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Topology and its Applications 142 (2004) 31–48
Topology and its Applications
www.elsevier.com/locate/topol

On spaces without non-trivial subcontinua and the dimension of their products

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Abstract

We introduce splintered and strongly splintered spaces. They are generalizations of both almost zero-dimensional spaces and weakly 1-dimensional spaces. We prove that there are n-dimensional strongly splintered spaces for every n, and that there is a 1-dimensional splintered space X such that $\dim X^n = n$ for every n. This solves a problem in the literature. Finally, we correct a flaw in an argument of Tomaszewski in his product formula for the dimension of the product of a weakly n-dimensional and a weakly m-dimensional space.

MSC: 54F45

Keywords: Almost zero-dimensional space; Weakly 1-dimensional space; L-embedded subspace; Splintered space; Tomaszewski's theorem

1. Introduction

All spaces under discussion are separable and metrizable.

A subset X of a compactum K is *L-embedded* in K if for every open cover $\mathcal{U}$ of K there is a neighborhood V of X in K such that every subcontinuum of K which is a subset of V is contained in an element of $\mathcal{U}$. This notion is due to Levin and Pol [4], who proved that an L-embedded subspace of a compact space is at most 1-dimensional.

A space X is called *almost zero-dimensional* if it has an open base $\mathcal{B}$ such that every $B \in \mathcal{B}$ has the property that $X \setminus \overline{B}$ is the union of clopen subsets of X. This notion was

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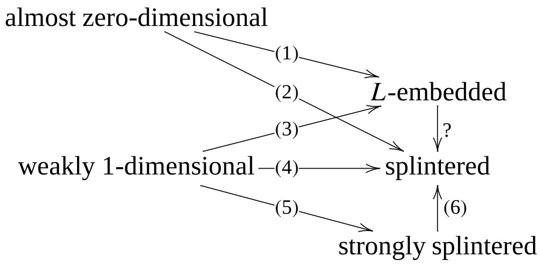
0166-8641/$ – see front matter
doi:10.1016/j.topol.2003.12.010

introduced by Oversteegen and Tymchatyn [11]. They proved that almost zero-dimensional spaces are at most 1-dimensional, and used this result to conclude that the homeomorphism groups of various important spaces such as Sierpiński's Carpet and Menger's Universal Curve, are 1-dimensional. The standard example of an almost zero-dimensional space which is not zero-dimensional, is Erdős space [8, Exercise 3.2.7].

If X is an n-dimensional space then its *dimensional kernel* $\Lambda(X)$ is the set of all points in X at which the dimension of X is n. It is known that $\Lambda(X)$ is an F_σ-subset of X which is at least of dimension $n-1$. This is due to Menger [6], see also [8, Lemma 3.11.1], who called a space X *weakly n-dimensional* if it is n-dimensional, but its dimensional kernel is of dimension $n-1$. The first examples of weakly n-dimensional spaces were constructed by Sierpiński [14] ($n=1$) and Mazurkiewicz [5] (for arbitrary n). Simpler construction can be found in Tomaszewski [15] and van Mill and Pol [9]. We are particularly interested here in the class of all weakly 1-dimensional spaces.

A space X is *splintered* if every open cover $\mathcal{U}$ of X has *countable* refinement by pairwise disjoint closed sets. Observe that the Sierpiński theorem that no continuum can be partitioned into countably many pairwise disjoint closed and nonempty sets implies that every compact subspace of a splintered space is zero-dimensional. For a space X, we let $X_{(0)}$ denote the subspace of all points of X at which the dimension is 0. That is, $x \in X_{(0)}$ if and only if x has arbitrarily small clopen neighborhoods in X. Observe that $X_{(0)}$ is a G_δ-subset of X. We call a space X *strongly splintered* if there are closed sets F_i in X for $i \in \mathbb{N}$ such that $X = \bigcup_{i=1}^{\infty}(F_i)_{(0)}$. It is clear that every strongly splintered space is countable dimensional (but not conversely).

In the following diagram we display the basic relations between the above notions:



(1) is due to Levin and Pol [4], (2) is [8, Exercise 3.2.8], (3) was proved by the authors of the present paper in [8, Theorem 3.11.11], (5) is trivial and (6) is Corollary 3.2 below (there is a simple direct proof that every weakly 1-dimensional space is splintered). We do not know whether every L-embedded subspace of a compact space is splintered. In Section 7 we will demonstrate that for the above notions there are no other implications than the ones shown in the diagram.

It is well known that the statement 'X is at most n-dimensional' has many equivalent formulations. See, e.g., [8, Theorem 3.2.5]. For example, a space X is at most n-dimensional if and only if every open cover $\mathcal{U}$ of X has a locally finite closed refinement $\mathcal{V}$ of order at most n. Since every open cover of X can be refined by the closed cover $\{\{x\}\colon x \in X\}$ of X, it is natural to ask whether the following property characterizes the class of all n-dimensional spaces:

$(*)_n$ for every open cover $\mathcal{U}$ of X there exists a *countable* closed refinement $\mathcal{V}$ of $\mathcal{U}$ with $\operatorname{ord}(\mathcal{V}) \leqslant n$.

So a space X has $(*)_0$ if and only if it is splintered. It is not true that $(*)_n$ characterizes all at most n-dimensional spaces since Erdős space is almost zero-dimensional and hence is both splintered and 1-dimensional. It was asked in [8, p. 160] whether spaces that satisfy condition $(*)_n$ are at most $(n+1)$-dimensional. This motivated us to define and study the class of all splintered spaces.

It is clear that an arbitrary product of almost zero-dimensional spaces is almost zero-dimensional, hence at most 1-dimensional by the Oversteegen–Tymchatyn theorem. A similar result was proved by Tomaszewski [15]. He showed that the product of two weakly 1-dimensional spaces is 1-dimensional which gave a negative answer to a question of Menger [7]. These results suggest the question whether similar results can be proved for the (strongly) splintered spaces and the L-embedded subspaces of compact spaces. It is a trivial observation that 'L-embeddedness' is a productive property. Indeed, we will show that if X_i is L-embedded in the compact space K_i for every i then $\prod_i X_i$ is L-embedded in $\prod_i K_i$. Hence by the result of Levin and Pol [4], the product of an arbitrary family of L-embedded subspaces of compact spaces is at most 1-dimensional. Hence by (3) it follows that the product of an arbitrary family of weakly 1-dimensional spaces is 1-dimensional, which improves the Tomaszewski theorem. The situation for the (strongly) splintered spaces is quite different. We shall construct a 1-dimensional splintered space X such that $\dim X^n = n$ for every n and X^∞ is not countable-dimensional. Since products of splintered spaces are splintered (Corollary 3.2), this answers, in particular, the question whether every splintered space is at most 1-dimensional. A much stronger negative answer to this question follows from our result that for each $\alpha < \omega_1$, there is a strongly splintered space of small transfinite dimension α. We shall also show that the product of two 1-dimensional strongly splintered spaces can be 2-dimensional (even if one of the factors is weakly 1-dimensional).

