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Almost disjoint families of connected sets

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Abstract

We investigate the question which (separable metrizable) spaces have a 'large' almost disjoint family of connected (and locally connected) sets. Every compact space of dimension at least 2 as well as all compact spaces containing an 'uncountable star' have such a family. Our results show that the situation for 1-dimensional compacta is unclear.

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1. Introduction

All topological spaces under discussion are separable and metrizable.

Let X be a set and $\mathcal{A}$ be a family of subsets of X. As usual we say that $\mathcal{A}$ is *almost disjoint* if $\mathcal{A} \subseteq [X]^{|X|}$ and for all distinct $A, B \in \mathcal{A}$ we have $|A \cap B| < |X|$. It is well known and easy to prove that every infinite set X has an almost disjoint family of subsets $\mathcal{A}$ with $|\mathcal{A}| > |X|$; in particular, $\mathbb{R}$ has an almost disjoint family of size greater than $\mathfrak{c}$. The full binary tree of height $\mathfrak{c}$ has cardinality $2^{<\mathfrak{c}}$ and $2^{\mathfrak{c}}$ cofinal branches, hence $2^{<\mathfrak{c}} = \mathfrak{c}$

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(in particular the Continuum Hypothesis) implies that there is an almost disjoint family of maximal size $2^{\mathfrak{c}}$ in $[\mathfrak{c}]^{\mathfrak{c}}$.

If $C \subseteq \mathbb{R}$ is connected and nontrivial (meaning that it has cardinality at least 2) then it contains an interval with rational endpoints. Hence $\mathbb{R}$ has a family of countably many nontrivial connected sets $\mathcal{B}$ such that every nontrivial connected set C contains a member of $\mathcal{B}$ (so $\mathcal{B}$ is sort of a π-base for the connected sets). For $\mathbb{R}^2$ a family with similar properties must have size at least $\mathfrak{c}$, as the collection $\{\{x\} \times \mathbb{R}\colon x \in \mathbb{R}\}$ clearly demonstrates. Can there be such a family of size $\mathfrak{c}$? We will answer this question in the negative by proving the existence of an almost disjoint family of more than $\mathfrak{c}$ connected and locally connected subsets of $\mathbb{R}^2$. This motivated us to study the following question: which spaces have 'large' almost disjoint families of nontrivial connected sets? Here 'large' means: any size greater than $\mathfrak{c}$ (for which there exists an almost disjoint family of subsets of $\mathbb{R}$). Let Γ denote the class of spaces X for which such families can be found. We prove that if X is compact and $\dim X \geqslant 2$ then $X \in \Gamma$. Moreover, if X contains a certain uncountable 'star' then $X \in \Gamma$. These results suggest that the rational continua are precisely the continua that are not in Γ. We disprove this by showing that the Sierpiński triangular curve which is rational is in Γ and that there is a certain continuum which is not rational at a dense open set of points is not in Γ. We conclude by asking for a transparent characterization of all the 1-dimensional compacta that are in Γ.

2. Preliminaries

If X is a set then $\mathcal{P}(X)$ denotes its powerset. A space X is locally of cardinality κ if each nonempty open subset U of X has size κ. A set T is a *transversal* for a collection of sets $\mathcal{A}$ provided that $|A \cap T| = 1$ for every $A \in \mathcal{A}$.

Let $\mathbb{AD}(\mathfrak{c})$ be the set of all cardinals $\kappa > \mathfrak{c}$ for which there exist κ almost disjoint subsets of $\mathbb{R}$. As noted above, we have $\mathfrak{c}^+ \in \mathbb{AD}(\mathfrak{c})$. We denote by Γ the class of all spaces X for which there exists for every $\kappa \in \mathbb{AD}(\mathfrak{c})$ an almost disjoint family of κ connected sets in X. In addition, Δ is the collection of all spaces in which there exists for every $\kappa \in \mathbb{AD}(\mathfrak{c})$ an almost disjoint family of size κ of connected and locally connected sets.

