \to \mathrm{St}(\mathbb{B})$ dual to e.

Assume first that this can be accomplished and let h be an automorphism of the Boolean algebra $\mathbb{B}$ that can be extended to an automorphism $\bar{h}$. Let H denote the autohomeomorphism of $St(\mathbb{B})$ dual to h and similarly, $\bar{H}$ denotes the autohomeomorphism of $\mathbb{N}^*$ dual to $\bar{h}$. Note first that, for every $\alpha < \omega_1$ there is a $\beta < \omega_1$ such that $H(x_\alpha) = x_\beta$ as by (2) and (5) the points x_α are exactly those points $x \in \mathrm{St}(\mathbb{B})$ for which $E^{-1}(x)$ has non-empty interior. Then, however, $\bar{H} \restriction E^{-1}(x_\alpha)$ is a homeomorphism between $E^{-1}(x_\alpha)$ and $E^{-1}(x_\beta)$ and furthermore $\bar{H}[E^{-1}(x_\alpha) \cap A^s_{\alpha]} = E^{-1}(x_\beta)) \cap A^s_\beta$. By (4) this implies that $H(x_\alpha) = x_\alpha$ for all $\alpha < \omega_1$ and as $\{x_\alpha : \alpha < \omega_1\}$ is dense in $\mathrm{St}(\mathbb{B})$ (by (5)), it follows that $h = \mathrm{id}$.

Next we will specify the promises which will ensure that the conditions (1)–(5) are satisfied. To that end enumerate $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ as $\{X_\alpha : \alpha < \omega_1\}$. At stage α (when we build $\mathbb{B}_{\alpha+1}$ and $e_{\alpha+1}$, and choose x_α) we look, for every $\beta < \alpha$, at the family $\{e_\alpha(B) \cap A^r_\beta \cap X_\alpha : B \in x_\beta \cap \mathbb{B}_\alpha\}$ and diagonalize it (if possible) by $R_{\alpha,\beta} \in \mathcal{P}(\mathbb{N})/\mathrm{fin}^+$. Otherwise, if $e_\alpha(B) \cap A^r_\beta \cap X_\alpha = \mathbf{0}$ for some $B \in x_\beta \cap \mathbb{B}_\alpha$, we will set $R_{\alpha,\beta} = \mathbf{0}$. Similarly, $R_{\beta,\alpha}$ will be a diagonalization (if possible) of the family $\{e_\alpha(B) \cap A^r_\alpha \cap X_\beta : B \in x_\alpha \cap \mathbb{B}_\alpha\}$ for $\beta \leqslant \alpha$. The promise is that

$$(A) \qquad (\forall \beta \leqslant \alpha)(\forall B \in x_\beta)(R_{\alpha,\beta} < e(B))$$

This will ensure that $E^{-1}(x_\beta) \cap A^r_\beta = \mathrm{cl} \bigcup_{\beta \leqslant \alpha < \omega_1} R_{\alpha,\beta} = R_\beta$ is regular closed. We will use Lemma 3.4 to choose a closed separable P-set $S_\alpha \subseteq A^s_\alpha$ (non-homeomorphic to any of the previous choices S_β, $\beta < \alpha$, and disjoint from them) and its decreasing neighborhood base (a P-filter base on $\mathcal{P}(\mathbb{N})/\mathrm{fin}$) $\{T_{\alpha,\gamma} : \gamma \geqslant \alpha\}$ with $T_{\alpha,\alpha} = A^s_\alpha$ and promise that

$$(B) \qquad (\forall \gamma \geqslant \alpha)(\forall B \in \mathbb{B}_\gamma \cap x_\alpha)(S_\alpha \subseteq e(B) \cap A^s_\alpha) \text{ and}$$

$$(\forall \gamma \geqslant \alpha)(\exists B \in \mathbb{B}_\gamma \cap x_\alpha)(e(B) \cap A^s_\alpha \subseteq T_{\alpha,\gamma})$$

to ensure that $E^{-1}(x_\alpha) \cap A^s_\alpha = S_\alpha$. Further, we promise

$$(\exists \beta \leqslant \alpha)(S_\beta \cap X_\alpha \neq^* \varnothing \text{ or } R_{\beta,\alpha} > \mathbf{0})$$