We mentioned above the result of Tomaszewski [15] that the product of two weakly 1-dimensional spaces is at most 1-dimensional. Tomaszewski claimed that from this result by an inductive argument one obtains the following more general and interesting inequality: if X is weakly n-dimensional, and Y is weakly m-dimensional, then

$$\dim(X \times Y) \leqslant \dim X + \dim Y - 1. \tag{T}$$

So the weakly n- and m-dimensional spaces demonstrate that the product formula does not hold in general (for all possible values). The interesting thing about (T) is that it holds for spaces with natural point-set topological properties. We will correct a flaw in Tomaszewski's arguments.

2. Preliminaries

If $\mathcal{A}$ and $\mathcal{B}$ are collections of sets then we say that $\mathcal{A}$ *refines* $\mathcal{B}$ if for every $A \in \mathcal{A}$ there is an element $B \in \mathcal{B}$ such that $A \subseteq B$.

Let X be a topological space with subset A. Then $\overline{A}$ and $\operatorname{Fr}(A)$ denote its *closure* and *boundary*, respectively.

For all undefined notions in dimension theory, we refer the reader to Engelking [1] and van Mill [8].

The following result generalizes the result obtained earlier by the authors in [8, Exercise 3.5.6].

Lemma 2.1. *If X_i is L-embedded in the compact space K_i for every $i \in \mathbb{N}$ then $\prod_{i=1}^{\infty} X_i$ is L-embedded in $\prod_{i=1}^{\infty} K_i$.*

Proof. For every $i \in \mathbb{N}$ let ϱ_i be an admissible metric for K_i which is bounded by 1. We endow $K = \prod_{i=1}^{\infty} K_i$ with the admissible metric

$$\varrho(x, y) = \sum_{i=1}^{\infty} 2^{-i} \varrho_i(x_i, y_i).$$

Let $\mathcal{U}$ be an open cover of K. By compactness, there is $\varepsilon > 0$ such that each set of ϱ-diameter less than ε is contained in some element of $\mathcal{U}$. Pick $N \in \mathbb{N}$ so large that $2^{-N} < \frac{1}{2}\varepsilon$. For every $i \leqslant N$ let U_i be a neighborhood of X_i in K_i such that every continuum in U_i has ϱ_i-diameter less than $\frac{1}{2}\varepsilon$. Put $U = \prod_{i=1}^{N} U_i \times \prod_{i=N+1}^{\infty} K_i$, and let $A \subseteq U$ be a continuum. Then $\pi_i(A)$ has ϱ_i-diameter less than $\frac{1}{2}\varepsilon$ for every $i \leqslant N$, where π_i is the projection onto the ith coordinate. So if $a, b \in A$ are arbitrary then

$$\varrho(a, b) \leqslant \sum_{i=1}^{N} 2^{-i} \varrho_i(a_i, b_i) + \sum_{i=N+1}^{\infty} 2^{-i} < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon,$$

as desired. $\square$

So by the result of Levin and Pol [4] cited in the introduction, we obtain:

Corollary 2.2. *Let X_i be L-embedded in the compactum K_i for every $i \in \mathbb{N}$. Then $\prod_{i=1}^{\infty} X_i$ is at most 1-dimensional.*

Let us note that Levin and Pol [4] proved that every almost zero-dimensional space is L-embedded in some compactification. It was shown by the authors in [8, Theorem 3.11.11] that the same result can be proved for weakly 1-dimensional spaces.

So we obtain:

Corollary 2.3. *Let X_i be weakly 1-dimensional for every $i \in \mathbb{N}$. Then $\prod_{i=1}^{\infty} X_i$ is at most 1-dimensional.*

We already noticed that almost zero-dimensional spaces are splintered. Weakly 1-dimensional spaces are obviously strongly splintered, and hence are splintered as well, cf., Proposition 3.3. This fact can also easily be established directly.

Question 2.4. Let X be L-embedded in some compact space K. Is X splintered?

3. Splintered and strongly splintered spaces

In this section we will make some preliminary observations on splintered and strongly splintered spaces.

As observed in Section 1, Erdős space E is splintered. Instead of Erdős space, one can also consider the so-called *complete Erdős space* F, i.e., the subspace of Hilbert space consisting of those points all of whose coordinates are irrational. It is almost zero-dimensional for the same reasons E is, and it is topologically complete being a G_δ-subset of Hilbert space. It is well known, and easy to prove, that one can topologize $G = F \cup \{p\}$, where $p \notin F$, in such a way that G is connected (and, clearly, topologically complete). Also, G is splintered because F is. Hence G is an example of a topologically complete, connected and splintered space. Observe that a splintered space which is topologically complete, connected and locally connected, must be a singleton. For otherwise it would contain an arc by the Mazurkiewicz theorem, and hence it would violate the Sierpiński theorem (cf., Section 1).

Lemma 3.1. *Let X be a subspace of a space Y. The following statements are equivalent*:

(1) *For every $\varepsilon > 0$ there is a countable closed collection $\mathcal{F}$ of Y such that*
   (a) $X \subseteq \bigcup \mathcal{F}$,
   (b) $\operatorname{mesh}(\mathcal{F}) < \varepsilon$,
   (c) *if $F, F' \in \mathcal{F}$ are distinct then $F \cap F' \cap X = \emptyset$.*

(2) *X is splintered.*

Proof. We only need to prove that (1) implies (2). To this end, for every n let $\mathcal{F}_n$ be a countable collection closed subsets of Y which satisfies (a), (b) and (c) for $\varepsilon = 1/n$.

Let $\mathcal{G} = \{F \cap F'\colon\ F \in \mathcal{F}_n,\ F' \in \mathcal{F}_{n+1}\}$. Pick arbitrary distinct elements $G_1, G_2 \in \mathcal{G}$. There are elements $F_1, F_2 \in \mathcal{F}_n$ and $F_1', F_2' \in \mathcal{F}_{n+1}$ such that

$$G_1 = F_1 \cap F_1', \qquad G_2 = F_2 \cap F_2'.$$

We may assume without loss of generality that $F_1 \neq F_2$. So

$$G_1 \cap G_2 \cap X \subseteq F_1 \cap F_2 \cap X = \emptyset.$$

These considerations show that we may assume that $\mathcal{F}_{n+1}$ refines $\mathcal{F}_n$ for every n. For every $x \in X$ and $n \in \mathbb{N}$ there is by (a) and (c) a unique element in $\mathcal{F}_n$ which contains x, say F_n^x.

