COMPLETE ERDŐS SPACE IS UNSTABLE

JAN J. DIJKSTRA, JAN VAN MILL, AND JURIS STEPRĀNS

Abstract. It is proved that the countably infinite power of complete Erdős space $\mathfrak{E}_{\mathrm{c}}$ is not homeomorphic to $\mathfrak{E}_{\mathrm{c}}$. The method by which this result is obtained consists of showing that $\mathfrak{E}_{\mathrm{c}}$ does not contain arbitrarily small closed subsets that are one-dimensional at every point. This observation also produces solutions to several problems that were posed by Aarts, Kawamura, Oversteegen, and Tymchatyn. In addition, we show that the original (rational) Erdős space does contain arbitrarily small closed sets that are one-dimensional at every point.

1. Introduction

In [10] Paul Erdős considered the space $\mathfrak{E}$ consisting all vectors in the Hilbert space ℓ^2 all of whose coordinates are rational and proved that this space is totally disconnected but not zero-dimensional. The space that is the primary subject of this note consists of all vectors in Hilbert space with only irrational coordinates and is referred to as *complete Erdős space* $\mathfrak{E}_{\mathrm{c}}$.

The spaces $\mathfrak{E}$ and $\mathfrak{E}_{\mathrm{c}}$ are important examples of almost zero-dimensional spaces, a concept that was introduced by Oversteegen and Tymchatyn [13] who proved that such a space is always at most one-dimensional. We will call a space X *almost zero-dimensional* if every point $x \in X$ has arbitrarily small neighbourhoods U that can be written as an intersection of clopen subsets of the space. Note that it is immediate that almost zero-dimensionality is hereditary and productive. The definition given here is easier to work with than the definition in [13] where there is the additional requirement that U have a dense interior. We verify in §6 that both definitions are equivalent.

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It is clear that $\mathfrak{E}_c$ is homeomorphic to its own square (hence $\mathfrak{E}_c^n \approx \mathfrak{E}_c$ for every $n \in \mathbb{N}$). The aim of this note is to prove that $\mathfrak{E}_c$ is not homeomorphic to its countably infinite power $\mathfrak{E}_c^{\mathbb{N}}$, that is, $\mathfrak{E}_c$ is unstable. We accomplish this by showing in §3 that $\mathfrak{E}_c$ does not contain arbitrarily small closed sets that are one-dimensional at every point. This result enables us also to prove that several questions in the literature can be answered negatively. For example, $\mathfrak{E}_c$ is not the only homogeneous, almost zero-dimensional, one-dimensional, topologically complete, and pulverized space. In addition, $\mathfrak{E}_c$ is not homeomorphic to the homeomorphism group of the hairy arc, and not homeomorphic to the homeomorphism groups of the universal Menger continua, see §4. These observations solve problems that were posed by Aarts, Kawamura, Oversteegen, and Tymchatyn, see [2, 11, 12]. Interestingly, we prove in §5 that the rational Erdős space $\mathfrak{E}$ does contain arbitrarily small closed sets that are one-dimensional at every point.

2. Preliminaries

Every topological space in this note is assumed to be separable and metrizable. We let $\mathbb{Q}$ denote the subspace of rational numbers of the real line $\mathbb{R}$. Let $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$, the space of irrational numbers, and let $\mathbb{I}$ stand for the interval $[0, 1]$.

It is easy to see that $\mathfrak{E}_c$ is almost zero-dimensional. We will now present the argument in a form that we will use in the proof of Theorem 3.1. Consider the topological vector space $\mathbb{R}^{\mathbb{N}}$ with the product topology and define the Hilbert norm from $\mathbb{R}^{\mathbb{N}}$ to $[0, \infty]$ by

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}$$

for any $x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}$. It is well-known that $\|\cdot\|$ is a lower semicontinuos (LSC) function on $\mathbb{R}^{\mathbb{N}}$, that is, $\{x \in \mathbb{R}^{\mathbb{N}} : \|x\| \le M\}$ is closed for every $M \in [0, \infty]$. Hilbert space ℓ^2 is defined as the vector space $\{x \in \mathbb{R}^{\mathbb{N}} : \|x\| < \infty\}$ equipped with the topology that is generated by the norm $\|\cdot\|$.

Let $\alpha \colon \ell^2 \to \mathbb{R}^{\mathbb{N}}$ be the continuous injection defined by $\alpha(x) = x$. Put

$$\mathfrak{E}_c = \{x \in \ell^2 : \alpha(x) \in \mathbb{P}^{\mathbb{N}}\}$$

and note that this space is a G_δ-subset of ℓ^2 and hence topologically complete. Since the norm is LSC on $\mathbb{R}^{\mathbb{N}}$ the closed neighbourhood

$$B_\varepsilon(x) = \{y \in \mathfrak{E}_c : \|y - x\| \le \varepsilon\}$$

has the property that $\alpha(B_\varepsilon(x))$ is closed in $\mathbb{P}^{\mathbb{N}}$ for each $x \in \mathfrak{E}_{\mathrm{c}}$ and $\varepsilon > 0$. Since $\mathbb{P}^{\mathbb{N}}$ is a zero-dimensional space this shows that $\mathfrak{E}_{\mathrm{c}}$ is almost zero-dimensional.

