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On the cardinality of power homogeneous compacta

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Abstract

We prove that if X is a power homogeneous compact space then $|X| \leqslant 2^{c(X)\cdot\pi_\chi(X)}$. This generalizes similar results of Arhangel'skiĭ, van Douwen and Ismail. We apply this result to get new estimates for the cardinality of (power) homogeneous compacta satisfying some special conditions.

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1. Introduction

For all undefined notions, see Engelking [7], Kunen [13] and Juhász [12]. Recall that $\pi_\chi(X)$, $\pi(X)$, $\mathrm{w}(X)$ and $c(X)$ denote the *π-character*, *π-weight*, *weight* and *cellularity* of X, respectively. *All spaces under discussion are Tychonoff.*

A space X is *power homogeneous* if X^κ is homogeneous for some κ. In [5], van Douwen proved the surprising result that if X is power homogeneous then $|X| \leqslant 2^{\pi(X)}$. Kunen [14] proved that no infinite compact F-space is power homogeneous. Bell [4] proved that if X is a continuous image of a compact ordered space and X is power homogeneous, then X is first countable. Dow and Pearl [6] proved that every first countable zero-dimensional space is power homogeneous. Results in the same spirit were obtained recently by Arhangel'skiĭ [3]. He proved that if X is Corson compact and power homogeneous then X is first countable. And also that a compact scattered power homogeneous space is countable.

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It was shown by Ismail [11, 1.6] that $|X| \leqslant |R(X)|^{\pi\chi(X)}$ if X is homogeneous. Here $R(X)$ is the collection of all regular open subsets of X. Arhangel'skiĭ [2, 1.5] observed that from this inequality and one of Šapirovskiĭ [19] one easily gets that $|X| \leqslant 2^{c(X)\cdot\pi\chi(X)}$ for every homogeneous space X. In view of van Douwen's Theorem quoted above, this suggests the question whether the same result holds for power homogeneous spaces. We were only able to answer this question for compact spaces.

Theorem 1.1. *Let X be compact and power homogeneous. Then $|X| \leqslant 2^{c(X)\cdot\pi\chi(X)}$.*

We prove that if X is compact and power homogeneous then $|X| \leqslant \mathrm{w}(X)^{\pi\chi(X)}$. This result is close to Ismail's Theorem 1.6 from [11]. Then by applying Šapirovskiĭ's inequality from [19] in precisely the same way as was done in Arhangel'skiĭ [2, 1.5], we obtain Theorem 1.1. Applications are that the cardinality of a homogeneous compact space which has countable spread or is hereditarily normal and satisfies the countable chain condition does not exceed $\mathfrak{c}$.

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2. Proof of Theorem 1.1

In [5], van Douwen introduced a technique, inspired by ideas of Frolík [9,10], which is useful for showing that certain spaces are not homogeneous. This technique will be used in the proof of our main result.

Let X be a compact space and let $\mathcal{B}(X)$ denote the collection of open F_σ-subsets of X. Observe that since every $B \in \mathcal{B}(X)$ is σ-compact,

$$\left|\mathcal{B}(X)\right| \leqslant \mathrm{w}(X)^{\omega},$$

and that $\mathcal{B}(X)$ is invariant under homeomorphisms, i.e., if $B \in \mathcal{B}(X)$ and $f : X \to X$ is a homeomorphism then $f(B) \in \mathcal{B}(X)$.

Throughout, let $\kappa \geqslant \omega$ be a fixed cardinal. For $\phi \in \mathcal{B}(X)^\kappa$ and $x \in X$ we define "the way ϕ clusters at x", $w(\phi, x)$, as follows:

$$w(\phi, x) = \left\{A \subseteq \kappa\colon\ x \in \overline{\bigcup\left\{\phi(\alpha)\colon\ \alpha \in A\right\}}\right\}.$$

Then

$$W(x, \kappa) = \left\{w(\phi, x)\colon\ \phi \in \mathcal{B}(X)^\kappa\right\}$$

is the set of all possible ways κ-sequences in $\mathcal{B}$ cluster at x. Observe that

$$\left|W(x, \kappa)\right| \leqslant \left|\mathcal{B}(X)\right|^\kappa \leqslant \mathrm{w}(X)^{\omega\cdot\kappa} = \mathrm{w}(X)^\kappa. \tag{†}$$

It is clear that if $x, y \in X$ and $f : X \to X$ is a homeomorphism with $f(x) = y$ then $W(x, \kappa) = W(y, \kappa)$. This yields:

Proposition 2.1 (van Douwen [5, 2.2]). *Let X be a homogeneous space. Then for every $p \in X$ and $\phi \in \mathcal{B}(X)^\kappa$, we have*

$$\left|\left\{w(\phi, x)\colon\ x \in X\right\}\right| \leqslant \left|W(p, \kappa)\right|.$$

Proof. If not, then we can find $x \in X$ such that $w(\phi, x) \notin W(p, \kappa)$. But then $W(p, \kappa) \neq W(x, \kappa)$, and so X is not homogeneous. $\square$

The following lemma, which is similar to van Douwen [5, 3.1], is the key to our results.