Let $\mathcal{U}$ be an open cover of X. Fix $x \in X$ for a moment. There is an element $U \in \mathcal{U}$ which contains x. Let $U' \subseteq Y$ be open such that $U' \cap X = U$. There is $\varepsilon > 0$ such that the open ball around x with radius ε is contained in U'. Since $\operatorname{mesh}(\mathcal{F}_n) \searrow 0$, this implies that there is an element $n \in \mathbb{N}$ such that $F_n^x \subseteq U'$. These considerations show that the integer

$$n(x) = \min\bigl\{n \in \mathbb{N}\colon\ (\exists\, U \in \mathcal{U})\bigl(F_n^x \cap X \subseteq U\bigr)\bigr\}$$

is well-defined.

Put

$$\mathcal{E} = \bigl\{F_{n(x)}^x \cap X\colon\ x \in X\bigr\}.$$

We first claim that $\mathcal{E}$ is pairwise disjoint. Assume that $(F^x_{n(x)} \cap X) \cap (F^y_{n(y)} \cap X) \neq \emptyset$ for certain $x, y \in X$. If $n(x) = n(y)$ then $F^x_{n(x)} = F^y_{n(y)}$ by (c). Suppose therefore that, e.g., $n(x) < n(y)$. Observe that $F^y_{n(y)} \cap X \subseteq F^x_{n(x)} \cap X$ since $\mathcal{F}_{n(y)}$ refines $\mathcal{F}_{n(x)}$ and $\mathcal{F}_{n(x)} \restriction X$ is pairwise disjoint. By minimality of $n(y)$ we therefore get $n(y) \leqslant n(x)$, contradiction. Since $\mathcal{E}$ is countable because every $\mathcal{F}_n$ is countable, this means that we are done. $\square$

Corollary 3.2. *Subspaces and countable products of splintered spaces are splintered.*

It is clear that the property of being strongly splintered is hereditary and finitely productive. It seems, however, to be a delicate question whether the infinite product of strongly splintered spaces must be strongly splintered.

Proposition 3.3. *Every strongly splintered space is splintered.*

Proof. Let us say that a disjoint collection of closed subsets $\mathcal{F}$ of X has property $(*)$ provided that every $F \in \mathcal{F}$ is clopen in the subspace $\overline{\bigcup \mathcal{F}}$ of X. Observe that each family with property $(*)$ is countable. In addition, a disjoint collection $\mathcal{F}$ of closed subsets of X is said to have property $(\mathrm{w}*)$ if $\mathcal{F}$ can be written as the union of finitely many subfamilies, each having property $(*)$.

Let $\mathcal{U}$ be an arbitrary open cover of X.

Claim 1. *If A and B are closed in X then there is a closed collection $\mathcal{F}$ of X such that*

(1) $B_{(0)} \setminus A \subseteq \bigcup \mathcal{F} \subseteq B \setminus A$,
(2) $\mathcal{F}$ *refines* $\mathcal{U}$,
(3) $\mathcal{F}$ *has property* $(*)$.

For every $x \in B_{(0)} \setminus A$ we may pick a relatively clopen subset $C_x \subseteq B$ which is contained in an element of $\mathcal{U}$ such that $x \in C_x \subseteq B \setminus A$. Countably many C_x's cover $B_{(0)} \setminus A$, say $C_1, C_2, \ldots$. So the collection

$$\mathcal{F} = \left\{ C_1, C_2 \setminus C_1, \ldots, C_n \setminus \bigcup_{i=1}^{n-1} C_i, \ldots \right\}$$

is as required.

Claim 2. *Let $\mathcal{F}$ be a collection of closed subsets of X with property* $(\mathrm{w}*)$. *Assume that $\mathcal{F}$ refines $\mathcal{U}$. Then for every closed subset M of X there is a disjoint closed collection $\mathcal{G}$ of X having the following properties*:

(4) $\mathcal{F} \subseteq \mathcal{G}$ *and* $\mathcal{G}$ *refines* $\mathcal{U}$,
(5) $M_{(0)} \subseteq \bigcup \mathcal{G}$,
(6) $\mathcal{G}$ *has property* $(\mathrm{w}*)$.

Write $\mathcal{F}$ as $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$, where $\mathcal{F}_i$ has property $(*)$ for every $i \leqslant n$, and put $C_i = \overline{\bigcup \mathcal{F}_i}$. For each (possibly empty) $A \subseteq \{1, 2, \ldots, n\}$, put

$$X(A) = \left(\bigcap_{i \in A} C_i\right) \setminus \bigcup_{i \notin A} C_i.$$

Observe that the sets $X(A)$ are pairwise disjoint and cover X. In addition, each $X(A)$ is the difference of two closed subsets of X. For $A \subseteq \{1, 2, \ldots, n\}$, let

$$Y(A) = X(A) \setminus \bigcup_{i \in A} \bigcup \mathcal{F}_i.$$

Observe that the $Y(A)$'s are a disjoint cover of $X \setminus \bigcup \mathcal{F}$. Now fix $A \subseteq \{1, 2, \ldots, n\}$ for a moment. If $W \in \mathcal{F}_i$ for certain $i \in A$ then W is clopen in C_i, hence $W \cap X(A)$ is clopen in $X(A)$. This shows that $Y(A)$ is a closed subspace of $X(A)$ and so we can write $Y(A)$ as $P \setminus Q$, where both P and Q are closed in X. Then $M \cap Y(A) = (M \cap P) \setminus Q$. Since

$$M_{(0)} \cap P \subseteq (M \cap P)_{(0)},$$

there exists by Claim 1 a closed collection $\mathcal{F}(A)$ of X such that

(7) $M_{(0)} \cap Y(A) = (M_{(0)} \cap P) \setminus Q \subseteq \bigcup \mathcal{F}(A) \subseteq (M \cap P) \setminus Q$,
(8) $\mathcal{F}(A)$ refines $\mathcal{U}$,
(9) $\mathcal{F}(A)$ has property $(*)$.

Now let $\mathcal{G}$ be the union of $\mathcal{F}$ and all the collections $\mathcal{F}(A)$ for $A \subseteq \{1, 2, \ldots, n\}$. Then $\mathcal{G}$ is clearly as required.

This completes the proof of the Proposition since Claim 2 can be used recursively to deal with the zero-dimensional parts of countably many closed sets, and hence with all of X since X is strongly splintered. □

As observed at the beginning of this section, there are connected and topologically complete spaces which are splintered. Such a space is not strongly splintered, as the next observation shows.

Lemma 3.4. *Suppose that X is Baire and strongly splintered. Then $X_{(0)}$ is dense in X* (*hence, X is not connected*).