Erdős' proof [10] of the one-dimensionality of the original Erdős space applies also to $\mathfrak{E}_{\mathrm{c}}$ and shows that the empty set is the only bounded clopen subset of $\mathfrak{E}_{\mathrm{c}}$, see also Dijkstra [8, Lemma 1]. This means that if we add a point ∞ to $\mathfrak{E}_{\mathrm{c}}$ whose neighbourhoods are the complements of bounded sets then the resulting space $\mathfrak{E}_{\mathrm{c}} \cup \{\infty\}$ is connected. We call a space *connectible* if it can be imbedded into a connected space in such a way that the remainder is a singleton. In [11] a totally disconnected but connectible space is called *pulverized*. A space is called *somewhere zero-dimensional* if it contains a point with a neighbourhood basis consisting of clopen sets. The following facts are easily verified: a connectible space cannot be somewhere zero-dimensional, an open subspace of a connectible space is connectible, and the product of any space with a connectible space is connectible.

3. $\mathfrak{E}_{\mathrm{c}}$ AND $\mathfrak{E}_{\mathrm{c}}^{\mathbb{N}}$

In this section we will show that $\mathfrak{E}_{\mathrm{c}}$ and $\mathfrak{E}_{\mathrm{c}}^{\mathbb{N}}$ are not homeomorphic.

Theorem 3.1. *Every bounded, closed, and nonempty subset of $\mathfrak{E}_{\mathrm{c}}$ is somewhere zero-dimensional.*

Proof. Let ρ be a complete metric on $\mathfrak{E}_{\mathrm{c}}$ such that $\operatorname{diam}_\rho \mathfrak{E}_{\mathrm{c}} \leq 1$. Let X be closed subset of $\mathfrak{E}_{\mathrm{c}}$ that is bounded and nonempty. Choose an $M > 0$ such that

$$X \subset B = \{x \in \mathfrak{E}_{\mathrm{c}} \colon \|x\| \leq M\}$$

and note that $\alpha(B)$ is a topologically complete space, being a closed subset of $\mathbb{P}^{\mathbb{N}}$. We construct by induction a sequence of nonempty clopen subsets $C_0 \supset C_1 \supset \cdots$ of X such that for each n, $\operatorname{diam}_\rho C_n \leq 2^{-n}$.

Put $C_0 = X$. Assume now that C_n has been found. The open set $\mathfrak{E}_{\mathrm{c}} \setminus C_n$ can be written as a union of a countable collection of closed balls $\{F_i \colon i \in \mathbb{N}\}$ so that each $\alpha(F_i)$ is closed in $\alpha(\mathfrak{E}_{\mathrm{c}})$, see §2. Consequently, $\alpha(C_n)$ is a G_δ-subset of $\alpha(B)$ and hence $\alpha(C_n)$ is topologically complete. Choose for every $x \in C_n$ an $\varepsilon(x) > 0$ such that $U(x) = C_n \cap B_{\varepsilon(x)}(x)$ has the property $\operatorname{diam}_\rho U(x) \leq 2^{-n-1}$. Select a countable set $\{x_i \colon i \in \mathbb{N}\}$ in C_n such that $C_n = \bigcup_{i=1}^{\infty} U(x_i)$. Observe that each $\alpha(U(x_i))$ is closed in $\alpha(C_n)$ because each $\alpha(B_\varepsilon(x))$ is closed in $\mathbb{P}^{\mathbb{N}}$. By the Baire Category Theorem we have that some $\alpha(U(x_i))$ has a nonempty interior in $\alpha(C_n)$. Since $\alpha(C_n)$ is zero-dimensional this means that $\alpha(U(x_i))$ contains a nonempty clopen subset K of $\alpha(C_n)$. Note that $C_{n+1} = \alpha^{-1}(K)$ is a nonempty clopen subset of C_n and hence of X. Since $C_{n+1} \subset U(x_i)$ we have $\operatorname{diam}_\rho C_{n+1} \leq 2^{-n-1}$ and the induction is complete.

Since ρ is complete and X is closed we have that $\bigcap_{n=0}^{\infty} C_n = \{x\}$ for some $x \in X$ and obviously the C_n's form a neighbourhood basis for x in X.

The topological property that we will use to distinguish $\mathfrak{E}_\mathrm{c}$ from a number of other pulverized topologically complete spaces is that every point in $\mathfrak{E}_\mathrm{c}$ has a neighbourhood U such that every nonempty closed subset of U is somewhere zero-dimensional, in particular, U fails to contain a closed copy of $\mathfrak{E}_\mathrm{c}$.

