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Note on function spaces with the topology of pointwise convergence

By

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Abstract. The note contains two examples of function spaces $C_p(X)$ endowed with the pointwise topology. The first example is $C_p(M)$, M being a planar continuum, such that $C_p(M)^m$ is uniformly homeomorphic to $C_p(M)^n$ if and only if $m = n$. This strengthens earlier results concerning linear homeomorphisms. The second example is a non-Lindelöf function space $C_p(X)$, where X is a monolithic perfectly normal compact space all linearly orderable closed subspaces of which are metrizable. This example is obtained under the additional set-theoretical axiom $\diamondsuit$. This solves a problem of Arhangel′skiĭ.

1. Introduction. For a space X we let $C_p(X)$ denote the space of continuous real-valued functions on X endowed with the topology of pointwise convergence. The natural uniform structure on $C_p(X)$ is the one determined by the pseudonorms

(1) $$\|f - g\|_K = \sup\{|f(x) - g(x)| : x \in K\},$$

where K is an arbitrary finite subset of X. A bijection $\Phi\colon C_p(X) \longrightarrow C_p(Y)$ is a *uniform homeomorphism* if both Φ and Φ^{-1} are uniformly continuous with respect to their natural uniform structures. A space X is *monolithic* if for every subset A of X we have $\mathrm{nw}(\overline{A}) \leqq |A|$. Here nw stands for *network weight*. We refer the reader to the book [3] and the articles [2] and [1, §3] by Arhangel′skiĭ, and to the article [11] by Marciszewski for a comprehensive treatment of these topics.

The aim of this note is to prove the following theorems.

Theorem 1.1. *There exists an infinite compact metrizable space M such that $C_p(M)^n$ is uniformly homeomorphic to $C_p(M)^m$ if and only if $n = m$.*

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This strengthens considerably earlier results in [14] dealing with linear homeomorphisms. The next result provides a negative answer to Problem 45 in Arhangel′skiĭ [2].

Theorem 1.2 **($\diamond$).** *There exists a monolithic perfectly normal compact space X every linearly orderable compact subspace of which is metrizable while $C_p(X)$ is not Lindelöf.*

The space $C_p(M)^k$ can be identified with the function space $C_p(M \oplus \ldots \oplus M)$, where $M \oplus \ldots \oplus M$ stands for the union of k disjoint copies of M.

Notice that for an infinite compact metrizable space K, the Banach space $C(K)$ of continuous functions on K is isomorphic to the Banach space $C(K \oplus K)$; hence the function space $C(K)$ endowed with the norm topology or the weak topology is linearly, hence uniformly, homeomorphic to its own square. To see that this is true, first observe that by the Milutin Theorem all the Banach spaces of the form $C(K)$ are linearly homeomorphic, where K is any uncountable compact metrizable space. If K is a countably infinite compact space then it is homeomorphic to an ordered space of ordinals. So for countable K the Bessaga-Pełczyński Theorem describing the isomorphism classes of spaces of continuous functions on countable ordinals applies.

Actually, we shall show that the assertion of Theorem 1.1 is satisfied by any *Cook continuum M*. A Cook continuum is a non-trivial metrizable continuum M having the property that for every subcontinuum H, every continuous function $f: H \longrightarrow M$ is either the identity or is constant. The first Cook continuum was constructed by Cook [4, Theorem 8]. Maćkowiak [10, Corollary 6.2] constructed a planar (in fact, chainable) Cook continuum. It was demonstrated in [14, §4] that for any Cook continuum M, $C_p(M)$ is not linearly homeomorphic to its square. To obtain the stronger result concerning uniform homeomorphisms we shall combine the construction in [14] and a theorem of Gul'ko (explained in §2) with the Borsuk-Ulam Antipodal Theorem. We do not know if $C_p(M)$ and $C_p(M) \times C_p(M)$ are homeomorphic.

The question which spaces X have the property that $C_p(X)$ is Lindelöf is still not quite solved, although some interesting partial results are known. The most general positive result is that the function space of every Corson compact space is Lindelöf (see [3, IV.2.22]). The most general negative result that we are aware of is the theorem of Nachmanson [13] (see also [3, IV.10]) that $C_p(X)$ is not Lindelöf if X is any nonmetrizable linearly orderable compactum. The referee of the paper kindly pointed out that this result was obtained independently by Sipacheva [15].