Proof. Let the sequence of closed sets F_i, $i \in \mathbb{N}$, witness that X is strongly splintered. We may assume without loss of generality that $(F_i)_{(0)}$ is dense in F_i for every i. Let U be a nonempty open subset of X. The closed collection $\{F_i \cap \overline{U}\colon i \in \mathbb{N}\}$ covers $\overline{U}$. Since X is Baire, for some i the interior of $F_i \cap \overline{U}$ is nonempty. So there is a nonempty open subset V of U such that $V \subseteq F_i$. Since $(F_i)_{(0)}$ is dense in F_i, there is a point $x \in V$ at which F_i is zero-dimensional. Since V is open in X and F_i is closed in X, this easily implies that X is zero-dimensional at x, i.e., $x \in U \cap X_{(0)}$. □

We shall close this section with one more observation, which will be useful in the next section.

Lemma 3.5. *Let $\varphi : T \to E$ be a Baire class* 1 *function from a closed subset T of the irrationals to a space E. Then the graph $G = \{(t, \varphi(t)):\ t \in T\} \subseteq T \times E$ is splintered.*

Proof. We shall show that

(†) if $F \subseteq G$ is closed and nonempty then $F_{(0)} \neq \emptyset$.

To that end, let us consider the projection A of F onto the first coordinate, and let B be the closure of A in T.

Since φ is of the first Baire class, the restriction of φ to B has a continuity point a.

Let $a_n \in A$, $a_n \to a$. Then $(a_n, \varphi(a_n)) \to (a, \varphi(a))$ and $(a_n, \varphi(a_n)) \in F$. Since F is closed, $c = (a, \varphi(a)) \in F$. Let us check that $c \in F_{(0)}$. Given an open neighborhood $U \times V$ of c in $T \times E$, one can find a clopen neighborhood W of a in T such that $\varphi(W \cap B) \subseteq V$. Then $W \times E$ is a clopen set containing c and $(W \times E) \cap F \subseteq U \times V$.

Having checked (†), let us show that this property implies the splinteredness of G.

Indeed, let $\mathcal{U}$ be an open cover of G. Define by transfinite induction disjoint closed sets H_α such that H_α is contained in some element of $\mathcal{U}$, and each union $\bigcup_{\alpha<\beta} H_\alpha$ is open in G. If H_α, $\alpha < \beta$, are already defined, let us consider $F = G \setminus \bigcup_{\alpha<\beta} H_\alpha$. If $F = \emptyset$, we stop. Otherwise, we pick $x \in F_{(0)}$ and we choose a clopen in F neighborhood H_β of x contained in an element of $\mathcal{U}$ containing x. Since G is separable, the process terminates at some $\lambda < \omega_1$. In effect we get a disjoint countable closed refinement $\{H_\alpha:\ \alpha < \lambda\}$ of $\mathcal{U}$. □

Remark 3.6. Let $f : E \to T$ be a perfect map from a complete space onto a closed subset of the irrationals. By the selection theorem of Kuratowski and Ryll-Nardzewski [3], there is a Baire class 1 function $\varphi : T \to E$ with $f(\varphi(t)) = t$ for $t \in T$. Let $S = \varphi(T)$. The map $\varphi(t) \to (f(\varphi(t)), \varphi(t))$ is a homeomorphism of S onto the graph $G = \{(t, \varphi(t)):\ t \in T\}$ of φ. Therefore, by Lemma 3.5, S is a splintered G_δ-selector for the decomposition of E into the fibers of f.

4. Examples of products of splintered and strongly splintered spaces

Let us recall that products of splintered spaces are splintered (Corollary 3.2). We shall use a construction of Rubin et al. [13], combined with an idea of Kulesza [2], to get the following

Theorem 4.1. *There is a complete* 1*-dimensional splintered space X such that* $\dim X^n = n$ *for every n and X^∞ is not countable-dimensional.*

Proof. Let $\mathbb{J} = [-1, 1]$, let $\Delta \subseteq \mathbb{J}$ be a Cantor set,

(1) $Z = \Delta \times \prod_{n=1}^{\infty} \mathbb{J}_n$,

and let

(2) $p_n : Z \to \Delta \times \mathbb{J}_n,\ \pi_n : Z \to \Delta \times \mathbb{J}_1 \times \cdots \times \mathbb{J}_n$

denote the projections

$$p_n(t, t_1, t_2, \ldots) = (t, t_n), \qquad \pi_n(t, t_1, t_2, \ldots) = (t, t_1, \ldots, t_n).$$

For any x in $\Delta \times \prod_{n \in \Gamma} \mathbb{J}_n$, $\Gamma \subset \mathbb{N}$, let $\pi(x)$ be the first coordinate of x. By [13] (see also the proof of Theorem 3.9.3 in [8]), there is a compact set

$$K \subseteq Z, \quad \pi(K) = \Delta$$

such that

(3) if $A \subseteq K$ and $\pi(A) = \Delta$ then A is not countable-dimensional,
(4) if $A \subseteq \pi_n(K)$ and $\pi(A) = \Delta$ then $\dim A = n$.

Let $\varphi : \Delta \to K$ be a Baire class 1 map such that $\pi(\varphi(t)) = t$, for $t \in \Delta$, cf., Remark 3.6. Then $\varphi_n = p_n \circ \varphi : \Delta \to p_n(K)$ is of the first Baire class and $\pi(\varphi_n(t)) = t$, for $t \in \Delta$. By Remark 3.6, each space

$$S_n = \varphi_n(\Delta) \subset p_n(K)$$

is complete and splintered. For every n, consider

$$A_n = \pi_n \circ \varphi(\Delta) \subseteq \pi_n(K).$$

By (4), $\dim A_n = n$. The map $(t, t_1, \ldots, t_n) \to ((t, t_1), \ldots, (t, t_n))$ embeds A_n in the product $S_1 \times \cdots \times S_n$, hence $\dim(S_1 \times \cdots \times S_n) \geqslant n$. Since $\dim S_i \leqslant 1$, we conclude that $\dim S_i = 1$, and that taking as X the topological sum of the spaces S_i we get a 1-dimensional, complete, splintered space with $\dim X^n = n$ for $n = 1, 2, \ldots$. To see that X^∞ is not countable-dimensional, let us consider

$$A_\infty = \varphi(\Delta)$$

which, by (3), is not countable-dimensional. Again, the map

$$(t, t_1, t_2, \ldots) \mapsto \big((t, t_1), (t, t_2), \ldots\big)$$

embeds A_∞ into X^∞, which completes the proof. $\square$

Before passing to the next example, let us notice that if $E \setminus E_{(0)}$ is a countable union of closed strongly splintered subspaces, then the space E is strongly splintered.

The example is based on constructions from [9] and [10], and as in Theorem 4.1, on [13] and an idea from [2].

