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SUMS OF ALMOST ZERO-DIMENSIONAL SPACES

MOHAMMAD ABRY, JAN J. DIJKSTRA, AND JAN VAN MILL

Abstract. Almost zero-dimensionality is a relatively new dimension theoretic concept that fits neatly between zero- and one-dimensionality. In this note we investigate to which extent familiar properties of dimension carry over to almost zero-dimensionality. We are particularly interested in sum theorems.

1. Introduction

All spaces in this note are assumed to be separable and metrizable. A subset of a space is called a *C-set* if it can be written as an intersection of clopen subsets of the space. Note that a space is zero-dimensional if and only if every closed subset is a C-set. A space is called *almost zero-dimensional* (*AZD*) if every point has a neighbourhood basis consisting of C-sets. This definition is due in essence to Oversteegen and Tymchatyn [13]; see also [5, Proposition 6.1]. It is shown in [13] that the dimension of these spaces is at most one; see [1] and [11] for simpler proofs. The standard example of an AZD space that is not zero-dimensional is Erdős space

$$\mathfrak{E} = \{(x_1, x_2, \dots) \in \ell^2 : x_i \in \mathbb{Q} \text{ for each } i \in \mathbb{N}\},$$

where ℓ^2 stands for the Hilbert space of square summable (real) sequences. The topology of this space is characterized by Dijkstra and van Mill in [3] and [4].

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It appears that almost zero-dimensionality is a dimension theoretic concept that fits neatly between zero- and one-dimensionality. In this note we investigate to which extent familiar properties of dimension carry over to almost zero-dimensionality.

2. Similarities between zero-dimensionality and almost zero-dimensionality

In this section we list properties of almost zero-dimensionality that correspond to familiar properties of zero-dimensionality.

A space X is called *totally disconnected* if for every two distinct points x and y there is a clopen subset of X that contains x and misses y. A space X is called *hereditarily disconnected* if its components are singletons. Every zero-dimensional space is AZD, every AZD space is totally disconnected, and every totally disconnected space is hereditarily disconnected.

Clearly, almost zero-dimensionality is hereditary. Also, countable products of AZD spaces are AZD and hence, inverse limits of AZD spaces are AZD. The class of zero-dimensional spaces has universal elements – the Cantor set, for instance. Since AZD spaces are totally disconnected every σ-compact AZD space is zero-dimensional so the class of AZD spaces cannot have a compact universal element. However, there is a complete universal space, for instance *complete Erdős space*:

$$\mathfrak{E}_{\mathrm{c}} = \{z \in \ell^2 : z_i \in \mathbb{R} \setminus \mathbb{Q} \text{ for each } i \in \mathbb{N}\}.$$

This fact is implicitly contained in [9] and [13]; see also [4, Theorem 5.13]. Thus, every AZD space has an AZD completion and hence, every AZD subspace of a space can be enlarged to a G_δ-subspace that is AZD (use Lavrientieff [10]). Another consequence is that every AZD space is imbeddable in $\mathbb{R}^2$, which makes the class of AZD spaces fit nicely between zero-dimensional and one-dimensional spaces which can be imbedded in $\mathbb{R}$, respectively $\mathbb{R}^3$.

If $X = E \cup F$, where $\dim E \le 0$ and $\dim F < n$, then $\dim X \le n$; see [6, Lemma 1.5.2]. Levin and Tymchatyn [12] proved that the union of an AZD space with a zero-dimensional space is at most one-dimensional. We have the following extension:

Theorem 2.1. *Let $n \in \mathbb{N}$. If $X = E \cup F$, where E is AZD and $\dim F < n$, then $\dim X \le n$.*

Proof: We use induction. The base case $n = 1$ is the Levin-Tymchatyn result. Assume that the theorem is valid for some $n \in \mathbb{N}$. Let $X = E \cup F$ where E is AZD and $\dim F \leq n$. Then we can write $F = F' \cup Z$, where $\dim F' < n$ and $\dim Z \leq 0$; see [6, Theorem 1.5.7]. By induction we have $\dim(E \cup F') \leq n$; thus, $\dim X \leq n + 1$. □

Using Theorem 2.1 inductively on finite unions of AZD spaces, we obtain:

Theorem 2.2. *Let $n \in \mathbb{N}$. If a space X can be covered by n AZD subspaces, then* $\dim X \leq n$.