Lemma 2.1. *Let $\kappa \in \mathbb{AD}(\mathfrak{c})$. If X is a set with $|X| = \mathfrak{c}$ then there are families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ such that*

(1) *$\mathcal{A}$ is almost disjoint and $\mathcal{B}$ is pairwise disjoint,*
(2) *$|\mathcal{A}| = \kappa$,*
(3) *$|\mathcal{B}| = \mathfrak{c}$ and $|B \cap A| = \mathfrak{c}$ for all $B \in \mathcal{B}$, $A \in \mathcal{A}$.*

Proof. Let $\mathcal{G} \subseteq \mathcal{P}(X)$ be almost disjoint with $|\mathcal{G}| = \kappa$. For every $G \in \mathcal{G}$ let $f_G : G \to X$ be a surjection such that $|f^{-1}(t)| = \mathfrak{c}$ for every $t \in X$. Let $E(G) = \{\langle x, f_G(x)\rangle\colon x \in G\}$, the graph of f_G in $G \times X$. It is clear that the family $\mathcal{E} = \{E(G)\colon G \in \mathcal{G}\}$ is almost disjoint in X^2 such that $|\mathcal{G}| = |\mathcal{E}|$. Moreover, $\mathcal{F} = \{X \times \{x\}\colon x \in X\}$ is pairwise disjoint, and $|F \cap E(G)| = \mathfrak{c}$ for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Since $|X|^2 = |X|$, we are done. $\square$

Corollary 2.2. *Let $\kappa \in \mathbb{AD}(\mathfrak{c})$. If X is a space with $|X| = \mathfrak{c}$ then for every pairwise disjoint family $\mathcal{E} \subseteq [X]^{\mathfrak{c}}$ there is a family $\mathcal{A} \subseteq \mathcal{P}(X)$ such that*

(1) *$\mathcal{A}$ is almost disjoint,*
(2) *$|\mathcal{A}| = \kappa$ and every $A \in \mathcal{A}$ is locally of size $\mathfrak{c}$,*
(3) *$|E \cap A| = \mathfrak{c}$ for all $E \in \mathcal{E}$, $A \in \mathcal{A}$.*

Proof. Let $\mathcal{A}', \mathcal{B} \subseteq \mathcal{P}(X)$ be as in Lemma 2.1. Since $|\mathcal{E}| \leqslant \mathfrak{c}$ and $|\mathcal{B}| = \mathfrak{c}$, there is a bijection $f : X \to X$ such that $f(E) \in \mathcal{B}$ for every $E \in \mathcal{E}$. So we may assume without loss of generality that $\mathcal{E} \subseteq \mathcal{B}$.

Fix an arbitrary $A' \in \mathcal{A}'$. The collection $\mathcal{U}$ of all relatively open subsets of A' of cardinality less than $\mathfrak{c}$ has a countable subcollection with the same union. Since $\mathfrak{c}$ has uncountable cofinality, this means that $|\bigcup \mathcal{U}| < \mathfrak{c}$. Hence $A = A' \setminus \bigcup \mathcal{U}$ is locally of cardinality $\mathfrak{c}$. The collection $\mathcal{A} = \{A\colon\ A' \in \mathcal{A}'\}$ is almost disjoint and $|\mathcal{A}| = |\mathcal{A}'|$. In addition, if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $|A \cap B| = \mathfrak{c}$ since $|A' \setminus A| < \mathfrak{c}$. $\square$

Remark 2.3. It is not true that for every almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$ there are two disjoint subsets of $\mathbb{R}$ both meeting each $A \in \mathcal{A}$. To prove this, let $\{A_\alpha\colon\ \alpha < 2^{\mathfrak{c}}\}$ be almost disjoint in $\mathbb{R}$ (observe that such a family exists, e.g., under the Continuum Hypothesis, see Section 1). Let $\{\langle B_\alpha^0, B_\alpha^1\rangle\colon\ \alpha < 2^{\mathfrak{c}}\}$ list all pairs of disjoint subsets of $\mathbb{R}$. For $\alpha < 2^{\mathfrak{c}}$ we will define sets $A'_\alpha \subseteq \mathbb{R}$, as follows. If $|A_\alpha \cap B_\alpha^0| = \mathfrak{c}$ then $A'_\alpha = A_\alpha \cap B_\alpha^0$. If this is not true then we put $A'_\alpha = A_\alpha \cap B_\alpha^1$ if this intersection has size $\mathfrak{c}$. Finally, if $|A_\alpha \cap (B_\alpha^0 \cup B_\alpha^1)| < \mathfrak{c}$ then $A'_\alpha = A_\alpha \setminus (B_\alpha^0 \cup B_\alpha^1)$. The family $\{A'_\alpha\colon\ \alpha < 2^{\mathfrak{c}}\}$ is almost disjoint and for every $\alpha < 2^{\mathfrak{c}}$ we have $B_\alpha^0 \cap A'_\alpha = \emptyset$ or $B_\alpha^1 \cap A'_\alpha = \emptyset$.