Every Corson compact space is monolithic and has the property that every linearly orderable closed subspace of it is metrizable. In view of Nachmanson's result, an Aronszajn continuum is an example of a first countable monolithic compact space whose function space is not Lindelöf. But this example trivially has a nonmetrizable linearly orderable closed subspace. There is an example of a compact monolithic space of countable tightness whose function space *is* Lindelöf and which is not Corson compact (see [3, §IV.7.1] and [2, 6.13] for details). This space has the property that every linearly orderable closed subspace of it is metrizable. These (known) remarks prompted Arhangel′skiĭ to ask in [2, Problem 45] whether $C_p(X)$ is Lindelöf provided that X is a monolithic compact space of countable tightness which has the additional property that every linearly orderable subspace of it is metrizable. Theorem 1.2 provides a strong negative answer to this question under the existence of a Souslin continuum.

2. Gul'ko's support maps. We shall denote by $\mathcal{F}(X)$ the space of finite nonempty subsets of X endowed with the Vietoris topology, cf. [9, §17].

Theorem 2.1 (Gul'ko)**.** *Let $\Phi: C_p(X) \longrightarrow C_p(Y)$ be a uniform homeomorphism, X and Y being compact metrizable spaces. Then there exists a function $K: Y \longrightarrow \mathcal{F}(X)$ such that, for any $y \in Y$,*

$$\sup\{|\Phi(f)(y) - \Phi(g)(y)| : \ \|f - g\|_{K(y)} \leqq 1\} < \infty. \tag{2}$$

In addition, each nonempty subset F of Y contains a relatively open nonempty subset V such that the restriction of K to V is continuous.

This is a special case of the results 1.10, 1.11, 1.13, 1.19 established by Gul'ko in [7]. For more information on this topic we refer the reader to [12].

Remark 2.2. Let $\Phi: C_p(X) \longrightarrow C_p(Y)$ be a homeomorphism such that $\Phi(\underline{0}) = \underline{0}$. Here $\underline{0}$ denotes the constant function with value 0 on a given space. We claim that the continuity of Φ^{-1} at $\underline{0}$ in $C_p(Y)$ guarantees that for any $x \in X$ there is a finite set $E(x) \subseteqq Y$ such that $|\Phi^{-1}(u)(x) - \Phi^{-1}(\underline{0})(x)| < 1/2$, whenever $u \in C_p(Y)$ vanishes on $E(x)$. This follows from the following simple reasoning. The function $\psi: C_p(Y) \to \mathbb{R}$ defined by $\psi(f) = \Phi^{-1}(f)(x)$ is clearly continuous. Observe that $\psi(\underline{0}) = 0$. There consequently is a neighborhood U of $\underline{0}$ in $C_p(Y)$ such that $|\psi(u)| < 1/2$ for every $u \in U$. Since $C_p(Y)$ is endowed with the topology of pointwise convergence, there is a finite subset $F \subseteqq Y$ such that if $u \in C_p(Y)$ and $\underline{0}$ agree on F then $u \in U$. It is clear that $E(x) = F$ is as required.

3. Proof of Theorem 1.1. Let us fix an arbitrary Cook continuum M, cf. Sec. 1. Let $n < m$ be natural numbers, $X = M \times \{1, \ldots, n\}$, $Y = M \times \{1, \ldots, m\}$, i.e., X and Y are respectively the unions of n or m disjoint copies of M. For $A \subseteqq M$, $x \in M$ and $i \leqq m$, we shall write

$$A_{(i)} = A \times \{i\}, \quad x_{(i)} = (x, i).$$

Similarly for $A \subseteqq M$, $x \in M$ and $j \leqq n$.

Striving for a contradiction, assume that $\Phi: C_p(X) \longrightarrow C_p(Y)$ is a uniform homeomorphism, and let $K: Y \longrightarrow \mathcal{F}(X)$ be the Gul'ko support map described in Theorem 2.1. We may assume without loss of generality that $\Phi(\underline{0}) = \underline{0}$.