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$\bigcup_{\beta < \omega_1} E^{-1}(x_\beta)$ is dense in ω^*. Finally, in order to make sure that the recursive construction will not prematurely terminate we will require that

$\mathrm{rng}(e) \cap \langle R_{\gamma,\beta} : \gamma, \beta < \alpha \rangle = \{\mathbf{0}, \mathbf{1}\}, (D) where \langle R_{\gamma,\beta} : \gamma, \beta < \alpha \rangle$ denotes the subalgebra of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ generated by $\{R_{\gamma,\beta} : \gamma, \beta < \alpha\}$.

It is easy to see that if we succeed in constructing algebras $\mathbb{B}_\alpha$ such that $\mathbb{B} = \bigcup_{\alpha < \omega_1} \mathbb{B}_\alpha$ and embeddings $e_\alpha : \mathbb{B}_\alpha \to \mathcal{P}(\mathbb{N})/\mathrm{fin}$, together with choosing $x_\alpha \in D$, A^s_α and A^r_α as well as $R_{\alpha,\beta}$ and $T_{\alpha,\beta}$ so that the above conditions are satisfied, then so are the conditions (1)–(5).

Assume that an increasing chain $\{\mathbb{B}_\beta : \beta \in \alpha\}$ of countable subalgebras of $\mathbb{B}$ and compatible embeddings $e_\beta : \mathbb{B}_\beta \to \mathcal{P}(\mathbb{N})/\mathrm{fin}$ have already been constructed and set $\mathbb{C} = \bigcup\{\mathbb{B}_\beta : \beta \in \alpha\}$ as well as $c = \bigcup\{e_\beta : \beta \in \alpha\}$. Let $\mathbb{C}^*$ be the subalgebra of $\mathbb{B}$ generated by $\mathbb{C} \cup \{B_\alpha\}$. Assume that $B_\alpha \notin \mathbb{C}$ and let

$$F = \{c(C) : C < B_\alpha\} \cup \{R_{\beta,\gamma} : \beta, \gamma < \alpha \;\&\; B_\alpha \in x_\beta\} \cup \{T_{\beta,\gamma_\beta} : \beta < \alpha \;\&\; B_\alpha \in x_\beta\},$$

$$G = \{c(C) : C > B_\alpha\} \cup \{{R_{\beta,\gamma}}^c : \beta, \gamma < \alpha \;\&\; B_\alpha \notin x_\beta\} \cup \{{T_{\beta,\gamma_\beta}}^c : \beta < \alpha \;\&\; B_\alpha \notin x_\beta\},$$

$$H = \{c(C) : C \not< B_\alpha \;\&\; C \not> B_\alpha\},$$

where γ_β is such that:

- $(\forall B \in \mathbb{C})(S_\beta \subseteq c(B) \to T_{\beta,\gamma_\beta} < c(B))$,
- The family $\{T_{\beta,\gamma_\beta} : \beta < \alpha\}$ is pairwise disjoint,
- $(\forall B \in \mathbb{C})(c(B) \notin \langle \{R_{\gamma,\beta} : \beta, \gamma < \alpha\} \cup \{T_{\beta,\gamma_\beta} : \beta < \alpha\} \rangle$.

It is now easy to check that $\mathcal{R}_\omega$ applies (here is where the technical requirement (D) comes in handy), and c can be extended to an embedding $c^* : \mathbb{C}^* \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$.