Recall that a space X can be covered by n zero-dimensional subspaces if and only if $\dim X < n$; see [6, Theorem 1.5.8]. In contrast, the converse of Theorem 2.2 is not valid – the following result shows that for instance $\mathbb{R}^n$ cannot be covered by n AZD subspaces.

Theorem 2.3. *Let $n \in \mathbb{N}$. If X is σ-compact then* $\dim X < n$ *if and only if X can be covered by n AZD subspaces.*

Proof: The "only if" part follows from the partition into zero-dimensional spaces. We prove the "if" part by induction. In the base case $n = 1$, X is a σ-compact AZD space; thus, $\dim X \leq 0$. Assume that the "if" part is valid for some $n \in \mathbb{N}$. Let $X = E_1 \cup \cdots \cup E_{n+1}$, where each E_i is AZD. As remarked above, we may assume that E_{n+1} is a G_δ-set in X. So $X \setminus E_{n+1}$ is a σ-compact space that can be covered by n AZD subspaces. Thus by induction, $\dim(X \setminus E_{n+1}) < n$. Theorem 2.1 now guarantees that $\dim X \leq n$ and the proof is complete. □

Theorem 2.2 is sharp – $\mathfrak{E}_c \times [0,1]^{n-1}$ is an n-dimensional complete space (see [8] or [6, Problem 1.9.E(b)]) that can be partitioned into n AZD subsets because the $(n-1)$-cell can be partitioned into n zero-dimensional spaces. This example was presented in [12] for the case $n = 2$.

In a zero-dimensional space, the retracts are precisely the non-empty closed subsets. This result was extended to AZD spaces by Dijkstra and van Mill [4, Theorem 5.16] as follows.

Theorem 2.4. *A non-empty subset of an AZD space is a C-set if and only if it is a retract of the space.*

This theorem and the following corollary are the main tools in our proofs in the remaining part of this note.

Corollary 2.5. *Let A be a C-set in an AZD space X. Every clopen subset of A can be extended to a clopen subset of X and every C-set in A is also a C-set in X.*

3. Finite sums of closed sets

In this section we show that almost zero-dimensionality is rather poorly behaved with respect to sums of closed sets.

Lemma 3.1. *Let $X = E \cup F$, where E and F are closed subsets of X.*

(1) *E is a C-set in X if and only if $E \cap F$ is a C-set in F.*
(2) *If E is a nonempty C-set in X and if F is AZD then E is a retract of X and hence every C-set in E is a C-set in X.*

Proof: For statement (1), note that the "only if" part is trivial. For the "if" part, note that if C is a clopen subset of F that does not meet E, then C is open in X because E is closed, and C is closed in X because F is closed.

For statement (2), if $E \cap F = \emptyset$ then (2) is a trivial statement, so assume that $E \cap F \neq \emptyset$. Since $E \cap F$ is a C-set in F, we have by Theorem 2.4 that there is a retraction $r\colon F \to E \cap F$. Extend r to an $\tilde{r}\colon X \to E$ by using the identity on E. Since E and F are closed we have that $\tilde{r}$ is continuous and a retraction. Let A be a C-set of E. Clearly, $A = E \cap \tilde{r}^{-1}(A)$ and hence A is a C-set in X as an intersection of two C-sets. $\square$

Theorem 3.2. *Let $X = E \cup F$, where E and F are AZD closed subsets of X. If E is a C-set in X then X is AZD.*

Proof: Let $x \in X$ and let V be an arbitrary neighbourhood of x in X. We consider three cases.

Case 1: $x \in E \setminus F$. Let U be a C-set neighbourhood of x in E such that $U \subseteq V$. Since F is closed, U is also a neighbourhood of x in X. By Lemma 3.1, U is a C-set in X.

Case 2: $x \in F \setminus E$. Since E is a C-set there exists a clopen subset C of X that contains x and misses E. Now there is a C-set neighbourhood U of x in F that is contained in $V \cap C$. Thus, U is a C-set in the clopen set C and hence a C-set in X.