We shall check that for any non-trivial continuum C in M and any pair (i, j), $i \leqq m$, $j \leqq n$, there is a non-trivial continuum $C' \subseteqq C$ and a finite set $D \subseteqq X$ such that

$$K(x_{(i)}) \cap M_{(j)} \subseteqq \{x_{(j)}\} \cup D \quad \text{for } x \in C'. \tag{3}$$

To begin with, we shall consider the continuum $C_{(i)}$ and use the properties of K to get a non-empty relatively open subset W of C such that K is continuous on $W_{(i)}$, and let H be a non-trivial continuum in W. The set $H' = \{x \in H : K(x_{(i)}) \cap M_{(j)} \neq \emptyset\}$ is open-and-closed in H, hence either $H' = \emptyset$ or $H' = H$. In the first case we just let $C' = C$ and $D = \emptyset$. In the second case we consider the continuous map $S: H \longrightarrow \mathcal{F}(M_{(j)})$ defined by

$S(x) = K(x_{(i)}) \cap M_{(j)}$. As in [14, §4], basic properties of Cook continua allow one to get a continuum C' and a finite set D satisfying (3). For the reader's convenience, we provide more details to that effect in Remark 3.1, following the proof.

Now, let σ be an enumeration of the pairs (i, j), $i \leqq m$, $j \leqq n$, and let us choose subsequently non-trivial continua $C_1 \supseteqq C_2 \supseteqq \ldots \supseteqq C_{m\cdot n}$, such that, whenever $\sigma(k) = (i, j)$, condition (3) is satisfied with $C' = C_k$ and $D = D_k$. Then $T = C_{m\cdot n}$ and $J = \bigcup\{D_k : k \leq m\cdot n\}$ are a non-trivial continuum in M and a finite set in X such that

(4) $$K(x_{(i)}) \subseteqq \{x_{(1)}, \ldots, x_{(n)}\} \cup J \quad \text{for } x \in T,\ i \leqq m.$$

By Remark 2.2, there is a finite set $E \subseteqq Y$ such that for any $u \in C_p(Y)$,

(5) $$u \restriction E = \underline{0} \ \text{ implies } \ \|\Phi^{-1}(u) - \Phi^{-1}(\underline{0})\|_J < 1/2.$$

Let us pick

(6) $$c \in T \ \text{ with } \ c_{(i)} \notin E \ \text{ for } \ i \leqq m,$$

and let $u_i \in C_p(Y)$ satisfy the conditions

(7) $$u_i(c_{(i)}) = 1, \ \ u_i \restriction E = \underline{0}, \ \ u_i \restriction M_{(j)} = \underline{0} \ \text{ for } \ i \neq j.$$

To reach a contradiction, we shall consider the continuous function

$$\phi\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

which is the composition of the map $(t_1, \ldots, t_m) \to \sum_{i=1}^{m} t_i u_i$ from $\mathbb{R}^m$ to $C_p(Y)$, the map $\Phi^{-1}\colon C_p(Y) \longrightarrow C_p(X)$, and the evaluation $u \to (u(c_{(1)}), \ldots, u(c_{(n)}))$ from $C_p(X)$ to $\mathbb{R}^n$, i.e.,

(8) $$\phi(t_1, \ldots, t_m) = \left(\Phi^{-1}\left(\sum_{i=1}^{m} t_i u_i\right)(c_{(j)})\right)_{j=1}^{n}$$

Let for $i \leqq m$,

$$\alpha_i = \sup\{|\Phi(f)(c_{(i)}) - \Phi(g)(c_{(i)})| : \ \|f - g\|_{K(c_{(i)})} \leqq 1\}.$$

Observe that by (2), $\alpha_i < \infty$ for every $i \leqq m$. Striving for a contradiction, assume that for some $i \leqq m$ we have $\alpha_i = 0$. Define $\xi\colon C_p(X) \to \mathbb{R}$ by $\xi(f) = \Phi(f)(c_{(i)})$. We claim that ξ is constant. This will give us the desired contradiction since Φ is surjective and the evaluation at c_i maps $C_p(Y)$ onto $\mathbb{R}$. Pick arbitrary elements $f, g \in C_p(X)$; we will prove that $\xi(f) = \xi(g)$. To this end, let F be the finite set $K(c_{(i)})$. Observe that our assumption $\alpha_i = 0$ implies that if $v, w \in C_p(X)$ are such that $\|v - w\|_F \leqq 1$ then $\xi(v) = \xi(w)$. The functions $f \restriction F$ and $g \restriction F$ both belong to the Euclidean space $\mathbb{R}^F$. There clearly are for some k elements $v_1, \ldots, v_k \in \mathbb{R}^F$ such that