For every $\beta < \alpha$ consider the family $\{c^*(B) \cap A^r_\beta \cap X_\alpha : B \in x_\beta \cap \mathbb{C}^*\}$. If possible, diagonalize it by $R_{\alpha,\beta} \in \mathcal{P}(\mathbb{N})/\operatorname{fin}^+$. Otherwise, fix $C_\beta \in x_\beta \cap \mathbb{C}^*$ such that $c^*(C_\beta) \cap A^r_\beta \cap X_\alpha = \mathbf{0}$ and set $R_{\alpha,\beta} = \mathbf{0}$. If there is a $\beta < \alpha$ such that $R_{\alpha,\beta} > \mathbf{0}$ or such that $X_\alpha \cap S_\beta \neq \varnothing$ let $\{D_\beta : \beta < \alpha\}$ be a pairwise disjoint family in $\mathbb{B}$ such that $D_\beta \in x_\beta$ (recall that all countable subsets of D are relatively discrete). If $R_{\alpha,\beta} = \mathbf{0}$ and $X_\alpha \cap S_\beta = \varnothing$ for all $\beta < \alpha$, require moreover that $D_\beta \leqslant C_\beta$. By picking the D_β's sufficiently small one can make sure that there is a $D_\alpha \in \mathbb{B}^+$ disjoint from all D_β, $\beta < \alpha$. Let $\mathbb{C}^{**}$ be the subalgebra of $\mathbb{B}$ generated by $\mathbb{C}^* \cup \{D_\beta : \beta < \alpha\}$. Enumerate α as $\{\beta_n : n \in \omega\}$ and extend c^* to $c^{**} : \mathbb{C}^{**} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ step by step, each time determining $c^{**}(D_{\beta_n})$ so that

- $(\forall \gamma \leqslant \alpha)(R_{\beta_n,\gamma} < c^{**}(D_{\beta_n}))$,
- $T_{\beta_n,\gamma_{\beta_n}} < c^{**}(D_{\beta_n})$,
- $(\forall i < n)(c^{**}(D_{\beta_i}) \wedge c^{**}(D_{\beta_n}) = \mathbf{0})$,
- $(\forall i \neq n)(\forall \gamma \leqslant \alpha)(R_{\beta_i,\gamma} \wedge c^{**}(D_{\beta_n}) = \mathbf{0})$,
- $(\forall i \neq n)(\forall \gamma \leqslant \alpha)(T_{\beta_i,\gamma_{\beta_i}} \wedge c^{**}(D_{\beta_n}) = \mathbf{0})$,
- $(\forall B \in \langle \mathbb{C}^* \cup \{D_{\beta_i} : i < n\}\rangle)(D_{\beta_n} < B \ \rightarrow c^{**}(D_{\beta_n}) < c^{**}(B))$,
- $c^{**}(D_{\beta_n}) \wedge A^s_{\beta_n} \leqslant T_{\beta_n,\alpha}$,
- $\operatorname{rng}(c^{**}) \cap \langle R_{\beta,\gamma} : \beta, \gamma \leqslant \alpha \rangle = \{\mathbf{0}, \mathbf{1}\}$.

These are all compatible demands. Pick an ultrafilter u on $\mathbb{C}^{**}$ containing ${D_\beta}^c$ for all $\beta < \alpha$ such that the family $\{X_\alpha\} \cup \{c^{**}(B) : B \in u\}$ is centered. Choose A_α below $X_\alpha \wedge c^{**}(B)$ for $B \in u$ and split it into two pieces A^s_α and A^r_α. Use Lemma 3.4 to choose a closed separable P-set $S_\alpha \subseteq A^s_\alpha$ non-homeomorphic to any of the previous choices S_β, $\beta < \alpha$ and let $\{T_{\alpha,\gamma} : \gamma \geqslant \alpha\}$ with $T_{\alpha,\alpha} = A^s_\alpha$ be a decreasing neighborhood base of S_α. Finally, pick $x_\alpha \in D$ extending u (this can be done as D is G_δ-dense) and determine $R_{\alpha,\beta}$ for $\beta \leqslant \alpha$.