It is known that $\mathfrak{E}_\mathrm{c}$ is a homogeneous space (cf. Proposition 4.3) and Kawamura, Oversteegen, and Tymchatyn pose the following question in [11, Problem 1]: is every almost zero-dimensional, one-dimensional, topologically complete, pulverized, homogeneous space homeomorphic to $\mathfrak{E}_\mathrm{c}$? The following result shows that the answer is no.

Corollary 3.2. $\mathfrak{E}_\mathrm{c}^{\mathbb{N}}$ *is not homeomorphic to* $\mathfrak{E}_\mathrm{c}$.

Proof. We have that every subset of $\mathfrak{E}_\mathrm{c}^{\mathbb{N}}$ with a nonempty interior contains closed copies of the space itself namely sets of the form $\{(x_1, x_2, \ldots, x_n)\} \times \mathfrak{E}_\mathrm{c}^{\mathbb{N}}$.

It can be derived from results in [11] and [13] that $\mathfrak{E}_\mathrm{c}$ is a universal space for the class of almost zero-dimensional spaces, cf. [9]. Let X be an arbitrary nonempty almost zero-dimensional complete space. We then can identify X with a subspace of $\mathfrak{E}_\mathrm{c}$ and we can write $X = \bigcap_{i=1}^{\infty} O_i$ where every O_i is open in $\mathfrak{E}_\mathrm{c}$. X can now be imbedded as a closed subset (namely the diagonal) of $\prod_{i=1}^{\infty} O_i$. This product is homeomorphic to $\mathfrak{E}_\mathrm{c}^{\mathbb{N}}$ because it was proved in [11, Theorem 4] that every nonempty open subset of $\mathfrak{E}_\mathrm{c}$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$. In conclusion, every subset of $\mathfrak{E}_\mathrm{c}^{\mathbb{N}}$ with a nonempty interior contains a closed copy of every almost zero-dimensional complete space, where as $\mathfrak{E}_\mathrm{c}$ obviously does not contain a closed copy of $\mathfrak{E}_\mathrm{c}^{\mathbb{N}}$. This means that $\mathfrak{E}_\mathrm{c}^{\mathbb{N}}$ is "more universal" than $\mathfrak{E}_\mathrm{c}$ and a better candidate for being the "maximal" element of the class of almost zero-dimensional complete spaces.

It is obvious that $\mathfrak{E}_\mathrm{c} \times \mathfrak{E}_\mathrm{c}$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$. The Hilbert product $\ell^2(\mathfrak{E}_\mathrm{c})$ of $\mathfrak{E}_\mathrm{c}$ is defined as the set $\{x \in \mathfrak{E}_\mathrm{c}^{\mathbb{N}} \colon \sum_{i=1}^{\infty} \|x_i\|^2 < \infty\}$ equipped with the topology that is generated by the metric $\sqrt{\sum_{i=1}^{\infty} \|x_i - y_i\|^2}$. Note that $\ell^2(\mathfrak{E}_\mathrm{c})$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$.

The following result answers a question that was also posed in [11, p. 98]: is every pulverized and dense G_δ-subset of $\mathfrak{E}_\mathrm{c}$ homeomorphic to $\mathfrak{E}_\mathrm{c}$?

Corollary 3.3. *There exists a connectible and dense G_δ-subset of $\mathfrak{E}_c$ that is not homogeneous.*

Proof. Let a be some fixed element of $\mathfrak{E}_c$. Consider the dense G_δ-subset $G \times \mathfrak{E}_c$ of $\mathfrak{E}_c \times \mathfrak{E}_c$, where $G = \{x \in \mathfrak{E}_c \colon 1/\|x - a\| \notin \mathbb{N}\}$. Since $\mathfrak{E}_c$ is connectible we have that $G \times \mathfrak{E}_c$ is connectible as well. Note that for each $n \in \mathbb{N}$, the set $V_n = \{x \in \mathfrak{E}_c \colon 1/(n+1) < \|x-a\| < 1/n\}$ is clopen in G and open in $\mathfrak{E}_c$, so V_n is connectible. Thus every neighbourhood of a point (a, x) in $G \times \mathfrak{E}_c$ contains a connectible and closed $V_n \times \{x\}$ (and hence $G \times \mathfrak{E}_c$ is not homeomorphic to $\mathfrak{E}_c$). Since $(G \setminus \{a\}) \times \mathfrak{E}_c$ is open in $\mathfrak{E}_c \times \mathfrak{E}_c$ every point of that set has a neighbourhood that contains only closed nonempty sets that are somewhere zero-dimensional. So $G \times \mathfrak{E}_c$ is not homogeneous.