Lemma 2.2. *Let X be a compact space and let $Y \in [X]^{\leqslant\kappa}$ be such that $\pi\chi(x, X) \leqslant \kappa$ for every $x \in Y$. Then there is an element $\phi \in \mathcal{B}(X)^\kappa$ such that the function*

$$x \mapsto w(\phi, x)$$

is one-to-one on $\overline{Y}$.

Proof. For every $x \in Y$ let $\mathcal{U}_x \subseteq \mathcal{B}(X)$ be a local π-basis at x of size at most κ. Put $\mathcal{U} = \bigcup_{x \in Y} \mathcal{U}_x$, and let $\phi : \kappa \to \mathcal{U}$ be an arbitrary surjection. Pick arbitrary distinct $y, z \in \overline{Y}$. There is an open neighborhood V of y such that $z \notin \overline{V}$. Put

$$A = \{\alpha < \kappa \colon \phi(\alpha) \subseteq V\}.$$

If W is an arbitrary open neighborhood of y then $V \cap W$ contains an element x of Y, and hence contains a member of the collection $\mathcal{U}_x$. From this we conclude that

$$y \in \overline{\bigcup\{\phi(\alpha) \colon \alpha \in A\}},$$

i.e., $A \in w(\phi, y)$. But $z \notin \overline{V}$, hence $A \notin w(\phi, z)$. We conclude that $w(\phi, y) \neq w(\phi, z)$. $\square$

Van Douwen was particularly interested in proving that there are spaces X no power of which is homogeneous. For handling big powers of compact spaces, he proved the following result:

Proposition 2.3 (van Douwen [5, 3.7]). *Let X be compact, and let $\lambda \geqslant \kappa \geqslant \omega$. Then for $x \in X^\lambda$ we have*

$$W(x, \kappa) = \bigcup\{W(x \restriction A, \kappa) \colon A \subseteq \lambda,\ |A| = \kappa\}.$$

By using the technique in the proof of van Douwen [5, 4.1], this yields:

Corollary 2.4. *Let X be a compact power homogeneous space. If $Y \in [X]^{\leqslant\kappa}$ and $\pi\chi(x, X) \leqslant \kappa$ for every $x \in Y$, then $|\overline{Y}| \leqslant \mathrm{w}(X)^\kappa$.*

Proof. Let us first observe that

$$\mathrm{w}(X^\kappa)^\kappa = (\kappa \cdot \mathrm{w}(X))^\kappa = 2^\kappa \cdot \mathrm{w}(X)^\kappa = \mathrm{w}(X)^\kappa. \tag{‡}$$

Let λ be such that X^λ is homogeneous. We may assume without loss of generality that $\lambda \geqslant \kappa$. This is clear since if $\lambda \leqslant \kappa$ and X^λ is homogeneous, then so is $(X^\lambda)^\kappa \approx X^\kappa$.

Consider any $p \in X^\lambda$ which is constant (as a function from λ into X). Then by Proposition 2.3,

$$W(p, \kappa) = \bigcup\{W(p \restriction A, \kappa) \colon A \subseteq \lambda,\ |A| = \kappa\}.$$

But, clearly, $W(p\restriction A, \kappa) = W(p\restriction B, \kappa)$ for all $A, B \in [\lambda]^\kappa$. Hence

$$W(p, \kappa) = W(p\restriction\kappa, \kappa).$$

Observe that by (†) and (‡),

$$\left|W(x, \kappa)\right| \leqslant \left(\mathrm{w}\left(X^\kappa\right)\right)^\kappa = \mathrm{w}(X)^\kappa.$$

for every $x \in X^\kappa$. This shows by homogeneity of X^λ that $|W(y, \kappa)| \leqslant \mathrm{w}(X)^\kappa$ for every $y \in X^\lambda$.

Now let $Y \in [X]^{\leqslant\kappa}$ be such that $\pi_\chi(x, X) \leqslant \kappa$ for every $x \in Y$. Let $\phi \in \mathcal{B}(X)^\kappa$ be as in Lemma 2.2 for Y. Let $\pi : X^\lambda \to X$ be an arbitrary projection, and let $S \subseteq X^\lambda$ be such that $\pi \restriction S : S \to \overline{Y}$ is a bijection. Define $\psi : \kappa \to \mathcal{B}(X^\lambda)$ by $\psi(\alpha) = \pi^{-1}(\phi(\alpha))$. Since π is open, it is not hard to show that for every $x \in S$,

$$w(\psi, x) = w\left(\phi, \pi(x)\right)$$

(van Douwen [5, 3.2(a)]). So by Proposition 2.1 we get $|S| \leqslant \mathrm{w}(X)^\kappa$, and so, $|\overline{Y}| \leqslant \mathrm{w}(X)^\kappa$, as required. □

We now formulate the main result in this note.