$$f \restriction F = v_1, \quad g \restriction F = v_k,$$

while moreover

$$\|v_j - v_{j+1}\| \leqq 1$$

for every $j \leqq k-1$. Here $\|\cdot\|$ stands for the sup-norm on $\mathbb{R}^F$. Put $\tilde{v}_1 = f$, $\tilde{v}_k = g$, and for every $2 \leqq j \leqq k-1$ let $\tilde{v}_j \colon X \to \mathbb{R}$ be an arbitrary continuous function extending v_j. Observe that $\|\tilde{v}_j - \tilde{v}_{j+1}\|_F \leqq 1$ for every $j \leqq k$. As a consequence,

$$\xi(f) = \xi(\tilde{v}_1) = \cdots = \xi(\tilde{v}_k) = \xi(g),$$

which is as required.

Put

$$\alpha = \max\{\alpha_i : i \leqq m\}, \quad r = \alpha\sqrt{m}. \tag{9}$$

By the above considerations, $r \in (0, \infty)$. Let $\mathbb{S}^{m-1} = \{(t_1, \dots, t_m) : \sum_{i=1}^{m} t_i^2 = r^2\}$ be the r-sphere centered at zero in the Euclidean space $\mathbb{R}^m$. The mapping ϕ takes $\mathbb{S}^{m-1}$ into $\mathbb{R}^n$ and therefore, by the Borsuk-Ulam Theorem [5, XVI 6.2], ϕ identifies a pair of antipodal points, i.e., there exists $(t_1, \dots, t_m) \in \mathbb{S}^{m-1}$ such that

$$\phi(t_1, \dots, t_m) = \phi(-t_1, \dots, -t_m). \tag{10}$$

Since $\sum_{i=1}^{m} t_i^2 = r^2$, we infer from (9) that

$$|t_l| \geqq \alpha \quad \text{for some} \quad l \leqq m. \tag{11}$$

Let us consider

$$u = \sum_{i=1}^{m} t_i u_i, \quad f = \Phi^{-1}(u), \quad g = \Phi^{-1}(-u). \tag{12}$$

By (8) and (10), we have

$$f(c_{(j)}) = g(c_{(j)}) \quad \text{for} \quad j \leqq n. \tag{13}$$

Moreover, (7) and (12) show that $u \restriction E = \underline{0}$, and hence by (5),

$$\|f - g\|_J < 1. \tag{14}$$

Putting together (4), (13) and (14) we infer that $\|f - g\|_{K(c_{(l)})} < 1$ and hence, by (9),

$$|\Phi(f)(c_{(l)}) - \Phi(g)(c_{(l)})| \leq \alpha. \tag{15}$$

However, by (12), $\Phi(f) = u$, $\Phi(g) = -u$, and by (7), $u(c_{(l)}) = t_l$. But, by (11), this is impossible, and the contradiction ends the proof of Theorem 1.1.

Remark 3.1. We shall give some more details concerning condition (3) in the proof. Let $S: H \longrightarrow \mathcal{F}(M_{(j)})$ be a continuous map, where H is a non-trivial continuum in the Cook continuum M. As was noticed in [14], Lemma 3.2, the continuity of S yields a nonempty relatively open set G in H, pairwise disjoint open sets $V_1, \dots, V_k$ in $M_{(j)}$ and continuous mappings $s_p : G \longrightarrow V_p$ such that $S(t) = \{(s_1(t))_{(j)}, \dots, (s_k(t))_{(j)}\}$ for $t \in G$. Let C' be a non-trivial continuum in G. Then, $s_p : C' \longrightarrow M_{(j)}$ being continuous, the properties of the Cook continuum M imply that for each p, either $s_p(x) = x_{(j)}$ for all $x \in C'$, or there is d_p with $s_p(x) = d_p$ for all $x \in C'$. Then the continuum C' and the set D consisting of all points d_p satisfies property (3) in the proof.

4. Resolutions. In the proof of Theorem 1.2 we will replace each element of a subspace of a certain space by a copy of the unit interval $\mathbb{I}$ so that certain 'essential' properties are preserved. This can be done by a transfinite inverse limit construction. But it is easier to use the resolution method of Fedorchuk [6] (see also Watson [17]).