4. Homeomorphism groups

We will now consider some interesting homeomorphism groups that are almost zero-dimensional but not zero-dimensional. If X is a compact metric space then $\mathcal{H}(X)$ is the group of autohomeomorphisms of X with the topology of uniform convergence. Let 1_X denote the identity. If O is an open subset of X then $\mathcal{H}_O(X) = \{h \in \mathcal{H}(X) \colon h|X \setminus O = 1_{X \setminus O}\}$ is the closed subgroup of $\mathcal{H}(X)$ consisting of homeomorphisms that are supported on O.

In [1] Aarts and Oversteegen introduce a continuum $\mathfrak{H}$ called the hairy arc. The hairy arc is topologically unique and can be represented as follows. Let $l \colon \mathbb{I} \to \mathbb{I}$ be a function such that

(a) l is upper semicontinuous, that is, $\{x \in \mathbb{I} \colon l(x) < t\}$ is open for each $t \in \mathbb{I}$,

(b) $l(0) = l(1) = 0$ and the set $\{x \in \mathbb{I} \colon l(x) = 0\}$ and its complement are both dense in $\mathbb{I}$, and

(c) for each $x \in \mathbb{I}$ with $l(x) > 0$ there exist sequences $(a_n)_n$ and $(b_n)_n$ in $\mathbb{I}$ such that $a_n \nearrow x$ and $b_n \searrow x$ and $\lim l(a_n) = \lim l(b_n) = l(x)$.

Then $\mathfrak{H}_l = \{(x, y) \in \mathbb{I}^2 \colon y \leq l(x)\}$ and a *hairy arc* $\mathfrak{H}$ is any space that is homeomorphic to an $\mathfrak{H}_l$. According to [1, Theorem 3.2] all hairy arcs are homeomorphic to each other. The set of endpoints of hairs $E_l = \{(x, l(x)) \colon l(x) > 0\}$ of $\mathfrak{H}_l$ is dense in $\mathfrak{H}_l$ and Kawamura, Oversteegen, and Tymchatyn [11, §5] prove that E_l is homeomorphic to $\mathfrak{E}_c$ as an application of their Characterization Theorem.

Aarts and Oversteegen prove that $\mathcal{H}(\mathfrak{H})$ is almost zero-dimensional but not zero-dimensional and they ask in [2, Problem 2.9] whether the homeomorphism group of the hairy arc is homeomorphic to the set of endpoints of the hairy arc. We show that the answer is no.

Corollary 4.1. *The homeomorphism group of the hairy arc is not homeomorphic to $\mathfrak{E}_c$.*

Proof. Consider $\mathfrak{H}_l$ and its *base arc* $B = \mathbb{I} \times \{0\}$. Let $\mathcal{H}^+$ stand for the subgroup of $\mathcal{H}(\mathfrak{H}_l)$ consisting of homeomorphisms that fix the endpoints of B. Since every element of $\mathcal{H}(\mathfrak{H}_l)$ maps B onto B (see [2, Corollary 1.5]) $\mathcal{H}^+$ is a clopen subgroup of $\mathcal{H}(\mathfrak{H}_l)$ and hence one-dimensional at each element. We will show that every neighbourhood of the identity contains closed copies of $\mathcal{H}^+$ which means that neither $\mathcal{H}^+$ nor $\mathcal{H}(\mathfrak{H}_l)$ can be homeomorphic to $\mathfrak{E}_c$.

Let $\varepsilon > 0$ and select an $a \in (0, \varepsilon)$ such that $l(a) = 0$ and $l(x) < \varepsilon$ for all $x \in [0, a]$. Then $O = \mathfrak{H}_l \cap ([0, a) \times [0, \varepsilon))$ is an open subset of $\mathfrak{H}_l$ such that its closure $O \cup \{(a, 0)\}$ is a hairy arc (stretch the interval $[0, a]$ to $\mathbb{I}$). We obviously have that $\mathcal{H}^+$, $\mathcal{H}_O(\overline{O})$, and $\mathcal{H}_O(\mathfrak{H}_l)$ are all homeomorphic. Using the max metric on $\mathbb{I}^2$ we have that $\operatorname{diam} O < \varepsilon$ so every element of the closed group $\mathcal{H}_O(\mathfrak{H}_l)$ is ε-close to the identity and the proof is complete.

The *Lelek fan* is a space that can be obtained by identifying the base arc of the hairy arc to a point. A similar argument as we used for the hairy arc shows that the homeomorphism group of the Lelek fan is not homeomorphic to $\mathfrak{E}_c$ (use [7] or [5] instead of [1]).