Remark 2.4. Let $\mathcal{A}$ be a family of at most κ sets, each of cardinality κ. It is well known and easy to prove that for every $A \in \mathcal{A}$ there is a set $B(A) \subseteq A$ of size κ such that the family $\{B(A)\colon\ A \in \mathcal{A}\}$ is pairwise disjoint. This is called the 'Disjoint Refinement Lemma' and will be used frequently in the remaining part of this note.

Remark 2.5. A space is *analytic* if it is a continuous image of a complete space. It is an old result of Souslin that every analytic space is either countable or contains a Cantor set, [5, 1.5.12]. This implies that a complete space is either countable or of cardinality $\mathfrak{c}$. This will be used frequently without explicit reference in the remaining part of this note.

3. The first method for constructing almost disjoint families

In this section we will describe our first method for constructing almost disjoint families of connected sets. We will conclude that if X is compact and $\dim X \geqslant 2$ then $X \in \Gamma$ and if $n \geqslant 2$ then $\mathbb{R}^n \in \Delta$.

For a space X we let $\mathcal{S}(X)$ denote the collection of all closed subsets of X that separate X.

Theorem 3.1. *Let X be a space containing a set S of size less than $\mathfrak{c}$ such that*:

(1) *X is connected and nontrivial,*
(2) *every $A \in \mathcal{S}(X)$ that misses S has cardinality $\mathfrak{c}$.*

Then $X \in \Gamma$.

Assume that X is connected, locally connected and nontrivial, and that for every connected open $U \subseteq X$ we have

(3) *every $A \in \mathcal{S}(U)$ that misses S has cardinality $\mathfrak{c}$.*

Then $X \in \Delta$.

Proof. We may assume without loss of generality that S is dense in X. Let $\mathcal{B}$ be the collection of all F_σ-subsets of X that miss S and have cardinality $\mathfrak{c}$. Then, clearly, $|\mathcal{B}| \leqslant \mathfrak{c}$. (It is easy to see that $\mathcal{B} \neq \emptyset$.) The Disjoint Refinement Lemma gives us for every $B \in \mathcal{B}$ a set $E(B) \subseteq B$ of size $\mathfrak{c}$ such that the collection

$$\{E(B)\colon\ B \in \mathcal{B}\}$$

is pairwise disjoint. Let $\kappa \in \mathbb{AD}(\mathfrak{c})$ be arbitrary. By Corollary 2.2 there is an almost disjoint family $\mathcal{A}$ of subsets of X such that $|\mathcal{A}| = \kappa$ and $|E(B) \cap A| = \mathfrak{c}$ for every $B \in \mathcal{B}$ and $A \in \mathcal{A}$. So every $A \in \mathcal{A}$ intersects every $B \in \mathcal{B}$, hence to establish the first part of the theorem it clearly suffices to prove the following:

Claim 1. *If $Y \subseteq X$ intersects every $B \in \mathcal{B}$ then $Y \cup S$ is connected.*

Assume that $Y \cup S$ can be written as the union of the disjoint relatively open and non-empty sets E and F. Since $Y \cup S$ is dense in X there exist disjoint and open sets E' and F' in X such that $E' \cap (Y \cup S) = E$ and $F' \cap (Y \cup S) = F$. Then by (2), $B = X \setminus (E' \cup F') \in \mathcal{B}$. Since $Y \cap B \neq \emptyset$, this is a contradiction.