Theorem 2.5. *If X is compact and power homogeneous, then $|X| \leqslant \mathrm{w}(X)^{\pi_\chi(X)}$.*

Proof. We may assume without loss of generality that X is infinite. Let $\kappa = \pi_\chi(X)$ and observe that $\kappa \geqslant \omega$. By Corollary 2.4, we get:

Claim 1. If $Y \in [X]^{\leqslant\kappa}$ then $|\overline{Y}| \leqslant \mathrm{w}(X)^\kappa$.

Claim 2. $|X| \leqslant \mathrm{w}(X)^\kappa$.

There is a dense subset D of X with $|D| \leqslant \mathrm{w}(X)$. Fix an arbitrary $x \in X$, and let $\mathcal{B}_x$ be a local π-basis at x with $|\mathcal{B}_x| \leqslant \kappa$. For every $B \in \mathcal{B}_x$, pick an arbitrary element $d_B \in B \cap D$. Clearly,

$$x \in \overline{\{d_B\colon\ B \in \mathcal{B}_x\}}.$$

So for every $x \in X$ there is $A_x \in [D]^{\leqslant\kappa}$ such that $x \in \overline{A}_x$. From this we conclude by Claim 1 that $|X| \leqslant \mathrm{w}(X)^\kappa \cdot \mathrm{w}(X)^\kappa = \mathrm{w}(X)^\kappa$. □

We are now in a position to draw some interesting conclusions.

Corollary 2.6. *If X is compact and power homogeneous, then $|X| \leqslant 2^{c(X)\cdot\pi_\chi(X)}$.*

Proof. We may clearly assume that X is infinite. Let $\mathsf{RO}(X)$ denote the Boolean algebra of regular open subsets of X. It was proved by Šapirovskiĭ [19] (see also [12, 2.37]) that

$$\left|\mathsf{RO}(X)\right| \leqslant \pi_\chi(X)^{c(X)}.$$

So we conclude by Theorem 2.5 that

$$|X| \leqslant \mathsf{w}(X)^{\pi_\chi(X)} \leqslant \left|\mathsf{RO}(X)\right|^{\pi_\chi(X)} \leqslant \left(\pi_\chi(X)^{c(X)}\right)^{\pi_\chi(X)} = 2^{c(X)\cdot\pi_\chi(X)}$$

as required. □

Remark 2.7. We do not know whether there is a power homogeneous space X such that $|X| > 2^{\pi_\chi(X)\cdot c(X)}$.

Ismail [11, 1.13] proved that if X is a homogeneous compactum then $|X| = 2^{\chi(X)}$. This is a consequence of the classical Čech–Pospišil Theorem, see [12, 3.16], that if X is compact and if for some κ, $\chi(x, X) \geqslant \kappa$ for every $x \in X$, then $|X| \geqslant 2^\kappa$. So by Corollary 2.6 we get:

Corollary 2.8. *Let X be a homogeneous compact space. Then $2^{\chi(X)} \leqslant 2^{c(X)\cdot\pi_\chi(X)}$.*

This inequality is not very appealing. But, in the presence of the GCH, it looks much better.

Corollary 2.9 (GCH). *Let X be a compact power homogeneous space. Then $\chi(X) \leqslant c(X)\cdot\pi_\chi(X)$.*

Question 2.10. Let X be a compact homogeneous space. Is $\chi(X) \leqslant \pi_\chi(X)$ under GCH?

3. On the cardinality of homogeneous compacta

Ismail [11] observed that the cardinality of a hereditarily separable compact homogeneous space is at most $\mathfrak{c}$ (see also Arhangel'skiĭ [2], the discussion before Conjecture 1.18). This is relevant since Fedorčuk [8] constructed a hereditarily separable hereditarily normal compact space X with $|X| > \mathfrak{c}$ under $\diamondsuit$. By our results this can be generalized to power homogeneous compacta (with an identical proof).