Case 3: $x \in E \cap F$. We select with Lemma 3.1 a retraction $r : X \to E$. Let $U_1 \subseteq V$ be a C-set neighbourhood of x in E. Then $r^{-1}(U_1)$ is a (C-set) neighbourhood of x in X. Let U_2 be a C-set neighbourhood of x in F that is contained in $V \cap r^{-1}(U_1)$. We claim that the neighbourhood $U = U_1 \cup U_2$ of x is a C-set of X. Let $y \in X \setminus U$. If $y \in F \setminus E$, then there is a clopen subset C of X that contains y and misses E, and there is a clopen subset C' of F that contains y and does not intersect U_2. Note that the intersection $C \cap C'$ is a clopen subset of C and hence a clopen subset of X that contains y and misses U. If $y \in E$, then there exists a clopen set C in E that contains y that does not intersect U_1. Thus, the clopen set $r^{-1}(C)$ misses $r^{-1}(U_1)$ and its subset U_2. We have that $r^{-1}(C)$ is a clopen neighbourhood of y that misses U. In conclusion, U is a C-set in X. $\square$

Corollary 3.3 (Finite C-set Sum Theorem)**.** *If X can be covered by finitely many C-sets that are AZD, then X is AZD.*

Example 3.4. Erdős [7] proved in essence that the empty set is the only clopen and bounded subset in $\mathfrak{E}_{\mathrm{c}}$; see also [2]. (The term "bounded" refers to the standard norm on ℓ^2 that is given by $\|x\| = (\sum_{i=1}^{\infty} x_i^2)^{1/2}$.) Consequently, if we add a new point ∞ to $\mathfrak{E}_{\mathrm{c}}$ whose neighbourhoods are the complements of bounded sets, then the resulting space $\mathfrak{E}_{\mathrm{c}}^+ = \mathfrak{E}_{\mathrm{c}} \cup \{\infty\}$ is a connected space (this is well-known). Let $R_n = \{x \in \mathfrak{E}_{\mathrm{c}} : n \leq \|x\| \leq n+1\}$ for each $n \in \mathbb{N} \cup \{0\}$. Define the closed subsets $\mathcal{E}_{\mathrm{even}} = \bigcup_{k=0}^{\infty} R_{2k} \cup \{\infty\}$ and $\mathcal{E}_{\mathrm{odd}} = \bigcup_{k=0}^{\infty} R_{2k+1} \cup \{\infty\}$ of $\mathfrak{E}_{\mathrm{c}}^+$. The space $\mathcal{E}_{\mathrm{even}}$ is AZD because each R_{2n} is a clopen AZD subspace of $\mathcal{E}_{\mathrm{even}}$ and $\mathcal{E}_{\mathrm{even}}$ is clearly zero-dimensional at the point ∞. $\mathcal{E}_{\mathrm{odd}}$ is AZD for the same reason, and we have $\mathfrak{E}_{\mathrm{c}}^+ = \mathcal{E}_{\mathrm{even}} \cup \mathcal{E}_{\mathrm{odd}}$.

By the example we have that in Theorem 3.2 we cannot delete the requirement that E be a C-set.

Proposition 3.5. *There exists a non-trivial connected complete space X such that X can be written as a union of two AZD closed subsets.*

In the next example we show that the closedness of the subspace F in Theorem 3.2 is essential.

Example 3.6. Let N be the convergent sequence $\{0\}\cup\{\frac{1}{n} : n \in \mathbb{N}\}$. Consider the product space $\mathfrak{E}_c^+ \times N$ and its subspace

$$P = (\mathfrak{E}_c \times \{\tfrac{1}{n} : n \in \mathbb{N}\}) \cup \{(\infty, 0)\}.$$

Since every $\mathfrak{E}_c \times \{\frac{1}{n}\}$ is clopen in P, we have that $\{(\infty, 0)\}$ is a C-set in P, that $P \setminus \{(\infty, 0)\}$ is AZD, and that P is totally disconnected.