For the second part of the theorem, simply observe that if $U \subseteq X$ is open and connected and $A \in \mathcal{S}(U)$ then A is an F_σ-subset of X. Hence if such an A misses S then $A \in \mathcal{B}$ by (3). So the proof of Claim 1 can be repeated to show that if $Y \subseteq X$ intersects every $B \in \mathcal{B}$ then $U \cap (Y \cup S)$ is connected for every connected open set U. Consequently, $\{A \cup S\colon\ A \in \mathcal{A}\}$ is now an almost disjoint family of connected and locally connected sets. □

Remark 3.2. A space X is *punctiform* if it contains no nondegenerate continuum. Knaster and Kuratowski [3, pp. 236 and 237] proved that if $X \subseteq \mathbb{R}^2$ is punctiform then $\mathbb{R}^2 \setminus X$ is connected and locally connected. Their proof is similar to the argument in Claim 1 in the proof of Theorem 3.1 (observe that if $U \subseteq \mathbb{R}^2$ is connected and open and $B \in \mathcal{S}(U)$ then B contains a nondegenerate continuum). See also Jones [2] where similar arguments were used.

Corollary 3.3. *If $n \geqslant 2$ then $\mathbb{R}^n \in \Delta$.*

Proof. Observe that no countable set separates a connected open subset of $\mathbb{R}^n$ if $n \geqslant 2$. So we can apply Theorem 3.1 with $S = \emptyset$. $\square$

Remark 3.4. If E and F are two opposite faces of $\mathbb{I}^{n+1} \subseteq \mathbb{R}^{n+1}$, then the almost disjoint sets we get from the proof of Theorem 3.1 evidently meet each continuum that intersects both E and F. By the results in Rubin, Schori and Walsh [7] this means that they are all of dimension at least n. Since every $(n+1)$-dimensional subspace of $\mathbb{R}^{n+1}$ has nonempty interior, [5, 3.7.1], only countably many of them are of dimension $n+1$. So we conclude that in $\mathbb{R}^{n+1}$ there exists a 'large' almost disjoint family of connected and locally connected n-dimensional sets. Moreover, these sets can be assumed to be pairwise non-homeomorphic. This is clear since for every set $A \subseteq \mathbb{R}^n$ the collection $\{f(A)\colon\ f: A \to \mathbb{R}^n \text{ continuous}\}$ has size at most $\mathfrak{c}$ (use that A is separable).

A continuum X is *decomposable* if it is the union of two proper subcontinua. A continuum that is not decomposable, is called *indecomposable*. It is known that a continuum is indecomposable if and only if all of its proper subcontinua are nowhere dense, [5, 1.10.13].

The following lemma is well known. It can be proved by observing that every indecomposable continuum has uncountably many composants and that they are all dense and connected, [5, Exercise 1.10.23]. We include a somewhat simpler proof.

Lemma 3.5. *Let X be an indecomposable continuum. Then no countable closed subset of X separates X.*

Proof. Striving for a contradiction, let $K \subseteq X$ be a countable closed set such that $X \setminus K$ can be written as $E \cup F$, where E and F are disjoint nonempty open subsets of X. Let $\mathcal{E}$ be the collection of components of $E \cup K$. Since X is a continuum, and $E \cup K$ is a proper closed subset of X, every $E \in \mathcal{E}$ meets K, [5, A.10.5]. This means that $\mathcal{E}$ is countable. Similarly, the collection $\mathcal{F}$ of components of $F \cup K$ is countable. Since X is indecomposable, the countable closed cover $\mathcal{E} \cup \mathcal{F}$ of X consists of nowhere dense sets. But this violates the Baire Category Theorem for X. $\square$

Corollary 3.6. *Let X be a compact space with* $\dim X \geqslant 2$. *Then $X \in \Gamma$.*

Proof. It was shown by Bing that X contains a (hereditarily) indecomposable continuum K, [5, 3.8.3]. By Lemma 3.5 it follows that no countable closed subset of K separates K. This means that if $T \in \mathcal{S}(K)$ then T is uncountable, hence $|T| = \mathfrak{c}$. So $K \in \Gamma$ by Theorem 3.1 (let again $S = \emptyset$), hence $X \in \Gamma$. $\square$

4. The second method for constructing almost disjoint families

We will now present our second method that concerns uncountable stars, i.e., uncountable collections of subcontinua 'emanating' from a single common point.