Suppose that X is a topological space and that $\{Y_x : x \in X\}$ are topological spaces and, for each $x \in X$, $f_x : X \setminus \{x\} \to Y_x$ is a continuous function. We topologize

$$Z = \bigcup \{\{x\} \times Y_x : x \in X\}$$

as follows. If $x \in X$, $U \subseteq X$ is open such that $x \in U$, and $W \subseteq Y_x$ is open then

$$U \otimes W = (\{x\} \times W) \cup \bigcup \{\{x'\} \times Y_{x'} : x' \in U \cap f_x^{-1}[W]\}.$$

The collection of all sets of the form $U \otimes W$ is an open basis for Z. Topologized in this way, Z is called the *resolution* of X at each point $x \in X$ into Y_x by the mapping f_x. Let $\pi_0 : Z \to X$ be the 'projection'. Then π_0 is continuous by Watson [17, 3.1.35].

Suppose that for certain $x \in X$ we have that Y_x is a singleton. Then the topology of Y_x is irrelevant and so for convenience we may assume that $Y_x = \{x\}$. Then $(x, x) \in Z$ and a basic neighborhood of (x, x) has the form

$$\pi_0^{-1}[U] = \{(x, x)\} \cup \bigcup \{\{x'\} \times Y_{x'} : x' \in U\},$$

where U is an arbitrary open neighborhood of x in X.

Lemma 4.1. *Suppose that Z is the resolution of X at each point $x \in X$ into Y_x by the function f_x. Assume that $D \subseteq X$ is second countable and for all but countably many $d \in D$ we have that Y_d is a singleton. In addition, assume that Y_d is second countable for every $d \in D$. Then $\pi_0^{-1}[D]$ is second countable.*

Proof. Let $\mathcal{U}$ be a countable open base for D, and let E be the countable set of all points $d \in D$ for which Y_d is not a singleton. By the above, the countable collection

$$\{\pi_0^{-1}[U] : U \in \mathcal{U}\}$$

is a local base (in $\pi_0^{-1}[D]$) at every point of the form (x, x), where $x \in D \setminus E$. Let $\mathcal{V}_e$ be a countable base for Y_e for every $e \in E$. For a fixed $e_0 \in E$, the countable collection of sets of the form $U \otimes W$, where $U \in \mathcal{U}$ and $e \in E$ and $W \in \mathcal{V}_e$, contains a local base at every point of $\pi_0^{-1}(e_0)$. So we are done. $\square$

Let X and Y be spaces, and for certain $x \in X$, let $f: X \setminus \{x\} \to Y$ be a continuous function. The *boundary* $\partial_f Y$ of Y is the set of all elements $y \in Y$ having the following property: for every neighborhood U of x in X and for every neighborhood V of y in Y we have

$$U \cap f^{-1}[V] \neq \emptyset.$$

Assume that in our resolution the space Y_x is a singleton and x is not an isolated point of X. Then clearly $\partial_{f_x} Y_x = Y_x$.

Lemma 4.2. *Let X be a dense in itself space containing a point x such that $X \setminus \{x\}$ is normal and X is first countable at x. Then there is a continuous function $f: X \setminus \{x\} \to \mathbb{I}$ such that $\partial_f \mathbb{I} = \mathbb{I}$.*

Proof. (Standard.) Let D be a discrete sequence in $X \setminus \{x\}$ converging to x. In addition, let $g: D \to \mathbb{I}$ be such that $g[D]$ is dense and every fiber of g is infinite. Extend g to a continuous function $f: X \setminus \{x\} \to \mathbb{I}$. Then f is clearly as required. $\square$

We now summarize some basic properties of resolutions that are important to us.

It is known that if X and every Y_x for $x \in X$ is compact then so is Z. As noted above, $\pi_0: Z \to X$ is continuous. Finally, if $\partial_{f_x} Y_x = Y_x$ for every $x \in X$ then π_0 is ireducible (Watson [17, 3.1.33 and 3.1.35]).

A space in which every nowhere dense set is second countable is called *almost Luzin*. This concept is due to Kunen [8] who showed that every almost Luzin space without isolated points is hereditarily Lindelöf (abbreviated: HL).