Theorem 4.2. *There are subspaces X, Y of $[-1,1] \times [-1,1]$ such that X is weakly 1-dimensional, $Y \setminus Y_{(0)}$ is a countable union of closed weakly 1-dimensional spaces (in particular, $\dim X = \dim Y = 1$), and $\dim(X \times Y) = 2$.*

Proof. Let $\mathbb{J}_n = [-1, 1]$, $n = 1, 2$, let $\Delta \subseteq [-1, 1]$ be a Cantor set, and let

$$\pi : \Delta \times \mathbb{J}_1 \times \mathbb{J}_2 \to \Delta$$

be the projection onto the first coordinate. There is a compact set $K \subseteq \Delta \times \mathbb{J}_1 \times \mathbb{J}_2$ such that $\pi(K) = \Delta$ and

(1) if $A \subseteq K$ and $\pi(A) = \Delta$ then $\dim A = 2$,

cf. [13] and the proof of Theorem 3.9.3 in [8]. Let

$$p_n : K \to \Delta \times \mathbb{J}_n, \quad n = 1, 2,$$

be the projections $p_n(t, t_1, t_2) = (t, t_n)$, $L = p_1(K)$ and $\pi_L : L \to \Delta$ be the projection $\pi_L(t, t_1) = t$. Let $\Delta_0 = \{t \in \Delta\colon \dim \pi_L^{-1}(t) = 0\}$, $N \subseteq L$ a zero-dimensional G_δ-set with $\dim(L \setminus N) = 0$, and let us define

(2) $X = \pi_L^{-1}(\Delta_0) \cup (L \setminus N)$.

Then repeating a reasoning from [9], one checks that X is weakly 1-dimensional and, moreover,

(3) $\pi_L(X) = \Delta$ and $X \setminus \pi_L^{-1}(\Delta_0)$ is σ-compact.

We shall now start the construction of the space Y. To that end, let

(4) $M = p_2(p_1^{-1}(X))$ and $\pi_M : M \to \Delta$,

be the projection $\pi_M(t, t_2) = t$. By (3) and (4), $M \setminus \pi_M^{-1}(\Delta_0)$ is σ-compact, and therefore there are compact sets

(5) $T_i \subseteq M \setminus \pi_M^{-1}(\Delta_0)$, $\bigcup_{i=1}^{\infty} \pi_M(T_i) = \Delta \setminus \Delta_0$,

such that

(6) $\pi_M(T_i) \cap \pi_M(T_j) = \emptyset$ for $i \neq j$

(notice that any countable collection of compact sets in Δ can be refined by a disjoint countable collection of compact sets with the same union).

Let

$$\Delta_1 = \bigl\{t \in \Delta_0\colon \dim \pi_M^{-1}(t) = 0\bigr\}.$$

Then, cf. [9, Lemma 2.2], (2) and (3),

(7) $\pi_M^{-1}(\Delta_1) \subseteq (p_2(K))_{(0)}$.

Let $\mathbb{Q}$ be the rational numbers from $\mathbb{J}$. For each $q \in \mathbb{Q}$, we set

$$M_q = M \cap \bigl(\Delta \times \{q\}\bigr),$$

and let

(8) $Y = \pi_M^{-1}(\Delta_1) \cup \bigcup_{i=1}^{\infty} (T_i)_{(0)} \cup \bigcup_{q \in \mathbb{Q}} M_q$.

Then, by (7), $\pi_M^{-1}(\Delta_1) \subseteq Y_{(0)}$, M_q are zero-dimensional closed subsets of M, and

$$(Y \cap T_i) \setminus (T_i)_{(0)} \subseteq \bigcup_{q \in \mathbb{Q}} M_q,$$

cf. (5), (6). It follows that $Y \setminus Y_{(0)}$ is a countable union of closed sets which are either zero-dimensional or weakly 1-dimensional. Therefore, it remains to check the inequality

(9) $\dim(X \times Y) \geqslant 2$

(notice that (9) implies that $\dim(Y \setminus Y_{(0)}) = 1$, by the Tomaszewski theorem). To that end, let us notice that

(10) $\pi_M(Y) = \Delta$.

Indeed, if $t \in \Delta_0 \setminus \Delta_1$, then by (2) and (4), $\pi_M^{-1}(t)$ contains a non-trivial interval, and for $q \in \mathbb{Q}$ in this interval, $\pi_M^{-1}(t) \cap M_q \neq \emptyset$. It follows that $\Delta_0 \subseteq \pi_M(Y)$. Let $t \in \Delta \setminus \Delta_0$, and let $t \in \pi_M(T_i)$, cf., (5). If $\dim(\pi_M^{-1}(t) \cap T_i) = 0$, $\pi_M^{-1}(t) \subseteq (T_i)_{(0)} \subseteq Y$. Otherwise, $\pi_M^{-1}(t) \cap T_i$ contains a non-trivial interval, hence some $q \in \mathbb{Q}$, and $\pi_M^{-1}(t)$ intersects M_q.

Having checked (10), let us pick for each $t \in \Delta$ points $u(t), v(t) \in \mathbb{J}$ with $(t, v(t)) \in Y$ and $(t, u(t)) \in X$, cf., (3). Then the product $X \times Y$ contains a set homeomorphic to $A = \{(t, u(t), v(t)) \colon t \in \Delta\} \subseteq K$ which projects onto Δ. By (5), $\dim A = 2$, and we get (9), which completes the proof. $\square$

5. Higher-dimensional strongly splintered spaces

The aim of this section is to prove that for every $\alpha < \omega_1$ there is a topologically complete strongly splintered space of small transfinite dimension α. The proof is based on constructions in van Mill and Pol [10] and Pol [12].

Theorem 5.1. *For each $\alpha < \omega_1$ there is a strongly splintered topologically complete space E such that* $\operatorname{ind} E = \alpha$.

The main tool in the proof is the following result:

Proposition 5.2. *Let $f : X \to \mathbb{P}$ be a perfect map, where X is topologically complete and countable-dimensional. There exists a G_δ-subset Y of X which is strongly splintered and satisfies $f(Y) = f(X)$.*

We shall derive this proposition from the following:

Lemma 5.3. *Let $f : Z \to H$ be a perfect map from a space Z with $0 < \operatorname{ind} Z < \infty$ onto a zero-dimensional space H, and let*

$$Z_0 = Z \setminus f^{-1}\big(f(Z \setminus Z_{(0)})\big) \subseteq Z_{(0)}. \tag{1}$$

There exists for $i \geqslant 1$ a closed set Z_i in Z such that:

(2) *the collection $\{f(Z_i)\colon i \geqslant 0\}$ is pairwise disjoint,*
(3) $\operatorname{ind} Z_i < \operatorname{ind} Z$ *for every i,*
(4) $f(\bigcup_{i=0}^{\infty} Z_i) = f(Z)$.

Moreover, $\bigcup_{i=0}^{\infty} Z_i$ is a G_δ-subset of Z.

Proof. We shall follow the reasoning in the proof of Theorem 1.3 in [10].