Let a be a fixed point in $\mathfrak{E}_c$ and consider the closed subset $A = \{(a, \frac{1}{n}) : n \in \mathbb{N}\}$ of P. We claim that for every C-set neighbourhood U of $(\infty, 0)$ in P, the set $A \setminus U$ is finite (thus, P is not AZD and A is no C-set). Let U be a C-set neighbourhood of $(\infty, 0)$ in P. Then there is a neighbourhood V of ∞ in $\mathfrak{E}_c^+$ and an $n \in \mathbb{N}$ such that $V \times \{\frac{1}{k} : k \geq n\} \subset U$. Assume that $(a, \frac{1}{k}) \notin U$ for $k \geq n$. Select a clopen set C such that $(a, \frac{1}{k}) \in C \subset P \setminus U$. Note that $C' = \{x \in \mathfrak{E}_c : (x, \frac{1}{k}) \in C\}$ is a clopen subset of $\mathfrak{E}_c$ that is disjoint from V, and hence C' is bounded. Since $a \in C'$ we have a contradiction with Erdős [7].

Let P^+ stand for the space $P \cup \{(a, 0)\}$. Then P^+ is not totally disconnected because $(\infty, 0)$ cannot be separated from $(a, 0)$ by a clopen set. For if there is a clopen set C that contains $(\infty, 0)$ but not $(a, 0)$, then $C \cap P$ is a C-set neighbourhood of $(\infty, 0)$ in P such that $A \cap C$ is finite, contradicting the result above. Note that these two points are the only points that cannot be separated; thus, P^+ is hereditarily disconnected.

Proposition 3.7. *There exists a complete space X that is totally disconnected but not AZD with a C-subset E such that E is zero-dimensional and $X \setminus E$ is AZD.*

Proof: $X = P$ and $E = \{(\infty, 0)\}$. □

The following proposition shows that we still do not have a closed sum theorem for AZD spaces even if we know that the union is totally disconnected.

Proposition 3.8. *There exists a complete space X that is totally disconnected but not AZD, and that can be written as a union of two closed AZD subsets E and F.*

Proof: We combine the examples 3.4 and 3.6 as follows: $X = P$, $E = P \cap (\mathcal{E}_{\text{even}} \times N)$, and $F = P \cap (\mathcal{E}_{\text{odd}} \times N)$. □

We also have:

Proposition 3.9. *There exists a complete space X that is hereditarily disconnected but not totally disconnected, and that can be written as a union of two closed AZD subsets E and F.*

Proof: $X = P^+$, $E = P^+ \cap (\mathcal{E}_{\text{even}} \times N)$, and $F = P^+ \cap (\mathcal{E}_{\text{odd}} \times N)$. □

4. Countable sums of C-sets

In view of Corollary 3.3, it is natural to ask whether a countable sum of AZD C-sets is AZD.

Proposition 4.1. *There are complete spaces X and Y that can be written as a countable union of AZD C-subsets such that*

(1) *X is totally disconnected but not AZD, and*
(2) *Y is hereditarily disconnected but not totally disconnected.*

Proof: Let $X = P$ and $Y = P^+$ as in Example 3.6. □

Note that every union of hereditarily disconnected C-sets is trivially hereditarily disconnected.

We show that there is a sum theorem for locally finite collections of C-sets. We say that the space X is *locally finitely coverable* by a collection $\mathcal{C}$ of subsets of X if each point of X has a neighbourhood that is covered by finitely many elements from $\mathcal{C}$. Obviously, if $\mathcal{C}$ is a locally finite cover for X, then X is locally finitely coverable by $\mathcal{C}$.

Lemma 4.2. *Let X be locally finitely coverable by a collection $\mathcal{C}$ consisting of C-sets and let C be a nonempty element of $\mathcal{C}$. If each element of $\mathcal{C} \setminus \{C\}$ is AZD then C is a retract of X, and hence every C-set in C is a C-set in X.*

Proof: Without loss of generality we may assume that $\mathcal{C}$ is countable and represent it as $\{X_n : n \in \mathbb{N}\}$ with $C = X_1$. By Lemma 3.1, we can find a retraction $r'_n : \bigcup_{k=1}^{n+1} X_k \to \bigcup_{k=1}^{n} X_k$ for each $n \in \mathbb{N}$. Define $r_n = r'_1 \circ r'_2 \circ \cdots \circ r'_n$; note that it is a retraction from $\bigcup_{k=1}^{n+1} X_k$ to C. Observe that $r_{n+1} \restriction \bigcup_{k=1}^{n+1} X_k = r_n$ for each n; thus, $r = \bigcup_{n=1}^{\infty} r_n$ is a well-defined function from X to C that restricts to the identity on C. For each $x \in X$ there is an $n \in \mathbb{N}$ such that $U = \bigcup_{k=1}^{n+1} X_k$ is a neighbourhood of x. Since $r \restriction U = r_n$ and r_n is continuous, we have that r is continuous. □