Corollary 4.3. *Suppose that Z is the resolution of X at each point $x \in X$ into Y_x by the function f_x. Assume that*

(16) *X is compact, has no isolated points and is almost Luzin,*
(17) *Y_x is second countable for every $x \in X$.*
(18) *$\partial_{f_x} Y_x = Y_x$ for every $x \in X$,*
(19) *if $D \subseteq X$ is nowhere dense then for all but countably many $x \in D$ we have that Y_x is a singleton.*

Then Z is compact, has no isolated points and is almost Luzin (*hence is* HL).

Proof. We only need to prove that Z has no isolated points, and is almost Luzin.

To prove that Z has no isolated points, assume that

$$U \otimes W = (\{x\} \times W) \cup \bigcup \{\{x'\} \times Y_{x'} : x' \in U \cap f_x^{-1}[W]\}.$$

is a nonempty basic open subset of Z, where $U \subseteq X$ is open, $x \in U$ and $W \subseteq Y_x$ is open. By (18) we have that $U \cap f_x^{-1}[W]$ is nonempty, hence contains at least two distinct points, say x' and x''. So $U \otimes W$ contains the disjoint nonempty sets $\{x'\} \times Y_{x'}$ and $\{x''\} \times Y_{x''}$, hence is not a singleton.

Next, assume that D is a nowhere dense closed subset of Z. Then $E = \pi_0[D]$ is a nowhere dense closed subset of X since π_0 is irreducible. As a consequence, E is second countable since X is almost Luzin. So we are done by (19) and Lemma 4.1. $\square$

5. Proof of Theorem 1.2. Our construction is based on the existence of a Souslin continuum, i.e., an ordered continuum which satisfies the countable chain condition (abbreviated: ccc) but is not separable. It is well-known that such a continuum exists under $\diamondsuit$. For more information, see Todorčević [16].

Let S be a Souslin continuum. We assume that S has no separable nontrivial intervals. This implies that S has weight ω_1 and a subset of S is nowhere dense if and only if it is second countable. Hence S is almost Luzin.

There are second countable closed subsets T_α of S for $\alpha < \omega_1$ such that

(20) $\alpha < \beta < \omega_1 \;\to\; T_\alpha \subsetneqq T_\beta$,

(21) $\bigcup_{\alpha<\omega_1} T_\alpha = S$.

Observe that if $D \subsetneqq S$ is nowhere dense then D is separable and hence $D \subsetneqq T_\alpha$ for certain $\alpha < \omega_1$.

Pick a dense subset D in S of size ω_1 such that $D \cap T_\alpha$ is countable for every $\alpha < \omega_1$.

We plan to replace each element of D by the unit interval $\mathbb{I}$ without losing the 'essential' properties of S. It is easy to do this. By Lemma 4.2 there is for every $d \in D$ a continuous function $f_d \colon S \setminus \{d\} \to \mathbb{I}$ such that $\partial_{f_d}\mathbb{I} = \mathbb{I}$. For every $x \in S \setminus D$ let $Y_x = \{x\}$ (this also specifies the function f_x). Now let Z be the resolution of S at each point $x \in S$ into Y_x by the function f_x. We claim that Z is the space we are after. As usual, $\pi_0 \colon Z \to S$ is the 'projection'.

The space Z is compact and dense in itself and almost Luzin by Corollary 4.3. Hence Z is HL by Kunen's result cited in the previous section. By the same reasoning, the closure of any countable set in Z is second countable. Since Z has clearly weight at most ω_1, this implies that Z is monolithic.

Since $\pi_0 \colon Z \to S$ is a continuous surjection between compact spaces, the map adjoint to π_0 is a closed embedding of $C_p(S)$ into $C_p(Z)$. Hence $C_p(Z)$ is not Lindelöf since $C_p(S)$ is not Lindelöf by Nachmanson's result quoted in the introduction.

So it suffices to prove that if L is a linearly orderable closed subspace of Z then L is metrizable. To this end, let L be any linearly orderable closed subspace of Z. We may assume without loss of generality that L is not nowhere dense. So $\pi_0[L]$ is not nowhere dense since π_0 is irreducible. As a consequence, L contains the preimage of a nonempty open subset of S. Since D is dense, this means that L contains uncountably many pairwise disjoint copies of $\mathbb{I}$. Since L is orderable, all of those have nonempty interior in L. But this contradicts Z being HL.

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