Theorem 3.1. *Let X be a power homogeneous compactum. Then $|X| \leqslant 2^{s(X)}$.*

Proof. By results of Arhangel'skiĭ [1] and Šapirovskiĭ [20], it follows that $\pi_\chi(X) \leqslant t(X) \leqslant s(X)$ (see also [12, 3.12 and 3.14]). Clearly, $c(X) \leqslant s(X)$, hence

$$|X| \leqslant 2^{c(X)\cdot\pi_\chi(X)} \leqslant 2^{s(X)}$$

by Corollary 2.6. □

Theorem 3.2. *Let X be a hereditarily normal homogeneous compactum. Then $|X| \leqslant 2^{c(X)}$.*

Proof. Suppose first that there is a point $p \in X$ such that $\pi_\chi(p, X) \geqslant \omega_1$. By homogeneity it then follows that $\pi_\chi(x, X) \geqslant \omega_1$ for every $x \in X$. By deep results of Šapirovskiĭ [19] (see also [12, 3.18]), there is a continuous surjection $f : X \to \mathbb{I}^{\omega_1}$. But $\mathbb{I}^{\omega_1}$ is not hereditarily normal, [12, 3.21]. So we conclude that $\pi_\chi(X) \leqslant \omega$. Hence

$$|X| \leqslant 2^{c(X)\cdot\pi_\chi(X)} = 2^{c(X)}$$

by Corollary 2.6. □

We do not know whether Theorem 3.2 also holds for power homogeneous hereditarily normal compacta.

By arguing as in Corollaries 2.8 and 2.9, we get:

Corollary 3.3. *Let X be a homogeneous compactum. Then $2^{\chi(X)} \leqslant 2^{s(X)}$.*

Corollary 3.4 (GCH). *Let X be a homogeneous compactum. Then $\chi(X) \leqslant s(X)$.*

The question naturally arises whether spread in this corollary can be replaced by cellularity. Since 2^κ is homogeneous for every κ, and has countable cellularity, this cannot be done. It turns out that the presence of copies of certain Čech–Stone compactifications of discrete spaces is the determining factor.

Corollary 3.5 (GCH). *Let X be a homogeneous compactum, and let $\kappa = c(X)$. If X does not contain a copy of $\beta\kappa$, then $\chi(X) \leqslant c(X)$.*

Proof. Observe that $\beta\kappa$ is by GCH a subspace of $\mathbb{I}^{\kappa^+}$. Since $\beta\kappa$ is extremally disconnected, this means that X cannot be mapped onto $\mathbb{I}^{\kappa^+}$, cf., the proof of 3.22 in [12]. So there is a point $x \in X$ such that $\pi_\chi(x, X) < \kappa^+$ by [12, 3.20]. By homogeneity of X, this means that $\pi_\chi(X) \leqslant \kappa$. But then

$$|X| \leqslant 2^{c(X)\cdot\pi_\chi(X)} = 2^{c(X)}$$

by Corollary 2.6. So we can again apply the reasoning in the proof of Corollary 2.8 to conclude that under GCH we have $\chi(X) \leqslant c(X)$. □

4. Examples

In this section we will present some known examples that illustrate the sharpness of the obtained results and raise some questions.

(1) Let $\alpha\kappa$ denote the one-point compactification of κ (with the discrete topology). Then $\chi(\alpha\kappa) = \kappa$ and $\pi_\chi(\alpha\kappa) = \omega$. Hence the gap between character and π-character can be arbitrarily large.

(2) Jensen pointed out that it is easy to construct homogeneous compact Souslin lines from $\diamondsuit$. A related result is Kunen's construction under CH of a compact L-space which is even a right topological group, [15]. These spaces are first countable, have countable cellularity but have uncountable π-weight. So there are homogeneous spaces X for

which $c(X) \cdot \pi_\chi(X) = \omega$ but $\pi(X) = \omega_1$, hence Theorem 1.1 improves van Douwen's Theorem.

(3) There is an example under $\mathsf{MA} + \neg\mathsf{CH}$ of a compact homogeneous space X with $\pi(X) = \omega$ and $\chi(X) = \omega_1$ (van Mill [17]). So it is not true that a compact homogeneous space with countable π-weight is first countable. But X has uncountable spread since it contains a copy of 2^{ω_1}. This suggests the question whether Corollary 3.4 is true in ZFC, in particular, whether a compact homogeneous space with countable spread is first countable. This question is related to Conjecture 1.17 in Arhangel'skiĭ [2]: every homogeneous compactum of countable tightness is first countable.

(4) Theorem 3.1 is false if we drop compactness. There are (consistent) hereditarily separable topological groups of cardinality $2^{\mathfrak{c}}$. See Roitman [18] for details. De la Vega and Kunen [21] recently proved that there is a compact homogeneous S-space under CH.

(5) Theorem 3.2 implies that a hereditarily normal homogeneous compactum which satisfies the countable chain condition has cardinality at most $\mathfrak{c}$. There are hereditarily normal homogeneous compacta with cellularity and cardinality $\mathfrak{c}$ (Maurice [16]). These spaces are ordered, hence even monotonically normal, and first countable. The results in this note do not show 'why' these examples have cardinality at most $\mathfrak{c}$. So we ask whether *every* hereditarily normal homogeneous compactum has cardinality at most $\mathfrak{c}$.

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