This finishes the construction. It is not difficult to verify that the promises (A)–(D) are fulfilled, the D_β's being the witnesses required by (B). □

4. Concluding remarks

In this section we present some examples and open questions.

Definition 4.1. Given an embedding $e : \mathbb{B} \to \mathbb{A}$ of Boolean algebras, let
$E_e = \{h \in \operatorname{Aut}(\mathbb{B}) : (\exists \bar{h} \in \operatorname{Aut}(\mathbb{A}))(e \circ h = \bar{h} \circ e)\}$,
$R_e = \{h \in \operatorname{Aut}(\mathbb{A}) : h \restriction \operatorname{rng}(e) \in \operatorname{Aut}(\operatorname{rng}(e))\}$,
$K_e = \{h \in \operatorname{Aut}(\mathbb{A}) : h \restriction \operatorname{rng}(e) = \operatorname{id}\}$.

All three of these sets are easily seen to be groups, moreover, K_e is a normal subgroup of R_e and E_e is isomorphic to the quotient group R_e/K_e. In this notation, if $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ is the embedding described in Theorem 2.1, then K_e is a direct summand of E_e and therefore R_e is isomorphic to $E_e \oplus K_e$. A similar decomposition is trivially realized when e is a dense embedding, as in this case $K_e = \{\operatorname{id}\}$.

However, even in the particular case of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$, it is not immediately obvious whether any non-dense embedding leads to a trivial K_e. Clearly, the range of any such e has to form a splitting family in $\mathcal{P}(\mathbb{N})/\operatorname{fin}$. We will show next that there is (in ZFC) an embedding e of the free Boolean algebra on $\mathfrak{c}$ generators into $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ such that the only automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ which restricts to an automorphism of $\operatorname{CO}(2^{\mathfrak{c}})$ is the identity.

Recall that a family $\mathcal{I}$ of subsets of a set X is *independent* if $\bigcap \mathcal{F}_0 \setminus \bigcup \mathcal{F}_1 \neq \varnothing$ for all pairs $\mathcal{F}_0, \mathcal{F}_1$ of disjoint finite subsets of $\mathcal{I}$. It is well-known that there is an independent family of size $\mathfrak{c}$.

Theorem 4.2. *There is an embedding* $e : \mathrm{CO}(2^{\mathfrak{c}}) \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ *such that* $K_e = \{\mathrm{id}\}$.

Proof. Let $\{I_\alpha : \alpha < \mathfrak{c}\}$ be an independent family of subsets of $\mathbb{N}$. Enumerate all pairs of disjoint nowhere dense subsets of the rational numbers $\mathbb{Q}$ as $\{\langle M_\alpha, N_\alpha\rangle : \alpha < \mathfrak{c}\}$.

Set

$$J_\alpha = \left(M_\alpha \cup \bigcup_{n \in I_\alpha} [n, n+1) \cap \mathbb{Q} \right) \setminus N_\alpha$$

for every $\alpha < \mathfrak{c}$. Note that $\{J_\alpha : \alpha < \mathfrak{c}\}$ is an independent family of subsets of $\mathbb{Q}$. Identify $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ with $\mathcal{P}(\mathbb{Q})/\mathrm{fin}$ and set

$$e(\{f \in 2^{\mathfrak{c}} : f(\alpha) = 1\}) = [J_\alpha],$$

defining thus an embedding $e : \mathrm{CO}(2^{\mathfrak{c}}) \to \mathcal{P}(\mathbb{Q})/\mathrm{fin}$. All that remains to be checked is that $K_e = \{\mathrm{id}\}$. To that end let $h \in \mathrm{Aut}(\mathcal{P}(\mathbb{Q})/\mathrm{fin}) \setminus \{\mathrm{id}\}$, i.e., there is an infinite $A \subseteq \mathbb{Q}$ such that $h([A]) \neq [A]$. It is easy to find an infinite nowhere dense subset of A, say M, such that there is an $N \in h([M])$ nowhere dense, disjoint from M. The pair $\langle M, N\rangle$ was enumerated as $\langle M_\alpha, N_\alpha\rangle$ for some $\alpha < \mathfrak{c}$.