Theorem 4.3. *Let X be locally finitely coverable by a collection $\mathcal{C}$ which consists of AZD C-sets. Then X is an AZD space.*

Proof: Let V be a arbitrary neighbourhood of $x \in X$. We may assume that V is a subset of $\bigcup \mathcal{F}$ for some finite subcollection $\mathcal{F}$ of $\mathcal{C}$. By Corollary 3.3, the set $\bigcup \mathcal{F}$ is AZD so we can select a C-set neighbourhood U of x in $\bigcup \mathcal{F}$ with $U \subset V$. Applying Lemma 4.2 to the cover $\{\bigcup \mathcal{F}\} \cup \mathcal{C}$, we find that U is a C-set in X that is obviously a neighbourhood of x. □

5. Closed mappings and retractions

Concerning totally disconnected spaces we have the following observation.

Proposition 5.1. *For a space X the following statements are equivalent:*

(1) *X is totally disconnected,*
(2) *every singleton in X is a C-set in X, and*
(3) *every retract of X is a C-set in X.*

Proof: The equivalence of (1) and (2) is a triviality. Since singletons are retracts, we have (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3). Let $r\colon X \to A$ be a retraction and let x be an arbitrary point in $X \setminus A$. Thus, $r(x) \neq x$ and hence by (1), there is a clopen C in X with $x \in C$ and $r(x) \notin C$. Consider the clopen neighbourhood $D = C \setminus r^{-1}(C)$ of x and note that $D \cap A = \emptyset$. □

In view of Proposition 5.1 a natural question would be whether Theorem 2.4 and Corollary 2.5 are valid in the class of totally disconnected spaces. The following result shows that the answer is no.

Proposition 5.2. *There exists a totally disconnected complete space X with a C-subset E, such that E contains a clopen subset C that is no C-set in X (and hence, E is no retract of X).*

Proof: Consider Example 3.6. We let $X = P$, $E = A \cup \{(\infty, 0)\}$, and $C = A$. □

Question 5.3. Does there exist a space that is not AZD such that the nonempty C-sets are precisely the retracts of the space?

A closed map with a zero-dimensional range and zero-dimensional fibers has a zero-dimensional domain; see [6, Theorem 1.12.4]. Again we see that almost zero-dimensionality is poorly behaved in this respect:

Proposition 5.4. *There exists a perfect and open retraction from a complete space that is not totally disconnected onto an AZD subspace such that each fibre is finite.*

The proof is contained in the following example:

Example 5.5. Let $B = \{x \in \mathfrak{E}_c : \|x\| \leq 1\}$ and let $S = \{x \in \mathfrak{E}_c : \|x\| = 1\}$. Every nonempty clopen subset of B intersects S, because if C is a nonempty clopen set of B with $C \cap S = \emptyset$, then C is clopen and bounded in $\mathfrak{E}_c$, in violation of Erdős [7]. Consider the following equivalence relation on the space $B \times \{0, 1\}$:

$$(x, \varepsilon) \sim (y, \delta) \Leftrightarrow x = y \wedge (\varepsilon = \delta \vee \|x\| = 1).$$

Let $\mathcal{B} = (B \times \{0, 1\})/\sim$ be the quotient space with quotient map q. Let $B_\varepsilon = q(B \times \{\varepsilon\})$ for $\varepsilon = 0, 1$ and let $\tilde{S} = B_0 \cap B_1 = q(S \times \{0, 1\})$. Thus, $\mathcal{B}$ consists of two closed copies of B which are attached to each other by their unit spheres. Let h be the homeomorphism of $B \times \{0, 1\}$ that is given by the rule $h(x, \varepsilon) = (x, 1 - \varepsilon)$, and let $\tilde{h}$ be the homeomorphism of $\mathcal{B}$ that is defined by $q \circ h = \tilde{h} \circ q$. Note that $\rho = \mathrm{id}_{B_0} \cup (\tilde{h} \restriction B_1)$ is a retraction of $\mathcal{B}$ onto B_0. The map ρ is easily seen to be both open and closed.