Let μ^n, $n \in \mathbb{N}$, denote the *universal Menger continuum* of dimension n and let M_n^{n+1} denote the *n-dimensional Sierpiński carpet*, see [3] respectively [6]. Oversteegen and Tymchatyn [13, Theorem 5] show that $\mathcal{H}(\mu^n)$ and $\mathcal{H}(M_n^{n+1})$ are almost zero-dimensional and they conjecture that $\mathcal{H}(\mu^1)$ is homeomorphic to $\mathfrak{E}_c$, see [12, Conjecture 7.8]. We disprove this conjecture:

Corollary 4.2. *If $X = \mu^n$ for $n \in \mathbb{N}$ or $X = M_n^{n+1}$ for $n \in \mathbb{N} \setminus \{3\}$ then the homeomorphism group of X is not homeomorphic to $\mathfrak{E}_c$.*

Proof. Let $\varepsilon > 0$ and choose a nonempty open subset O of X such that $\operatorname{diam} O < \varepsilon$ with respect to some metric on X. In [8, Remarks 5 and 9] it is shown that $\mathcal{H}_O(X)$ is one-dimensional. (This result can be derived by combining [4, Theorem 2.1] with [3, 6]. However, there is a problem with the proof of [4, Theorem 2.1]; specifically, it is proved in [8, §3] that [4, Lemma 2.2] is false.) Since $\operatorname{diam} O < \varepsilon$ every element of $\mathcal{H}_O(X)$ is ε-close to the identity 1_X. So every neighbourhood of 1_X in $\mathcal{H}(X)$ contains a nonempty closed set that is one-dimensional and homogeneous and hence not somewhere zero-dimensional. We may conclude that $\mathcal{H}(X)$ is not homeomorphic to $\mathfrak{E}_c$.

We conclude this section by showing that the failure of $\mathcal{H}(\mathfrak{H})$ and $\mathcal{H}(\mu^n)$ to be homeomorphic to $\mathfrak{E}_c$ is not connected to the existence of a group structure. A group is *boolean* if every element equals its inverse.

Proposition 4.3. $\mathfrak{E}_c$ *admits the structure of a (boolean) topological group.*

Proof. Let C be the "middle third" Cantor set in $\mathbb{I}$. Every $a \in C$ has a unique representation as $a = \sum_{i=1}^{\infty} 2[a]_i 3^{-i}$, where $[a]_i \in \{0, 1\}$ for all $i \in \mathbb{N}$. Let $\triangle$ denote the standard boolean group operation on C: $[a \triangle b]_i = [a]_i + [b]_i \bmod 2$. Consider now the topological group $C^{\mathbb{N}}$ with $(x_1, x_2, \dots) \triangle (y_1, y_2, \dots) = (x_1 \triangle y_1, x_2 \triangle y_2, \dots)$. We define $\mathfrak{E}'_c = \{x \in \ell^2 \colon \alpha(x) \in C^{\mathbb{N}}\}$ and note that this space satisfies the Characterization Theorem [11, Theorem 3] and hence it is homeomorphic to $\mathfrak{E}_c$, see [8, Proposition 3].

Note that if $a, b, c, d \in C$ are such that $|a - b|, |c - d| < 3^{-n}$ then $[a]_i = [b]_i$ and $[c]_i = [d]_i$ for all $i \le n$ thus

$$|a \triangle c - b \triangle d| = \left| \sum_{i=n+1}^{\infty} 2([a \triangle c]_i - [b \triangle d]_i) 3^{-i} \right| \le \sum_{i=n+1}^{\infty} 2 \cdot 3^{-i} = 3^{-n}.$$

This implies that always

$$|a \triangle c - b \triangle d| \le 3 \max\{|a - b|, |c - d|\} \le 3(|a - b| + |c - d|).$$

Let $x, y, z, w \in \mathfrak{E}'_c$, so $x = (x_1, x_2, \dots)$ and so on. We have

$$\begin{aligned} \|x \triangle y - z \triangle w\| &= \sqrt{\textstyle\sum_{i=1}^{\infty} (x_i \triangle y_i - z_i \triangle w_i)^2} \\ &\le \sqrt{\textstyle\sum_{i=1}^{\infty} 9(|x_i - z_i| + |y_i - w_i|)^2} \\ &\le 3(\|x - z\| + \|y - w\|). \end{aligned}$$

This result means that $\mathfrak{E}'_c$ is closed under $\triangle$ and that the operation is continuous with respect to the norm topology. Since the group is boolean this suffices to show that $\mathfrak{E}'_c$ is a topological group.

If we do not require the group to be boolean then we can represent $\mathfrak{E}_c$ by a closed subgroup of $(\ell^2, +)$:

$$G = \{x \in \ell^2 \colon n x_n \in \mathbb{Z} \text{ for every } n \in \mathbb{N}\}.$$

Proposition 3 in [8] shows that also G is homeomorphic to $\mathfrak{E}_c$.

5. The space $\mathfrak{E}$

It is clear that the method used to prove Theorem 3.1 relies heavily on completeness and does not work for the original Erdős space $\mathfrak{E}$. In fact, we show that Theorem 3.1 is false for $\mathfrak{E}$:

Theorem 5.1. *Every nonempty open subset of $\mathfrak{E}$ contains a nonempty subset that is closed in $\mathfrak{E}$ and that is one-dimensional at every point.*

Proof. Since $\mathfrak{E}$ is a vector space over $\mathbb{Q}$ it suffices to prove that there exists a bounded, closed, and nonempty subset Y of $\mathfrak{E}$ that is one-dimensional at every point.