Then, however,

$$h([J_\alpha]) \geqslant h([M_\alpha]) = [N_\alpha] \text{ and } [J_\alpha] \wedge [N_\alpha] = \mathbf{0}$$

so $h([J_\alpha]) \neq [J_\alpha]$ and $h \notin K_e$. ☐

One might wonder to what extent are the very strong assumptions in Theorem 2.3 necessary. For instance, Theorem 2.3 only applies to Boolean algebras satisfying the countable chain condition for if a Boolean algebra $\mathbb{B}$ admits an embedding which lifts, it has to be c.c.c. The following example seems to indicate that one can not hope for a substantial weakening

of the assumptions. Let $\mathbb{A}(\kappa)$ denote the Boolean algebra of finite and co-finite subsets of κ or equivalently the algebra of clopen subsets of the one point compactification of a discrete space of size κ, i.e., the simplest Boolean algebra having an antichain of size κ.

Recall that a family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is *almost disjoint* if every two distinct members of $\mathcal{A}$ have finite intersection. An almost disjoint family $\mathcal{A}$ of size $\aleph_1$ is *Luzin* if there is no $B \subseteq \mathbb{N}$ such that $|\{A \in \mathcal{A} : |A \setminus B| \text{ is finite}\}| = |\{A \in \mathcal{A} : A \cap B \text{ is finite}\}| = \aleph_1$. Inspired by the ingenious construction of Hausdorff, N. Luzin ([10]) showed that there are Luzin almost disjoint families in ZFC.

Proposition 4.3. *There is an embedding* $e : \mathbb{A}(\omega_1) \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ *such that* $E_e \neq \mathrm{Aut}(\mathbb{A}(\omega_1))$.

Proof. Let $\mathcal{A}$ be a Luzin almost disjoint family of size $\aleph_1$. Identify $\mathbb{N}$ with $\mathbb{N} \times 3$ and let, for $X \subseteq \mathbb{N}$ and $i \in 3$, $X_i = X \times \{i\}$. Let $\mathbb{A}$ be the subalgebra of $\mathbb{P}(\mathbb{N} \times 3)/\operatorname{fin}$ generated by $[A_i]$, for $A \in \mathcal{A}$ and $i \in 3$. $\mathbb{A}$ is isomorphic to $\mathbb{A}(\omega_1)$. Let φ be a bijection between $\{A_1 : A \in \mathcal{A}\} \cup \{A_2 : A \in \mathcal{A}\}$ and $\{A_0 : A \in \mathcal{A}\}$ and let h be the induced automorphism of $\mathcal{A}$. Assume that h has an extension $H \in \mathrm{Aut}(\mathbb{P}(\mathbb{N} \times 3)/\operatorname{fin})$ and consider $H([\mathbb{N}_1])$. The sets $\{A \in \mathcal{A} : [A_0] \leqslant H([\mathbb{N}_1])\}$ and $\{A \in \mathcal{A} : [A_0] \wedge H([\mathbb{N}_1]) = \mathbf{0}\} = \{A \in \mathcal{A} : [A_0] \leqslant H([\mathbb{N}_2])\}$ are both uncountable contradicting that $\mathcal{A}$ was a Luzin family. ☐

A natural question is whether the assumption in Theorem 3.5 can be weakened to $\mathfrak{u}(\mathbb{B}) > \omega$, which would, of course make, Theorem 3.2 obsolete.