Let C be a clopen subset of $\mathcal{B}$. Since $\tilde{h}$ restricts to the identity on $\tilde{S}$ we have that $C \setminus \tilde{h}(C)$ is a clopen subset of $\mathcal{B}$ that is disjoint from $\tilde{S}$. By the remark above, this means that $C \setminus \tilde{h}(C) = \emptyset$. Since $\tilde{h} = \tilde{h}^{-1}$ we have that $\tilde{h}(C) = C$ for every clopen set C in $\mathcal{B}$. Consequently, no $x \in \mathcal{B} \setminus \tilde{S}$ can be separated from the distinct point $\tilde{h}(x)$ and hence $\mathcal{B}$ is not totally disconnected. Since B_0 is AZD and the fibers of ρ contain at most two points, we have that $\mathcal{B}$ is hereditarily disconnected. Note that we have found another proof of Proposition 3.9.

Example 5.5 shows that a space that is not totally disconnected may be the union of two of its AZD retracts. The following observations show that our example can not be strengthened, meaning that the union of two AZD retracts is either AZD or a hereditarily disconnected space that is not totally disconnected. It follows from

Corollary 3.3 and Proposition 5.1 that if X is a totally disconnected space that can be written as a finite union of AZD retracts of the space, then it is AZD.

Proposition 5.6. *If $X = \bigcup_{i \in I} F_i$, where each F_i is a totally disconnected retract of X and $|I| < 2^{\aleph_0}$, then X is hereditarily disconnected.*

Proof: Let C be a connected subset of X and let $i \in I$. If $C \cap F_i$ consists of at least two points then we can separate these points in X by a clopen set because F_i is a totally disconnected retract. Thus, $|C| < 2^{\aleph_0}$ and hence $|C| \leq 1$. □

6. Sums of open sets

We conclude this note by considering the following question:

Question 6.1. Is the union of two open AZD subspaces an AZD space?

Note that the answer is negative if we substitute totally disconnected for AZD:

Proposition 6.2. *There exists a complete space X that is not totally disconnected and that contains two distinct points x and y such that $X \setminus \{x\}$ is totally disconnected and $X \setminus \{y\}$ is AZD.*

Proof: Consider Example 3.6 and let $X = P^+$, $x = (a, 0)$, and $y = (\infty, 0)$. □

According to the next theorem such a simple counterexample does not exist for AZD spaces. Moreover, this theorem gives a partial answer to our question. Note that compacta in totally disconnected spaces are C-sets.

Theorem 6.3. *Let $X = O_1 \cup O_2$, where O_1 and O_2 are open AZD subsets. If $X \setminus O_1$ is a C-set in O_2 and $X \setminus O_2$ is compact, then X is AZD.*

Proof: Let $K = X \setminus O_2$. Let U be an open subset of X such that $K \subset U \subset \overline{U} \subset O_1$. By compactness there exists a C-set V_1 in O_1 such that $K \subset \operatorname{int} V_1 \subset V_1 \subset U$. We put $V_2 = X \setminus O_1$ and we claim that $V = V_1 \cup V_2$ is a C-set in X. Let $x \in X \setminus V$

and select for $i = 1, 2$ a clopen subset C_i of O_i such that $x \in C_i$ and $C_i \cap V_i = \emptyset$. Then C_i and $O_i \setminus C_i$ are open in X. Thus, $C_1 \cap C_2 = X \setminus ((O_1 \setminus C_1) \cup (O_2 \setminus C_2))$ is a clopen neighbourhood of x in X that misses V. Note that V is AZD as a topological sum of the AZD spaces V_1 and V_2. Observe that $\overline{X \setminus V}$ is contained in $X \setminus \operatorname{int} V_1 \subset \underline{O_2}$ and hence the set is AZD. Applying Theorem 3.2 to $X = V \cup \overline{X \setminus V}$, we find that the space is AZD. □

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Faculteit der Exacte Wetenschappen / Afdeling Wiskunde; Vrije Universiteit; De Boelelaan 1081a; 1081 HV Amsterdam; The Netherlands

E-mail address: mabry@cs.vu.nl

E-mail address: dijkstra@cs.vu.nl

E-mail address: vanmill@cs.vu.nl