Our construction takes place in the product space $\mathbb{Q}^{\mathbb{N}}$. If $x \in \mathbb{Q}^{\mathbb{N}}$ and $\varepsilon > 0$ then we define the closed set

$$F_\varepsilon(x) = \{y \in \mathbb{Q}^{\mathbb{N}} \colon \|x - y\| \le \varepsilon\}.$$

Recall that $\mathfrak{E} = \bigcup_{n=1}^{\infty} F_n(\mathbf{0})$ equipped with the norm topology, where $\mathbf{0}$ stands for the zero vector. For $A \subset \mathbb{Q}^{\mathbb{N}}$ we define $\operatorname{diam} A = \sup\{\|x - y\| \colon x, y \in A\}$. For $n \in \mathbb{N}$ we will identify $\mathbb{Q}^n$ with $\{x \in \mathbb{Q}^{\mathbb{N}} \colon x_i = 0$ for $i > n\}$ and we define the projection $\xi_n \colon \mathbb{Q}^{\mathbb{N}} \to \mathbb{Q}^n$ by $\xi_n(x) = (x_1, \dots, x_n, 0, 0, \dots)$. Put $D = \bigcup_{n=1}^{\infty} \mathbb{Q}^n$ and note that D is a countable dense subset (but not a subspace) of $\mathfrak{E}$. Let $\{(t_i, n_i) \colon i \in \mathbb{N}\}$ enumerate the set $D \times \mathbb{N}$ in such way that $t_1 = \mathbf{0}$ and for each $i \in \mathbb{N}$ we have $n_i \le i$ and $\xi_i(t_i) = t_i$.

We construct by induction two sequences of sets Y_i and U_i such that for every $i \in \mathbb{N}$,

(1) Y_i is a closed subset of $\mathbb{Q}^{\mathbb{N}}$,
(2) $\|x\| < 1$ for each $x \in Y_i$,
(3) $\xi_k(Y_i)$ is closed in $\mathbb{Q}^k$ for each $k \in \mathbb{N}$,
(4) $\xi_k(Y_i) \subset Y_i$ for each $k \ge i$,
(5) $\xi_1(Y_i)$ is finite and disjoint from $\bigcup_{j=1}^{i-1} \xi_1(Y_j)$,
(6) U_i is a closed subset of $\mathbb{Q}^{n_i}$,
(7) $\bigcup_{j=1}^{i} Y_j \cap \bigcup_{j=1}^{i} \xi_{n_j}^{-1}(U_j) = \emptyset$,
(8) if $\xi_{n_i}(t_i) \notin \bigcup_{j=1}^{i} \xi_{n_i}(Y_j)$ then U_i is a neighbourhood of $\xi_{n_i}(t_i)$ in $\mathbb{Q}^{n_i}$, and
(9) if $t_i \in \bigcup_{j=1}^{i-1} Y_j$ then there is an $r_i \in F_{1/i}(t_i) \cap Y_i$ such that every clopen neighbourhood C of r_i in Y_i with the norm topology has $\operatorname{diam} C \ge (1 - \|t_i\|)/2$.

For $i = 1$ we put $Y_1 = \{t_1\}$ and $U_1 = \emptyset$ and note that all hypotheses are trivially satisfied. Let us now assume that Y_i and U_i have been constructed.

Case I: $\xi_{n_{i+1}}(t_{i+1}) \notin \bigcup_{j=1}^{i} \xi_{n_{i+1}}(Y_j)$. Since by hypothesis (3) the set $A = \bigcup_{j=1}^{i} \xi_{n_{i+1}}(Y_j)$ is closed we can find a closed neighbourhood U_{i+1} of $\xi_{n_{i+1}}(t_{i+1})$ in $\mathbb{Q}^{n_{i+1}}$ that is disjoint from A. Putting $Y_{i+1} = \emptyset$ we note that all hypotheses are trivially satisfied for $i+1$.

