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A NOTE ON FORD'S EXAMPLE

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ABSTRACT. Ford gave an example of a homogeneous space that is not a coset space. This example is not metrizable. We present a separable metrizable space with similar properties.

1. Introduction

Unless stated otherwise, all spaces under discussion are separable and metrizable.

If G is a topological group acting on a space X then for every $x \in X$ we let $\gamma_x \colon X \to G$ be defined by $\gamma_x(g) = gx$. We also let $G_x = \{g \in G : gx = x\}$ denote the *stabilizer* of $x \in X$. Then G_x is evidently a closed subgroup of G.

A space X is a *coset space* provided that there is a topological group G with *closed* subgroup H such that X and $G/H = \{xH : x \in G\}$ are homeomorphic. Observe that G acts transitively on G/H and that the natural quotient map $\pi \colon G \to G/H$ is open. It is well-known, and easy to prove, that G/G_x is homeomorphic to X if γ_x is open. Observe that $H \subseteq G$ is the stabilizer of $H \in G/H$. So for a space X to be a coset space it is necessary and sufficient that there be a topological group G acting transitively on X such that for some $x \in X$ (equivalently: for all $x \in X$) the function $\gamma_x \colon G \to X$ is open.

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It is known that many homogeneous spaces are coset spaces. Ungar [11] proved that if X is homogeneous and locally compact then X is a coset space. This is a consequence of the Effros Theorem on transitive actions of Polish groups on Polish spaces (Effros [5]; see also Ancel [1], Hohti [7] and van Mill [9]).

Ford [6] gave an example of a homogeneous Tychonoff space that is not a coset space. This example is not metrizable. The aim of this note is to present a (separable metrizable) σ-compact space X that is not a coset space but on which some (separable metrizable) topological group acts transitively.

2. The example

Let $\mathcal{U}$ be a cover of a space X. If $A \subseteq X$ and $f\colon A \to X$ then we say that f is *limited by* $\mathcal{U}$ provided that for every $x \in A$ there is an element $U \in \mathcal{U}$ containing both x and $f(x)$.

Theorem 2.1. *Let X be a coset space. Then for every open cover $\mathcal{U}$ of X and every compact $K \subseteq X$ there is an open cover $\mathcal{V}$ of X with the following property: for all $V \in \mathcal{V}$ and $x, y \in V$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and $f \restriction K$ is limited by $\mathcal{U}$.*

Proof. Let G be a topological group acting transitively on X such that for every $x \in X$ we have that the function $\gamma_x\colon G \to X$ is open. For $x \in K$ let V_x be an open neighborhood of e in G such that $\gamma_x[V_x^2]$ is contained in an element of $\mathcal{U}$. There is a finite $F \subseteq K$ such that

$$K \subseteq \bigcup_{x \in F} \gamma_x[V_x].$$

Let $V = \bigcap_{x \in F} V_x$, and let W be a symmetric open neighborhood of e in G such that $W^2 \subseteq V$. Put $\mathcal{V} = \{\gamma_x[W] : x \in X\}$. Then $\mathcal{V}$ is an open cover of X, and we claim that it is as desired. To this end, pick arbitrary $z, p, q \in X$ such that $p, q \in \gamma_z[W]$. There are $h, g \in W$ such that $hz = p$ and $gz = q$. Then $\xi = gh^{-1} \in W^2$ and $\xi p = q$. So it suffices to prove that if $\alpha \in W^2$ and $y \in K$ are arbitrary then there exists $U \in \mathcal{U}$ containing both y and αy. Pick $x \in F$ such that $y \in \gamma_x[V_x] \subseteq \gamma_x[V_x^2]$. There is an element $h \in V_x$ such that $hx = y$. Since $\alpha y = (\alpha h)x \in \gamma_x[V_x^2]$ and $\gamma_x[V_x^2]$ is contained in an element of $\mathcal{U}$, this completes the proof. $\square$

Let $Q = \prod_{n=1}^{\infty} [-1,1]_n$ denote the Hilbert cube with admissible metric

$$\varrho(x,y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|.$$

For each i let

$$W_i = \prod_{j \neq i} [-1 + 2^{-i}, 1 - 2^{-i}]_j \times \{1\}_i \subseteq Q.$$

Then W_i is a 'shrunken' endface in the i-th coordinate direction.

It was shown by Anderson, Curtis and van Mill [2] that $Y = Q \setminus \bigcup_{i=1}^{\infty} W_i$ is homogeneous. It can be shown that Y is a coset space. We will prove that the σ-compact connected and locally connected space $W = \bigcup_{i=1}^{\infty} W_i$ is also homogeneous, but is not a coset space.

The following results were proved in Dijkstra [3].

(A) If h is a homeomorphism between compacta in Y then it has an extension $\bar{h} \in \mathcal{H}(Q)$ such that for each i, $\bar{h}[W_i] = W_i$ ([3, Corollary 4.3.7]).

(B) There is $g \in \mathcal{H}(Q)$ such that for every i, $g[W_i] = W_{i+1}$ ([3, Lemma 4.4.4]).