If X is compact then $\mathcal{C}(X)$ denotes the hyperspace of subcontinua of X with the Vietoris topology. It is known that $\mathcal{C}(X)$ is compact, [5, 1.11.14]. The topology of $\mathcal{C}(X)$ can also be described in terms of a metric. Indeed, if ϱ is an admissible metric for X then the Hausdorff metric ϱ_H is an admissible metric for $\mathcal{C}(X)$.

Lemma 4.1. *Let $\mathcal{A} \subseteq \mathcal{C}(X)$. If $\mathcal{D} \subseteq \mathcal{A}$ is dense in $\mathcal{A}$ in $\mathcal{C}(X)$ then $\bigcup \mathcal{D}$ is dense in $\bigcup \mathcal{A}$ in X.*

Proof. Let $x \in \bigcup \mathcal{A}$. If U is an open neighborhood of x then the collection $\mathcal{U} = \{A \in \mathcal{A}\colon A \cap U \neq \emptyset\}$ is a nonempty open subset of $\mathcal{A}$. So there is an element $D \in \mathcal{D} \cap \mathcal{U}$. Clearly, $D \cap U \neq \emptyset$, hence $\bigcup \mathcal{D}$ meets U. $\square$

Theorem 4.2. *Let X be a compact space containing a point p and an uncountable family $\mathcal{C}$ of subcontinua such that for all distinct $C, C' \in \mathcal{C}$ we have $C \cap C' = \{p\}$. Then $X \in \Gamma$.*

Proof. We may clearly assume that X is a continuum. Since $\mathcal{C}$ is an uncountable subspace of $\mathcal{C}(X)$, it contains by the Cantor–Bendixson Theorem, [5, Exercise A.2.13], an uncountable dense-in-itself subspace. So we may assume without loss of generality that $\mathcal{C}$ is dense-in-itself.

Let ϱ be an admissible metric for X. For $x \in X$ and $\varepsilon > 0$ we let $U_\varepsilon(x)$ denote the open ball $\{y \in X\colon \varrho(x, y) < \varepsilon\}$. If $C \in \mathcal{C}(X)$ and $\varepsilon > 0$ then we let $\mathcal{V}_\varepsilon(C)$ denote the closed ball

$$\left\{C' \in \mathcal{C}(X)\colon \varrho_H(C, C') \leqslant \varepsilon\right\}$$

in $\mathcal{C}(X)$. Our aim is to construct a Cantor set $\mathcal{K}$ in $\mathcal{C}(X)$ with certain additional desirable properties. Start with two distinct elements $C_0, C_1 \in \mathcal{C}$. Observe that

$$C_0 \setminus U_{2^{-1}}(p), \qquad C_1 \setminus U_{2^{-1}}(p)$$

are (possibly empty) disjoint compacta in X. Pick disjoint open sets E and F in X such that $C_0 \setminus U_{2^{-1}}(p) \subseteq E$, $C_1 \setminus U_{2^{-1}}(p) \subseteq F$ and $\overline{E} \cap \overline{F} = \emptyset$. There exists $\delta_0 > 0$ such that $\delta_0 < 2^{-1}$, $\mathcal{V}_{\delta_0}(C_0) \cap \mathcal{V}_{\delta_0}(C_1) = \emptyset$, and

$$\bigcup \mathcal{V}_{\delta_0}(C_0) \subseteq E \cup U_{2^{-1}}(p), \qquad \bigcup \mathcal{V}_{\delta_0}(C_1) \subseteq F \cup U_{2^{-1}}(p).$$

Observe that this implies that if $\varrho_H(C_0, A) \leqslant \delta_0$ and $\varrho_H(C_1, B) \leqslant \delta_0$ then $A \cap B \subseteq U_{2^{-1}}(p)$. This completes the first step. In the second step, pick pairwise distinct elements $C_{00}, C_{01}, C_{10}, C_{11}$ in $\mathcal{C}$ such that for $i, j = 0, 1$, $\varrho_H(C_i, C_{ij}) < \delta_0$. This is possible since $\mathcal{C}$ is dense in itself. By the same argument as the one above we find $0 < \delta_1 < 2^{-2}$ such that the closed balls

$$\left\{\mathcal{V}_{\delta_1}(C_{ij})\colon i, j = 0, 1\right\}$$

are pairwise disjoint, while moreover for all distinct $i_0 j_0$ and $i_1 j_1$ and elements $A \in \mathcal{V}_{\delta_1}(C_{i_0 j_0})$ and $B \in \mathcal{V}_{\delta_1}(C_{i_1 j_1})$ we have

$$A \cap B \subseteq U_{2^{-2}}(p).$$

Continuing in this way in the standard manner clearly yields a Cantor set $\mathcal{K}$ in $\mathcal{C}(X)$ such that for all distinct $A, B \in \mathcal{K}$ we have $A \cap B = \{p\}$.