Question 4.4. *Assume* CH. *Let* $|\mathbb{B}| = \mathfrak{u}(\mathbb{B}) = \omega_1$. *Is there an embedding* $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ *such that no element of* $\mathrm{Aut}(\mathbb{B})$ *other than the identity can be extended to an automorphism of* $\mathcal{P}(\mathbb{N})/\operatorname{fin}$*?*

Note that for the embeddings constructed in Theorem 3.2 and Theorem 3.5, $E_e = \{\mathrm{id}\}$ but $R_e \neq \{\mathrm{id}\}$. This suggests the following question:

Question 4.5. *Assume* CH. *For which Boolean algebras $\mathbb{B}$ is there an embedding $e : \mathbb{B} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ such that no element of* $\operatorname{Aut}(\mathcal{P}(\mathbb{N})/\operatorname{fin})$ *other than the identity restricts to an automorphism of $\mathbb{B}$?*

To this question we have a very partial answer.

Theorem 4.6 (CH)**.** *There is an embedding $e : \mathcal{P}(\mathbb{N})/\operatorname{fin} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ such that $R_e = \{\mathrm{id}\}$.*

Proof. Enumerate $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ as $\{B_\alpha : \alpha < \omega_1\}$ and let $p \in \mathbb{N}^*$ be a P-point. Let $T(p)$ be the type of p, i.e., the set of those $q \in \mathbb{N}^*$ which can be sent to p by a permutation of $\mathbb{N}$. Enumerate $T(p)$ as $\{p_\alpha : \alpha < \omega_1\}$ and note that it is a $G_\delta-$dense subset of $\mathbb{N}^*$ every countable subset of which is relatively discrete. List as $\{X_\alpha, Y_\alpha, \mathcal{X}_\alpha, \mathcal{Y}_\alpha\}$ all quadruples such that

- $X_\alpha, Y_\alpha \in \mathcal{P}(\mathbb{N})/\operatorname{fin}$, $X_\alpha \subseteq Y_\alpha$,
- $\mathcal{X}_\alpha, \mathcal{Y}_\alpha \in [\mathcal{P}(\mathbb{N})/\operatorname{fin}]^\omega$, $\mathcal{X}_\alpha \cup \mathcal{Y}_\alpha$ is a pairwise disjoint family,
- $(\forall Y \in \mathcal{Y}_\alpha)(Y \cap X_\alpha = \mathbf{0})$,
- $(\forall X \in \mathcal{X}_\alpha)(X \subseteq Y_\alpha)$.

As before we will construct the embedding $e : \mathcal{P}(\mathbb{N})/\operatorname{fin} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ as an increasing union of a chain of embeddings $\{e_\alpha : \alpha < \omega_1\}$, $e_\alpha : \mathbb{B}_\alpha \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ where $\mathbb{B}_\alpha$ is a countable subalgebra of $\mathbb{B}$ containing $\{B_\beta : \beta < \alpha\}$.

At stage $\alpha < \omega_1$ of the construction we will use Lemma 3.4 to recursively choose a rigid separable P-set $S_\alpha \subseteq \mathbb{N}^*$ non-homeomorphic to and disjoint from all of the previous choices S_β, $\beta < \alpha$ and its decreasing neighborhood base $\{T_{\alpha,\gamma} : \gamma \geqslant \alpha\}$, and $D_\alpha \in \mathcal{P}(\mathbb{N})/\operatorname{fin}^+$ promising that:

1. $(\forall \gamma \geqslant \alpha)(\forall B \in \mathbb{B}_\gamma \cap p_\alpha)(S_\alpha \subseteq e_\gamma(B))$,
2. $(\forall \gamma \geqslant \alpha)(\exists B \in \mathbb{B}_\gamma \cap p_\alpha)(e_\gamma(B) \subseteq T_{\alpha,\gamma})$,
3. $(\forall \beta \leqslant \alpha)(\forall \gamma \geqslant \alpha)(B_\beta$ does not split $S_\gamma)$,
4. $X_\alpha \cup \bigcup\{X : X \in \mathcal{X}\} \subseteq e_\alpha(D_\alpha) \subseteq Y_\alpha \setminus \bigcup\{Y : Y \in \mathcal{Y}\}$,
5. $(\exists \gamma \geqslant \alpha)(S_\alpha \cap B_\gamma \neq \mathbf{0})$.