First observe that Z_0 is a G_δ-subset of Z because $Z_{(0)}$ is and the map f is closed. Let

$$Z \setminus Z_0 = \bigcup_{i=1}^{\infty} A_i,$$

where each A_i is closed in Z. In addition, let $\{U_i\colon i \in \mathbb{N}\}$ be an open base for Z such that

$$\operatorname{ind} \operatorname{Fr} U_i < \operatorname{ind} Z$$

for every i; put $B_i = \operatorname{Fr} U_i$. For $i, j \in \mathbb{N}$ put

$$T_{ij} = A_i \cap B_j.$$

The sets $f(T_{ij})$ are closed in the zero-dimensional space H. There consequently are pairwise disjoint closed sets H_{ij} in H with $H_{ij} \subseteq f(T_{ij})$, while moreover

$$\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} H_{ij} = \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} f(T_{ij}).$$

For $i, j \in \mathbb{N}$, put

$$Z_{ij} = f^{-1}(H_{ij}) \cap T_{ij},$$

and arrange the Z_{ij}'s into the sequence $Z_1, Z_2, \ldots$. The conditions (2) and (3) are clearly satisfied.

For (4), pick t from $f(Z)$, and consider the fiber $f^{-1}(t)$. If $\dim f^{-1}(t) = 0$ then $f^{-1}(t) \subseteq Z_{(0)}$ by Lemma 2.1 from [9], hence $t \in f(Z_0)$. So assume that $\dim f^{-1}(t) > 0$. Then $f^{-1}(t)$ is a compactum with positive dimension and hence contains a non-trivial continuum, say C [8, Exercise 3.2.4]. Hence C must meet some boundary B_i, and since $C \subseteq Z \setminus Z_{(0)} \subseteq Z \setminus Z_0$, the intersection $B_i \cap C$ must meet some A_j. Hence $f^{-1}(t)$ must intersect some T_{ij}, i.e., t is in some H_{kl} and so in $f(Z_m)$, where m corresponds to the pair (k, l).

To finish the proof, observe that

$$\bigcup_{i=0}^{\infty} Z_i = Z \setminus \left(\bigcup_{i=1}^{\infty} f^{-1}\big(f(Z_i)\big) \setminus Z_i\right).$$

This shows that $\bigcup_{i=0}^{\infty} Z_i$ is indeed a G_δ-subset of Z. □

Proof of Proposition 5.2. Since X is complete and countable-dimensional we have that the transfinite small inductive dimension $\operatorname{ind} X$ of X is smaller than ω_1. We shall proceed by transfinite induction on $\operatorname{ind} X$. Let us suppose that the assertion is true for all spaces with $\operatorname{ind} < \alpha$, and let $f : X \to \mathbb{P}$ be as in Proposition 5.2. Let Z_i, $i = 0, 1, \ldots$, be as in Lemma 5.3 (with $Z = X$). By the inductive assumption, cf. (3), each Z_i contains a G_δ-subset Y_i of Z_i which is strongly splintered and $f(Y_i) = f(Z_i)$. Let

$$Y = Z_0 \cup \bigcup_{i=1}^{\infty} Y_i.$$

Then $Z_0 \subseteq Y_{(0)}$ and Y_i are strongly splintered closed subsets of Y, hence Y is strongly splintered. Moreover, $X \setminus Y = \bigcup_{i=1}^{\infty}(Z_i \setminus Y_i)$ is an F_σ-set, cf. (2), and $f(Y) = f(X)$. $\square$

Proof of Theorem 5.1. It is enough to check the assertion for non-limit α since the theorem for limit ordinals can be proved from the theorem for non-limit ordinals by taking topological sums.

Let $\pi : \mathbb{P} \times \mathbb{I}^\infty \to \mathbb{P}$ denote the projection. We shall use the following result from [12, Comment 6.2, p. 266]: there is an F_σ-subset $F \subseteq \mathbb{P} \times \mathbb{I}^\infty$ such that $\pi(F) = \mathbb{P}$ while moreover for every $D \subseteq F$ with $\pi(D) = \mathbb{P}$ we have $\operatorname{ind} D = \alpha$.

Since F is an F_σ-set, there are for every i disjoint closed sets X_i in F such that the sets $\pi(X_i)$ are pairwise disjoint and cover $\mathbb{P}$. Now apply Proposition 5.2 to the maps $\pi \restriction X_i : X_i \to \pi(X_i)$, getting strongly splintered G_δ-subsets E_i of X_i with $\pi(E_i) = \pi(X_i)$. Then

$$E = \bigcup_{i=1}^{\infty} E_i$$

is a G_δ set in $\mathbb{P} \times \mathbb{I}^\infty$, which is strongly splintered. Moreover, $\operatorname{ind} E = \alpha$, as E is a subset of F projecting onto $\mathbb{P}$. $\square$

6. Tomaszewski's theorem

Tomaszewski's approach to (T) in Section 1 is to use induction on $n+m$. The inequality (T) is true if $n = m = 1$. Then he proceeds on [15, p. 5], as follows. Assume that (T) is true for all n and m with $n + m \leqslant k - 1 \geqslant 2$. Consider a weakly n-dimensional space X, and a weakly m-dimensional space Y with $n + m = k$. Consider two points of the form (x_1, y) and (x_2, y) in $X \times Y$ such that $x_1 \neq x_2$. Tomaszewski then claims that there is a partition L in X between x_1 and x_2 such that either

(a) $\dim L \leqslant n - 2$, or
(b) L is weakly $(n-1)$-dimensional.

If so, then $L' = L \times Y$ is a partition between (x_1, y) and (x_2, y) such that $\dim L' \leqslant (n-2) + m = n + m - 2$ if (a) is true, and $\dim L' \leqslant (n-1) + m - 1 = n + m - 2$ if (b) is true (by the inductive hypothesis). However, if $n = 1$ then (b) is not defined, and (a) need not always hold because X is 1-dimensional. In fact, if $\dim L = 0$ then $\dim(L \times Y) = m > 1 + m - 2$. So this proof breaks down if $n = 1$; it is correct if both n and m are greater than 1. We will prove that (T) holds for $n = 1$ and arbitrary m by refining Tomaszewski's proof for the product of two weakly 1-dimensional spaces. The basic idea of our proof follows [15], although our approach is more direct and elementary.

Lemma 6.1. *Let X be weakly n-dimensional, $n \geqslant 2$. There exists a zero-dimensional F_σ-subset $N \subseteq X$ such that $X \setminus N$ is weakly $(n-1)$-dimensional.*

Proof. Let $\mathcal{B}_0$ be a countable open collection of X which is a base at all points of $\Lambda(X)$ while moreover $\dim \operatorname{Fr} B \leqslant n-1$ for every $B \in \mathcal{B}_0$. In addition, let $\mathcal{B}_1$ be a countable open collection of X which is a base at all points of $X \setminus \Lambda(X)$ while moreover $\dim \operatorname{Fr} B \leqslant n-2$ for every $B \in \mathcal{B}_1$.