Let $\mathcal{Y} \subseteq \mathcal{K}$ be locally of cardinality $\mathfrak{c}$, and put $Y = \bigcup \mathcal{Y}$. Let $\mathcal{B}$ be the collection of all closed subsets B of Y such that $p \notin B$ and

$$\left|\{C \in \mathcal{Y}\colon\ C \cap B \neq \emptyset\}\right| = \mathfrak{c}.$$

Since $|\mathcal{B}| \leqslant \mathfrak{c}$, there is a transversal $T_{\mathcal{Y}}$ for the collection $\{C \setminus \{p\}\colon\ C \in \mathcal{Y}\}$ such that $T_{\mathcal{Y}} \cap B \neq \emptyset$ for every $B \in \mathcal{B}$. We claim that $T_{\mathcal{Y}} \cup \{p\}$ is connected.

Let U and V be disjoint nonempty relatively open subsets of $T_{\mathcal{Y}} \cup \{p\}$ such that $p \in U$. It suffices to prove that $T_{\mathcal{Y}} \setminus (U \cup V) \neq \emptyset$. Let U' and V' be disjoint open subsets of Y such that $U' \cap (T_{\mathcal{Y}} \cup \{p\}) = U$ and $V' \cap (T_{\mathcal{Y}} \cup \{p\}) = V$, [5, A.8.2]. Put $B = Y \setminus (U' \cup V')$. Then B is closed in Y and does not contain p. Suppose first that $\mathcal{E} = \{C \in \mathcal{Y}\colon\ C \cap B = \emptyset\}$ is dense in $\mathcal{Y}$. Then by Lemma 4.1, the dense connected set $\bigcup \mathcal{E}$ in Y is contained in U (since $p \in U$ and it misses B). But this would imply that $U = Y$, which contradicts the fact that $V \neq \emptyset$. Hence $\mathcal{Y} \setminus \mathcal{E}$ contains a nonempty open subset of $\mathcal{Y}$ and hence is of size $\mathfrak{c}$. This shows that $B \in \mathcal{B}$ and hence, by construction, $T_{\mathcal{Y}} \cap B \neq \emptyset$ or equivalently $T_{\mathcal{Y}} \setminus (U \cup V) \neq \emptyset$.

Let $\kappa \in \mathbb{AD}(\mathfrak{c})$, and let $\{\mathcal{Y}_\alpha\colon\ \alpha < \kappa\}$ be an almost disjoint family of subsets of $\mathcal{K}$ consisting of sets that are all locally of cardinality $\mathfrak{c}$ (Corollary 2.2). Then the collection

$$\left\{T_{\mathcal{Y}_\alpha} \cup \{p\}\colon\ \alpha < \kappa\right\}$$

is clearly almost disjoint and consists of connected sets. □

Corollary 4.3. *Let X be a continuum containing a point p such that $X \setminus \{p\}$ has uncountably many components. Then $X \in \Gamma$.*

Proof. Let K be a component of $X \setminus \{p\}$. We claim that K is not compact. For if it is, then we can find an open neighborhood of K such that $K \subseteq U \subseteq \overline{U} \subseteq X \setminus \{p\}$. It is clear that K is a component of the compact space $\overline{U}$. Since components and quasi-components in a compact space agree, [5, A.10.1], there is a relatively clopen subset C of $\overline{U}$ such that $K \subseteq C \subseteq U$. Since C is open in U, it is open in X. And since C is compact, it is closed in X. This contradicts X being connected. (Alternatively, use the Boundary Bumping Theorem, [6, Theorem 5.4].)

So if $\mathcal{K}$ is the family of all components of $X \setminus \{p\}$ then $K \cup \{p\}$ is a continuum for every $K \in \mathcal{K}$. Hence we are in a position to apply Theorem 4.2. □

5. Examples

A space X is said to be *rational* at the point $x \in X$ if x has arbitrarily small neighborhoods the boundaries of which are countable. A space is *rational* if it is rational at any of its points.