Case II: $t_{i+1} \in \bigcup_{j=1}^{i} Y_j$. Note that this case is incompatible with Case I and hypothesis (8) is satisfied for $i+1$ no matter what choice we make for Y_{i+1}. We put $U_{i+1} = \emptyset$. By hypothesis (2) we have $\delta = 1 - \|t_{i+1}\| > 0$. Let $\varepsilon = \min\{\delta/4, 1/(i+1)\}$ and select an $r_{i+1} \in F_\varepsilon(t_{i+1})$ that differs from t_{i+1} only in the first coordinate which has been chosen from the complement of the finite set $\bigcup_{j=1}^{i} \xi_1(Y_j)$. Since the set $V = \bigcup_{j=1}^{i} \xi_{n_j}^{-1}(U_j)$ is closed and does not contain t_{i+1} by hypothesis (7) we may assume that $r_{i+1} \notin V$. Note that $\xi_{i+1}(r_{i+1}) = r_{i+1}$ because t_{i+1} has this property. Define

$$Y_{i+1} = \{x \in \mathbb{Q}^{\mathbb{N}} \colon \xi_{i+1}(x) = r_{i+1} \text{ and } \|x - r_{i+1}\| \leq \delta/2\}$$

and note that this set satisfies hypothesis (1) because the norm is LSC. The choice of the first coordinate of r_{i+1} guarantees that hypothesis (5) is also satisfied. If $x \in Y_{i+1}$ then $\|t_{i+1} - x\| \leq \frac{3}{4}\delta < 1 - \|t_{i+1}\|$ so hypothesis (2) is satisfied. Note that if $k \leq i+1$ then $\xi_k(Y_{i+1}) = \{\xi_k(r_{i+1})\}$ and if $k \geq i+1$ then $\xi_k(Y_{i+1}) = Y_{i+1} \cap \mathbb{Q}^k$ which means that hypotheses (3) and (4) are satisfied. Since $\xi_{i+1}(Y_{i+1}) = \{r_{i+1}\}$, $r_{i+1} \notin V$, and for every $j \leq i$, $n_j < i+1$, we have $Y_{i+1} \cap V = \emptyset$. Since moreover $U_{i+1} = \emptyset$ we may conclude that hypothesis (7) is satisfied for $i+1$.

We will now verify hypothesis (9). Obviously, we have $r_{i+1} \in F_{1/(i+1)}(t_{i+1}) \cap Y_{i+1}$. Recall that Erdős [10] proved that every clopen nonempty subset of $\mathfrak{E}$ is unbounded. This implies that if C is a clopen nonempty subset of, say, $B = \{x \in \mathfrak{E} \colon \|x\| \leq \delta/2\}$ then C contains a point x with $\|x\| = \delta/2$. Let C be a clopen neighbourhood of r_{i+1} in Y_{i+1} with the norm topology. Note that Y_{i+1} is an isometric copy of B where r_{i+1} plays the role of the zero vector so C contains an x with $\|x - r_{i+1}\| = \delta/2$ and hence $\operatorname{diam} C \geq \delta/2$.

Case III: neither Case I nor Case II. We can choose both Y_{i+1} and U_{i+1} to be empty.

The induction being complete we put $Y = \bigcup_{i=1}^{\infty} Y_i$ and note that every element of Y has norm less than 1 and hence Y is a bounded subset of $\mathfrak{E}$. Y is nonempty because it contains the zero vector t_1.

Claim 1. *Y is a closed subset of $\mathbb{Q}^{\mathbb{N}}$.*

Proof. Let x be an arbitrary element of $\mathbb{Q}^{\mathbb{N}}$. We consider two cases.

Case I: there is a $k \in \mathbb{N}$ such that $\xi_k(x) \notin \xi_k(Y)$. Let $i \in \mathbb{N}$ be such that $\xi_k(x) = t_i$ and $k = n_i$. Thus $\xi_{n_i}(t_i) = t_i \notin \xi_{n_i}(Y)$ and hence by hypothesis (8) U_i is a neighbourhood of t_i in $\mathbb{Q}^{n_i}$. We now have that $\xi_{n_i}^{-1}(U_i)$ is a neighbourhood of x that is disjoint from Y by hypothesis (7).

Case II: $\xi_k(x) \in \xi_k(Y)$ for every $k \in \mathbb{N}$. Then $\xi_1(x) \in \xi_1(Y_m)$ for some m. By hypothesis (5) we have that this implies that $\xi_k(x) \in \xi_k(Y_m)$ for every k. Since Y_m is closed in $\mathbb{Q}^{\mathbb{N}}$ we have that $x \in Y_m \subset Y$.

Claim 2. *For every $x \in Y$ and every clopen neighbourhood C of x in Y with the norm topology we have* $\operatorname{diam} C \geq (1 - \|x\|)/2$.

Proof. Let m be such that $x \in Y_m$ and let $\varepsilon > 0$ be such that $F_\varepsilon(x) \cap Y \subset C$. Select a $k \in \mathbb{N}$ such that $k > m$, $1/k < \varepsilon/2$, and $\|x - \xi_k(x)\| < \varepsilon/2$ and let i be such that $\xi_k(x) = t_i$ and $k = n_i$. Since $i \geq n_i = k > m$ we have by hypothesis (4) that $t_i \in Y_m$ and by hypothesis (9) that $r_i \in Y_i$ and $\|t_i - r_i\| < 1/i \leq 1/k < \varepsilon/2$. Thus $\|x - r_i\| < \varepsilon$ and $r_i \in C$. This means that $\operatorname{diam} C \geq \operatorname{diam}(C \cap Y_i) \geq (1 - \|t_i\|)/2 \geq (1 - \|x\|)/2$.