This can be accomplished in a very similar way to the appropriate part of the proof of Theorem 3.5. Note the double role the B_α's play. Assume that the construction has been successfully completed and let $E : \mathbb{N}^* \to \mathbb{N}^*$ be the dual map to the embedding $e = \bigcup_{\alpha<\omega_1} e_\alpha$.

- $(\forall \alpha < \omega_1)(E^{-1}(p_\alpha) = S_\alpha)$

This follows easily from (1) and (2). Now assume that $q \in \mathbb{N}^*$ is a P-point of a different type that p. We will show that

- $|E^{-1}(q)| = 1$

Aiming for a contradiction assume that $q = E(r) = E(s)$ for some $r \neq s \in \mathbb{N}^*$. Let $Y \in s$ be a neighborhood of s disjoint from r. By (3) the set $\mathcal{A} = \{\beta : S_\beta$ is split by $Y\}$ is countable. As q is a P-point there is a $Q \in q$ disjoint from $\{p_\beta : \beta \in \mathcal{A}\}$. Then $X = Y \cap e(Q)$ is a neighborhood of s such that $X \cap S_\beta = \varnothing$ for every $\beta \in \mathcal{A}$. Let $\mathcal{B} = \{\beta : S_\beta$ is split by $X\}$. Clearly, $\mathcal{B}$ is also countable. As $\mathcal{A} \cup \mathcal{B}$ is a pairwise disjoint collection of P-sets, there is a "swelling" of $\mathcal{A}$ and $\mathcal{B}$ respectively to families $\mathcal{Y}$ and $\mathcal{X}$ of pairwise disjoint clopen sets

such that $Y' \cap X = \mathbf{0}$ for every $Y' \in \mathcal{Y}$ and $X' \subseteq Y$ for every $X' \in \mathcal{X}$. Now, the quadruple $\{X, Y, \mathcal{X}, \mathcal{Y}\}$ was enumerated as $\{X_\alpha, Y_\alpha, \mathcal{X}_\alpha, \mathcal{Y}_\alpha\}$ for some $\alpha < \omega_1$. Consider D_α. By (4) $e(D_\alpha) \in s \setminus r$ and so $E(s) \neq E(r)$.

Let $h \in R_e$ and let H be the dual autohomeomorphism of $\mathbb{N}^*$, let $\tilde{h}$ be the restriction of h onto the range of e and let $\tilde{H}$ be the map dual to $\tilde{h}$. Note first that any homeomorphism sends P-points to P-points. In particular, $\tilde{H}(p_\alpha)$ has to be a P-point, and moreover, $H \restriction E^{-1}(p_\alpha)$ has to be a homeomorphism between $E^{-1}(p_\alpha)$ and $E^{-1}(\tilde{H}(p_\alpha))$. It follows that $\tilde{H}(p_\alpha) = p_\alpha$ for every $\alpha < \omega_1$ as $E^{-1}(p)$ is not homeomorphic to $E^{-1}(p_\alpha)$ for any other P-point in $\mathbb{N}^*$. By density of $\{p_\alpha : \alpha < \omega_1\} = T(p)$, $\tilde{H} = \mathrm{id}$ and consequently, $H(r) = s$ implies that $E(r) = E(s)$. To conclude the argument recall that all sets $S_\alpha = E^{-1}(p_\alpha)$ were rigid, hence $H \restriction E^{-1}(p_\alpha) = \mathrm{id}$ for every $\alpha < \omega_1$. By (5), $\bigcup_{\alpha<\omega_1} E^{-1}(p_\alpha)$ is dense in $\mathbb{N}^*$ so $H = \mathrm{id}$. □

As with most results contained here the situation becomes quite different if the assumption of the Continuum Hypothesis is dropped. One might have problems even formulating reasonable conjectures for at least two distinct reasons: (1) There may not be an embedding of the Boolean algebra into $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ [4], and (2) there may not be enough automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ [13]. For instance, assuming the Proper Forcing Axiom, Theorem 4.6 no longer holds:

Theorem 4.7 (PFA). *$E_e \neq \{\mathrm{id}\}$ for every embedding $e : \mathcal{P}(\mathbb{N})/\mathrm{fin} \to \mathcal{P}(\mathbb{N})/\mathrm{fin}$.*

Proof. Follows immediately from a result of I. Farah (see [5], Theorem 3.8.1). He has shown that, assuming PFA, if $e : \mathcal{P}(\mathbb{N})/\mathrm{fin} \to \mathcal{P}(\mathbb{N})/\mathrm{fin}$ is an embedding then there is an $A \in [\mathbb{N}]^\omega$ and a one-to-one map $f : A \to \mathbb{N}$ such that $e([B]) = [f[B]]$ for every $B \subseteq A$. □

References

[1] B. Balcar, R.Frankiewicz and C. Mills, *More on nowhere dense closed P-sets*, Bull. Acad. Polon. Sci. Sr. Sci. Math. 28, 1980, , no. 5-6, 295–299

[2] E. van Douwen and T. Przymusiński, *Separable extensions of first countable spaces*, Fund. Math 95, 1980, 147–158

[3] A. Dow, A. V. Gubbi and A. Szymański, *Rigid Stone spaces within ZFC*, Proc. Amer. Math. Soc. 102, 1988, no.3, 745–748

[4] A. Dow and K. P. Hart, *The measure algebra does not always embed*, Fund. Math 163, 2000, 163–176

[5] I. Farah, *Analytic Quotients*, Memoirs Amer. Math. Soc., vol. 148, no. 702, 2000

[6] A. Giarlotta, V. Pata and P. Ursino, *Combinatorial and topological aspects of measure preserving functions* , to appear in Topology Proceedings.

[7] M. Grzech, *On automorphisms of Boolean algebras embedded in $P(\omega)/fin$* , Fund. Math 150, 1996, 127–147

[8] S. Koppelberg, *An introduction to Boolean algebras* , Handbook of Boolean Algebra, J. D. Monk editor, North-Holland, Amsterdam,1989.

[9] K. Kunen, *Set theory. An introduction to independence proofs* , Studies in Logic and the Foundations of Mathematics, 102, North-Holland, Amsterdam-New York,1980.

[10] N. Luzin, *Sur les parties de la suite naturelle des nombres entiers*, Comptes Redus (Doklady) de l'Académie des Sciences de l'URSS 40, 1943, 175–178

[11] J. van Mill, *An introduction to $\beta\omega$*, Handbook of Set-Theoretic Topology, K. Kunen and J. E. Vaughan editors, North-Holland, Amsterdam,1984.

[12] P. Štěpánek, *Automorphisms and embeddings*, Handbook of Boolean Algebra, J. D. Monk editor, North-Holland, Amsterdam,1989.

[13] J. Steprāns and S. Shelah, *PFA implies all automorphisms are trivial*, Proc. Amer. Math. Soc. 104, 1988, no. 4, 1220-1225

(A. Bella, P. Ursino) Dipartimento di Matematica, Università di Catania, 95125 Catania, Italy
E-mail address: `bella@dmi.unict.it, ursino@dmi.unict.it`

(A. Dow) Mathematics Department, UNC Charlotte, 9201 University City Blvd., Charlotte, NC 28223, USA
E-mail address: `adow@uncc.edu`

(K. P. Hart) Faculty of Information Technology and Systems, TU Delft, Postbus 5031, 2600 GA Delft, the Netherlands
E-mail address: `k.p.hart@its.tudelft.nl`

(M. Hrušák, J. van Mill) Faculty of Sciences, Division of Mathematics, Vrije Universiteit, De Boelelaan 1081[A], 1081 HV Amsterdam, The Netherlands
E-mail address: `michael@cs.vu.nl, j.van.mill@cs.vu.nl`