In view of the results in Section 4, one would be tempted to conjecture that for a continuum X we have $X \in \Gamma$ if and only if X is not rational. The following examples demonstrate that this is not true.

Example 5.1. There is a continuum X which is not rational at a dense open set of points but which does not belong to Γ.

Proof. Let $(K_n)_n$ be a sequence of pairwise disjoint Cantor sets in $\mathbb{I}$ such that every non-degenerate interval (r,s) in $\mathbb{I}$ contains one of the K_n. Let $Z = \mathbb{I} \setminus \bigcup_n K_n$, and put

$$X = \bigl(\mathbb{I} \times \{0\}\bigr) \cup \bigcup_{n\geqslant 1} \bigl(K_n \times [0, 1/n]\bigr).$$

It is clear that X is closed in $\mathbb{I}^2$, and is connected. We claim that X is not rational at any point of the dense open set $X \setminus (\mathbb{I} \times \{0\})$. To prove this, observe that if $\langle x, y\rangle \in X \setminus (\mathbb{I} \times \{0\})$, say $\langle x, y\rangle \in K_n \times (0, 1/n]$, then there are a clopen neighborhood C of x in K_n and a positive real number $s \in (0, 1/n]$ with $s < y$ such that $V = C \times [s, 1/n]$ is a neighborhood of $\langle x, y\rangle$ in X. The boundary ∂ of V is clearly equal to $C \times \{s\}$. Let U be an arbitrary open neighborhood of $\langle x, y\rangle$ in X such that $\overline{U} \subseteq V \setminus \partial$. We claim that $\overline{U} \setminus U$ is uncountable. If not, then $B = \pi(\overline{U} \setminus U)$ is also countable, where $\pi : V \to C$ is the projection along the y-axis. By connectivity, if $p \in C \setminus B$ then either $\{p\} \times [s, 1/n] \subseteq U$ or $\{p\} \times [s, 1/n] \subseteq V \setminus \overline{U}$. But, clearly, $\{p\} \times [s, 1/n] \nsubseteq U$ since $\overline{U}$ misses ∂. Since C is a Cantor set and B is countable, $C \setminus B$ is dense in C. Hence $V \setminus \overline{U}$ contains the set $(C \setminus B) \times [s, 1/n]$ that is clearly dense in V. This, however, contradicts the fact that U is a nonempty open subset of V.

Next we show that X has no large almost disjoint family of connected sets, hence $X \notin \Gamma$. Striving for a contradiction, assume that $\mathcal{A}$ is an almost disjoint family of connected subsets of X such that $|\mathcal{A}| > \mathfrak{c}$. For every $A \in \mathcal{A}$ let $D(A) \subseteq A$ be countable and dense. Since X has only $\mathfrak{c}$ countable subsets, there is a subcollection $\mathcal{A}'$ of $\mathcal{A}$ of size greater than $\mathfrak{c}$ such that $D(A) = D(A')$ for all $A, A' \in \mathcal{A}'$. This observation shows that we may assume without loss of generality that there is a subcontinuum Y of X such that every $A \in \mathcal{A}$ is a dense subset of Y.

Put $Z' = Y \cap (Z \times \{0\})$.

Claim 1. *Z' is countable.*

Assume indirectly that Z' is uncountable. Hence Z' is an uncountable G_δ-subset of Y, and consequently has cardinality $\mathfrak{c}$. Observe that at most two elements of Z' are endpoints of Y (the possible 'minimum' and 'maximum' of Z'). So Z' contains a subset Z'' of cardinality $\mathfrak{c}$ which consists entirely of cutpoints of Y. Pick two distinct elements $A, A' \in \mathcal{A}$. Since Z'' consists of cutpoints of Y and both A and A' are dense in Y and connected, we clearly have $Z'' \subseteq A \cap A'$. But this contradicts $\mathcal{A}$ being almost disjoint.