Since the norm topology is stronger than the product topology Claim 1 implies that Y is closed in $\mathfrak{E}$. Claim 2 shows that Y with the norm topology is not zero-dimensional at any point.

Coming attraction: in [9] Dijkstra and van Mill prove that $\mathfrak{E}$ is in fact homeomorphic to $\mathfrak{E}^{\mathbb{N}}$.

6. Equivalent notions of almost zero-dimensionality

We conclude by showing that the definition of almost zero-dimensionality that we use in this note is equivalent to the original definition in [13].

Proposition 6.1. *If X is almost zero-dimensional then there exists an (open) basis $\mathcal{O}$ for the topology of X such that $\overline{O}$ can be written as an intersection of clopen subsets of X for each $O \in \mathcal{O}$.*

Proof. Let X be almost zero-dimensional. Since X is separable metric we can find a countable collection $\mathcal{B} = \{B_i \colon i \in \mathbb{N}\}$ that satisfies the following conditions:

(1) for every $x \in X$ and every neighbourhood U of x there is an $i \in \mathbb{N}$ such that $x \in \operatorname{int} B_i \subset B_i \subset U$ and
(2) every $B_i \in \mathcal{B}$ is an intersection of clopen subsets of X.

Note that every B_i can be written as $\bigcap_{j=1}^{\infty} D_{ij}$ where every D_{ij} is clopen in X. Then $\{D_{ij}, X \setminus D_{ij} \colon i, j \in \mathbb{N}\}$ forms a subbasis for a separable metric, zero-dimensional topology on X that is weaker than the original topology. Call X with this zero-dimensional topology Z and let $\beta\colon X \to Z$ be the identity map. Note that every $\beta(B_i)$ is closed in Z.

Consider an open set U in X and an $x \in U$. We will construct a set $W \subset \overline{U}$ that is an intersection of clopen sets and with a dense interior that contains x. Put $F_0 = \emptyset$ and let $(F_n)_{n=1}^\infty$ be an enumeration of the elements of $\{B_i\colon B_i \cap \overline{U} = \emptyset\}$. Note that $X \setminus \overline{U} = \bigcup_{n=1}^\infty F_n$ and that every $\beta(F_n)$ is closed. We construct inductively a sequence $G_0, G_1, \ldots$ of subsets of U and a sequence $C_0, C_1, \ldots$ of clopen subsets of Z such that every $\beta(G_n)$ is a closed set in Z and $C_n \cap \beta(F_n) = \emptyset$. Select a B_k such that $x \in \operatorname{int} B_k \subset B_k \subset U$ and put $G_0 = B_k$ and $C_0 = Z$. Assume that G_{n-1} and C_{n-1} have been found and consider the open set

$$V_n = \operatorname{int} B_n \cap \bigcap_{i=0}^{n-1} \beta^{-1}(C_i).$$

If $V_n \cap U = \emptyset$ then we put $G_n = \emptyset$ and if $V_n \cap U \neq \emptyset$ then we put $G_n = B_m$ for some m such that $B_m \subset V_n \cap U$ and $\operatorname{int} B_m \neq \emptyset$. Note that $\bigcup_{i=0}^n \beta(G_i)$ and $\beta(F_n)$ are disjoint closed subsets of the zero-dimensional space Z so there is a clopen $C_n \subset Z$ with $\bigcup_{i=0}^n \beta(G_i) \subset C_n$ and $C_n \cap \beta(F_n) = \emptyset$.

Put $W = \bigcap_{n=0}^\infty \beta^{-1}(C_n)$ so W is an intersection of clopen sets. Note that $W \cap F_n = \emptyset$ for each n so $W \subset \overline{U}$. By the construction, $G_i \subset C_j$ for all i, j thus the open set $O = \bigcup_{n=0}^\infty \operatorname{int} G_n$ is contained in $\operatorname{int} W$ and $x \in \operatorname{int} G_0 \subset \operatorname{int} W$. It now suffices to show that $W \subset \overline{O}$. Let $y \in W$ and let B_n be arbitrary such that $y \in \operatorname{int} B_n$. Note that V_n is an open set that contains y and since $y \in \overline{U}$ we have that $V_n \cap U \neq \emptyset$. So $B_n \cap U$ contains the nonempty set $\operatorname{int} G_n$ and since B_n can be chosen arbitrarily small we have that $y \in \overline{O}$.

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Faculteit der Exacte Wetenschappen / Afdeling Wiskunde, Vrije Universiteit, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands

E-mail address: `dijkstra@cs.vu.nl`

E-mail address: `vanmill@cs.vu.nl`

Department of Mathematics, York University, 4700 Keele Street, Toronto, Ontario, Canada M3J 1P3

E-mail address: `steprans@yorku.ca`