Now for every $B \in \mathcal{B}$ with $\dim \operatorname{Fr} B \geqslant 0$ let $F(B) \subseteq \operatorname{Fr} B$ be a zero-dimensional F_σ-subset such that

$$\dim\bigl(\operatorname{Fr} B \setminus F(B)\bigr) \leqslant \dim \operatorname{Fr} B - 1.$$

In addition, since $\dim \Lambda(X) = n-1$, by the same reason there is a zero-dimensional F_σ-subset $F \subseteq \Lambda(X)$ such that $\dim \Lambda(X) \setminus F \leqslant n-2$. Since $\Lambda(X)$ is an F_σ-subset of X, the set F is an F_σ-subset of X as well. Put

$$N = \bigcup_{B \in \mathcal{B}} F(B) \cup F.$$

Then $\dim N \leqslant 0$ by the Countable Closed Sum theorem and it is easy to see that N is as required. $\square$

Lemma 6.2. *Let X be weakly n-dimensional. If $Y \subseteq X$ is n-dimensional then Y is weakly n-dimensional.*

Proof. It is clear that $\Lambda(Y) \subseteq \Lambda(X)$. So we are done since $\Lambda(X)$ is $(n-1)$-dimensional. $\square$

Corollary 6.3. *Let X be weakly n-dimensional, $n \geqslant 2$. Then for every pair A, B of disjoint closed subsets of X there is a partition D between A and B such that either $\dim D \leqslant n-2$ or D is weakly $(n-1)$-dimensional.*

Proof. Let N be the zero-dimensional F_σ-subset of X we get from Lemma 6.1. There is a partition D between A and B which misses N. So $\dim D \leqslant n-1$. If $\dim D = n-1$ then D is weakly $(n-1)$-dimensional by Lemma 6.2. $\square$

Theorem 6.4. *If X is weakly 1-dimensional and Y is weakly m-dimensional, then*

$$\dim(X \times Y) \leqslant m.$$

We will prove this by induction on m. It is true for $m = 1$, so assume that $m > 1$.

Let U and V be arbitrary open subsets of X and Y, respectively, such that $x \in U$ and $y \in V$. Our aim is to construct an open subset $E \subseteq X \times Y$ such that $(x, y) \in E \subseteq U \times V$ while moreover $\dim \operatorname{Fr} E \leqslant m - 1$. We will do this in two steps. We first construct an open neighborhood L of (x, y) such that $L \subseteq X \times V$ and $\dim \operatorname{Fr} L \leqslant m - 1$. Then we construct an open neighborhood K of (x, y) such that $K \subseteq U \times Y$ and $\dim \operatorname{Fr} K \leqslant m - 1$. Then $E = K \cap L$ is a neighborhood of (x, y) such that $E \subseteq U \times V$, and

$$\operatorname{Fr} E \subseteq \operatorname{Fr} K \cup \operatorname{Fr} L$$

and hence is at most $(m - 1)$-dimensional by the Countable Closed Sum theorem.

First observe that the construction of L is simple. Indeed, Let P be a neighborhood of y in Y such that $P \subseteq V$ while moreover $\dim \operatorname{Fr} P \leqslant m - 2$ or $\operatorname{Fr} P$ is weakly $(m - 1)$-dimensional (Corollary 6.3). Put $L = X \times P$. Then

$$\operatorname{Fr} L = X \times \operatorname{Fr} P,$$

hence

$$\dim \operatorname{Fr} L \leqslant m - 2 + 1 = m - 1$$

in the first case, and

$$\dim \operatorname{Fr} L \leqslant m - 1$$

in the second case by our inductive hypothesis.

The construction of K is more complicated. Let U' be an open neighborhood of x such that $\overline{U'} \subseteq U$ and $\operatorname{Fr} U' \subseteq X \setminus \Lambda(X)$. It is possible to pick U' since $\Lambda(X)$ is zero-dimensional. Put $A = \operatorname{Fr} U'$ and let U'' be an open subset of X such that $A \subseteq U'' \subseteq \overline{U''} \subseteq U$.

We claim that for every $n \in \mathbb{N}$ there exist pairwise disjoint clopen subsets $U_{1n}, U_{2n}, \ldots$ of X such that

(1) $U_{in} \cap A \neq \emptyset$ for every i,
(2) $\operatorname{diam} U_{in} < 1/n$ for every i,
(3) $A \subseteq \bigcup_{i=1}^{\infty} U_{in} \subseteq \overline{\bigcup_{i=1}^{\infty} U_{in}} \subseteq U''$,
(4) $\operatorname{Fr}(\bigcup_{i=1}^{\infty} U_{in}) \subseteq \Lambda(X)$.

Indeed, for every $x \in X \setminus \Lambda(X)$ pick an clopen neighborhood C_x of diameter at most $1/n$ such that either $C_x \cap A = \emptyset$ or $C_x \subseteq U''$. A countable subcollection of the C_x cover $X \setminus \Lambda(X)$, say $\mathcal{C}$. Since the C_x are clopen we may assume without loss of generality that $\mathcal{C}$ is pairwise disjoint. At most countably many elements of $\mathcal{C}$ intersect A, say $\{C_{x_i}\colon i \in \mathbb{N}\}$. An easy check shows that the sets $U_{in} = C_{x_i}$, $i \in \mathbb{N}$, are as required.

Since $\Lambda(Y)$ is an F_σ-subset of Y, there are closed subsets B_n of Y for every n such that

$$\Lambda(Y) = \bigcup_{n=1}^{\infty} B_n.$$

Now for every $n \in \mathbb{N}$ let $\mathcal{E}_n$ denote the collection of all open subsets E of Y having the following properties:

– $\overline{E} \cap B_n = \emptyset$,
– $\dim \operatorname{Fr} E \leqslant m - 2$.

Observe that $\mathcal{E}_n$ covers $Y \setminus \Lambda(Y)$ since the dimension at every point of $Y \setminus \Lambda(Y)$ is at most $m-1$ and B_n is closed in Y. Pick a countable subcollection $\mathcal{F}_n \subseteq \mathcal{E}_n$ with $\bigcup \mathcal{F}_n = \bigcup \mathcal{E}_n$. Enumerate it as $\{F_{in}: i \in \mathbb{N}\}$ and put

$$V_{in} = Y \setminus \bigcup_{j=1}^{i} \overline{F}_{jn}$$

for every $i \in \mathbb{N}$. Observe that V_{in} is open for every i, that the sequence $(V_{in})_i$ is decreasing, and that

$$B_n \subseteq \widehat{B}_n = \bigcap_{i=1}^{\infty} V_{in} \subseteq \Lambda(Y).$$

Moreover, if $i \in \mathbb{N}$ then $\operatorname{Fr} V_{in} \subseteq \bigcup_{j=1}^{i} \operatorname{Fr} F_{jn}$ is at most $(m-2)$-dimensional by the Countable Closed Sum theorem.