So by Claim 1, Y is covered by the countable disjoint collection

$$\bigl\{Y \cap \bigl(K_n \times [0, 1/n]\bigr)\colon\ n \in \mathbb{N}\bigr\} \cup \bigl\{\{z\}\colon\ z \in Z'\bigr\}.$$

By the Sierpiński Theorem, [5, A.10.6], at most one element of that collection is nonempty. Since that element clearly is not a singleton, this means that there is a unique n such that $Y \subseteq K_n \times [0, 1/n]$. But this means that Y is contained in a topological copy of $\mathbb{I}$. This clearly leads to a contradiction (see the remarks in Section 1). □

Example 5.2. There is a rational continuum S such that $S \in \Delta$.

Proof. Let S be the Sierpiński triangular curve, [4, p. 276]. It is a locally connected rational plane continuum containing a countable dense set of local cutpoints D such that no point in $X \setminus D$ locally separates X. (It is even rim-finite, i.e., it has a base for the open sets the boundary of each element of which is finite.) We claim that if U is open and connected in S and $K \in \mathcal{S}(U)$ then K is uncountable provided that $K \cap D = \emptyset$. Striving for a contradiction, assume that there exists such a set K that is countable. Write $U \setminus K$ as the union of two disjoint nonempty open subsets E and F. Pick arbitrary $x \in E$ and $y \in F$. Then K separates between $\{x\}$ and $\{y\}$ in the (connected and) locally connected space U. Since U is locally connected, we may assume that $K \cap U$ is irreducible, i.e., no proper closed subset $K' \subseteq K$ separates between $\{x\}$ and $\{y\}$ in U, [5, Exercise 3.10.2]. Since K is countable and locally compact, K has an isolated point, say p. If $p \notin \overline{E \cup F}$ then $V = U \setminus \overline{E \cup F}$ is a nonempty open subset of U which is contained in K and contains p. Since p is isolated in K, we then get that p is isolated in U which contradicts the connectivity of U. So we may assume without loss of generality that $p \in \overline{E}$. Suppose that $p \notin \overline{F}$. Let W be a neighborhood of p such that $W \subseteq U \setminus \overline{F}$ and $W \cap K = \{p\}$. Then $E \cup W$ is open, misses F and $U \setminus (E \cup W \cup F)$ is equal to $K \setminus \{p\}$. But this contradicts the fact that K is irreducible. So it follows that $p \in \overline{F}$. Let $X \subseteq U$ be a connected open neighborhood of p such that $X \cap K = \{p\}$. Since $X \cap E \neq \emptyset$ and $X \cap F \neq \emptyset$, p is a local cutpoint of X, which contradicts the fact that K misses D.

So we conclude from Theorem 3.1 that $S \in \Delta$. □

6. Remarks

(1) Let $p \in \mathbb{R} \times (0, \infty)$ be an arbitrary point in the upper half-plane. For $x \in \mathbb{R}$ we let $[x, p]$ denote the straight-line segment in $\mathbb{R}^2$ connecting $\langle x, 0\rangle$ and p. For $A \subseteq \mathbb{R}$ let

$$\triangle_A^p = \bigcup_{x \in A} [x, p].$$

A set $T \subseteq \triangle_A^p \setminus \{p\}$ is called a *transversal* for $\triangle_A^p$ if $|T \cap [x, p]| = 1$ for every $x \in A$. The following result, which motivated Theorem 4.2 and which can be proved easily along the same lines, is inspired by the classical Knaster–Kuratowski fan from [3].

Proposition 6.1. *If $p \in \mathbb{R} \times (0, \infty)$ and $A \subseteq \mathbb{R}$ is locally of cardinality $\mathfrak{c}$ then there is a transversal T for $\triangle_A^p$ such that $T \cup \{p\}$ is connected.*

If A is zero-dimensional then clearly any transversal T for $\triangle_A^p$ is totally disconnected, i.e., any pair of points of T can be separated by a clopen set. It is clear however that the transversal T given by proposition Proposition 6.1 cannot be zero-dimensional since $T \cup \{p\}$ is connected, hence T is totally disconnected but not zero-dimensional. That there exist totally disconnected spaces X that can be embedded in a connected space Y such that $|Y \setminus X| = 1$ is well known, see e.g., [1, Problem 6.3.24]. Our construction seems to be an efficient way of getting such an example.

(2) The examples in Section 5 demonstrate that the 1-dimensional continua that are in Γ form a rather peculiar class. Can they be characterized in a transparent way?

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