Theorem 2.2. *W is homogeneous, but not a coset space.*

Proof. Let $x, y \in W$, say $x \in W_i$ and $y \in W_j$. We may assume without loss of generality that $j = \max(i,j)$. Let $\alpha = g^{-j}$, where g is as in (B). The disjoint collection Hilbert cubes

$$\{\alpha[W_1], \ldots, \alpha[W_j]\}$$

is contained in Y. Since Q is homogeneous, we may select $h \in \mathcal{H}(\bigcup_{k=1}^{j} \alpha[W_k])$ such that:

(1) $h \circ \alpha[W_k] = W_k$ if $k \notin \{i,j\}$,

(2) $h \circ \alpha[W_i] = W_j$, $h \circ \alpha[W_j] = W_i$ and $h \circ \alpha(x) = \alpha(y)$.

By (A) we may extend h to an element $\bar{h} \in \mathcal{H}(Q)$ such that for every i, $\bar{h}[W_i] = W_i$. Now put $\beta = \alpha^{-1} \circ \bar{h} \circ \alpha$. Then $\beta[W] = W$ and $\beta(x) = y$. Hence W is homogeneous.

We next prove that W does not satisfy the conclusion of Theorem 2.1, i.e., W is not a coset space.

The Sierpiński Theorem states that no continuum is the countable infinite union of disjoint nonempty compacta, [10, A.10.6].

This easily implies that any homeomorphism of W permutes the collection $\{W_i : i \in \mathbb{N}\}$. Let $\varepsilon = 1/8$. Since the diameter of W_i is equal to $2 - 2^{1-2i}$, any homeomorphism α of W which moves the points of W_1 less than ε has the property that $\alpha[W_1] = W_1$. Now let V be any neighborhood in W of $x = (0, 0, 0, \dots) \in W_1$. Pick $i \geq 2$ for which there exists $y \in V \cap W_i$. Each homeomorphism f of W such that $f(x) = y$ has the property that $f[W_1] = W_i$ and consequently moves a point of W_1 more than ε. This is clearly as desired. $\square$

The results in this section suggest the following question.

Question 2.3. Let X be a space which is homogeneous and and has the following property: for every open cover $\mathcal{U}$ of X and every compact $K \subseteq X$ there is an open cover $\mathcal{V}$ of X such that for all $V \in \mathcal{V}$ and $x, y \in V$ there is a homeomorphism $f \colon X \to X$ with $f(x) = y$ and $f \restriction K$ is limited by $\mathcal{U}$. Is X a coset space?

Observe that Q is a compactification of W with the following property: for all $x, y \in W$ there exists $h \in \mathcal{H}(Q)$ such that $h(x) = y$ and $h[W] = W$. This implies that there is a topological group G acting transitively on W. Simply let

$$G = \{g \in \mathcal{H}(Q) : g[W] = W\}.$$

Here $\mathcal{H}(Q)$ is the group of homeomorphisms of Q endowed with the (separable metrizable) compact-open topology. So there are spaces on which some group acts transitively but that are not coset spaces.

3. More questions

If X is a coset space then some topological group acts transitively on X, and if X admits a transitive group action then X is homogeneous. We saw that a transitive group action need not imply that the space under consideration is a coset space. This leaves the following basic problem open.

Question 3.1. Let X be a homogeneous space. Is there a topological group G acting transitively on X?

See also Question 3 of Ancel [1]. Observe that this question is non-trivial since all spaces and hence all topological groups under discussion are separable and metrizable. We believe that the answer to it is in the negative, but we have no idea how to construct a counterexample.

A sufficient condition for a transitive action is the existence of a 'homogeneous' compactification (as we observed in §2, Q is such a compactification of W). This suggests the following question.

Question 3.2. Let X be a homogeneous space. Is there a compactification γX of X such that for all $x, y \in X$ there is a homeomorphism $h\colon \gamma X \to \gamma X$ such that $h(x) = y$ and $h[X] = X$?

Observe that 'yes' to Question 3.2 implies 'yes' to Question 3.1.

The Čech-Stone compactification βX is a compactification of X which has the property that every homeomorphism $f\colon X \to X$ extends to a homeomorphism $\beta f\colon \beta X \to \beta X$. However, βX is not metrizable if X is a non-compact separable metrizable space. Let $\mathbb{Q}$ denote the space of rational numbers. It was shown by van Douwen [4] that $\beta\mathbb{Q}$ is the unique compactification $\gamma\mathbb{Q}$ of $\mathbb{Q}$ having the property that every homeomorphism of $\mathbb{Q}$ can be extended to a homeomorphism of $\gamma\mathbb{Q}$. This explains why in Question 3.2 we do not ask for a compactification with the property that all homeomorphisms extend.

4. Remarks

We finish this note by making a few remarks. By 'space' we mean here *Tychonoff space*. Compactifications such as the ones asked for in Question 3.2 surface at several places in the literature. If G is a topological group acting on a space X then X admits a compactification γX such that the action of G on X can be extended to an action of G on γX if and only if the so-called *right-uniformly continuous functions* on X separate the points and the closed subsets of X (such a compactification is called *equivariant*). See de Vries [13] for details. Observe that for an equivariant compactification γX we have that for every $g \in G$ the homeomorphism $x \mapsto gx$ of X can be extended to the homeomorphism $y \mapsto gy$ of γX. For locally compact G acting on X an equivariant compactification of X exists (de Vries [13]). Similarly if the action is transitive, the group is $\aleph_0$-bounded and the space is of the second category. See Uspenskiĭ [12] for details (I am indebted to Michael Megrelishvili for informing me about this result). As was shown by Megrelishvili [8], not all actions can be 'equivariantly compactified', even if the group and the space under consideration are both Polish.

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