Now put

$$W = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} U_{in} \times V_{in}. \tag{6}$$

We will first show that

$$A \times \Lambda(Y) \subseteq W. \tag{7}$$

Indeed, pick an arbitrary point $(a,b) \in A \times \Lambda(Y)$. There exists $n \in \mathbb{N}$ such that $b \in B_n \subseteq \widehat{B}_n$. Since $a \in A \subseteq \bigcup_{i=1}^{\infty} U_{in}$ there also exists $i \in \mathbb{N}$ such that $a \in U_{in}$. We conclude that $(a,b) \in U_{in} \times \widehat{B}_n \subseteq U_{in} \times V_{in} \subseteq W$.

Observe that if $i, n \in \mathbb{N}$ are arbitrary then since U_{in} is clopen, we have

$$\operatorname{Fr}(U_{in} \times V_{in}) = U_{in} \times \operatorname{Fr} V_{in},$$

and hence that

$$\dim \operatorname{Fr}(U_{in} \times V_{in}) \leqslant 1 + m - 2 = m - 1.$$

In addition, clearly,

$$\dim\bigl(\Lambda(X) \times \operatorname{Fr} V_{in}\bigr) \leqslant 0 + m - 2 = m - 2.$$

We will show that

$$\operatorname{Fr} W \subseteq \bigl(\Lambda(X) \times \Lambda(Y)\bigr) \cup (A \times Y) \cup \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \operatorname{Fr}(U_{in} \times V_{in}) \\ \cup \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \Lambda(X) \times \operatorname{Fr} V_{in}. \tag{8}$$

To this end, let $(a,b) \in \operatorname{Fr} W$ and let $(a_k, b_k) \in W$, $k \in \mathbb{N}$, be a sequence converging to (a,b). For every $k \in \mathbb{N}$ pick $i_k, n_k \in \mathbb{N}$ such that

$$(a_k, b_k) \in U_{i_k n_k} \times V_{i_k n_k}.$$

Let us first assume that the set $\{n_k\colon k \in \mathbb{N}\}$ is infinite. Then by (1) and (2) it follows that for infinitely many $k \in \mathbb{N}$ we have $\varrho(a_k, A) < 1/k$, whence $(a,b) \in A \times Y$.

Assume next that the set $\{n_k\colon k \in \mathbb{N}\}$ is finite. By passing to a subsequence if necessary, we may even assume that it consists of a single element, say n. If the set $Z = \{i_k\colon k \in \mathbb{N}\}$ is finite as well, then we may assume by the same argument that it consists of a single element, say i. But then

$$(a_k, b_k) \in U_{in} \times V_{in}$$

for every k, and since clearly $(a,b) \notin U_{in} \times V_{in}$ since $U_{in} \times V_{in}$ is an open subset of W, we obtain

$$(a,b) \in \operatorname{Fr}(U_{in} \times V_{in}),$$

as required. So we may assume without loss of generality that Z is infinite. But this means that we may assume without loss of generality that the function $k \mapsto i_k$ is one-to-one.

So by (4) we get $a \in \operatorname{Fr}(\bigcup_{i=1}^{\infty} U_{in}) \subseteq \Lambda(X)$.

If $b \in \widehat{B}_n \subseteq \Lambda(Y)$ then we are obviously done since then $(a,b) \in \Lambda(X) \times \Lambda(Y)$. So assume that $b \notin \widehat{B}_n$, and pick $i \in \mathbb{N}$ such that $b \notin V_{in}$. Since the set Z is infinite and the sequence $(V_{in})_i$ is decreasing, it follows that all but finitely many element of the sequence $(b_k)_k$ belong to V_{in}. Since $b \notin V_{in}$ this implies that $b \in \operatorname{Fr} V_{in}$ and hence that

$$(a,b) \in \Lambda(X) \times \operatorname{Fr} V_{in},$$

as required.

Now put

$$K = W \cup \bigl(U' \times Y\bigr). \tag{9}$$

Then K is an open neighborhood of (x,y) and by (3) we find that $\overline{K} \subseteq U \times Y$. We claim that $\operatorname{Fr} K$ is at most $(m-1)$-dimensional. First observe that

$$\operatorname{Fr} K \subseteq \operatorname{Fr} W \cup \operatorname{Fr}\bigl(U' \times Y\bigr) = \operatorname{Fr} W \cup (A \times Y). \tag{10}$$

Put $T_0 = \operatorname{Fr} K \cap (A \times Y)$ and $T_1 = \operatorname{Fr} K \setminus T_0$, respectively. Since by (7), $A \times \Lambda(Y) \subseteq W$ and $\operatorname{Fr} K \cap W = \emptyset$, it follows that

$$T_0 \subseteq A \times \bigl(Y \setminus \Lambda(Y)\bigr),$$

which is at most $(m-1)$-dimensional. We conclude that T_0 is a closed subspace of $\operatorname{Fr} E$ with $\dim T_0 \leqslant m-1$. In addition, (10) implies that

$$T_1 \subseteq \operatorname{Fr} W \setminus (A \times Y).$$

Hence by (8), T_1 is contained in an at most $(m-1)$-dimensional F_σ-subset of $X \times Y$. We conclude that T_1 is at most $(m-1)$-dimensional as well. Since T_0 is closed, the Countable Sum theorem now easily gives us that $\dim \operatorname{Fr} K \leqslant m-1$, as desired.

7. The diagram

Let us return to the diagram in Section 1. Erdős space E is almost zero-dimensional but neither weakly 1-dimensional nor strongly splintered (since it is nowhere zero-dimensional).

A space is rimcompact if it has base every element of which has compact boundary. In addition, a space is totally disconnected if the empty set is a partition between any two distinct points. It is not difficult to see that a rimcompact totally disconnected space is zero-dimensional. Since every almost zero-dimensional space is evidently totally disconnected, it follows that no 1-dimensional almost zero-dimensional space is rimcompact. There are weakly 1-dimensional rimcompact spaces by [9]. These spaces are consequently weakly 1-dimensional and hence strongly splintered, but not almost zero-dimensional.

These examples also show that neither (1) nor (3) can be reversed. The higher-dimensional (strongly) splintered spaces constructed in this paper demonstrate that (2), (4) and (5) cannot be reversed. They also show by [4] that not every (strongly) splintered space is L-embedded. Since there are connected splintered spaces, (6) cannot be reversed as well.

So with respect to the diagram in Section 1, the only open question that remains is whether every L-embedded subspace of a compact space is splintered.

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