Intermediate dimensions of products

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Abstract

We construct under the Continuum Hypothesis an example of a compact space no finite power of which contains an infinite closed subset that is of finite dimension greater than 0. This is a partial answer to a question of the third author. © 2006 Elsevier B.V. All rights reserved.

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1. Introduction

In 1965, D.W. Henderson [12] constructed the first infinite-dimensional metrizable compactum with no closed subspaces of finite positive dimension. Simpler examples were presented subsequently by R.H. Bing [3], D.W. Henderson [13], A.V. Zarelua [25], and others. In 1979, J.J. Walsh [23] improved Henderson’s result by the construction of a metrizable compactum all of whose subspaces are either zero- or infinite-dimensional (see also Rubin [18]).

For non-metrizable spaces, there are similar results. In 1973, V.V. Fedorchuk [9] constructed for any integer \( n \geq 2 \) a first countable separable \( n \)-dimensional compactum all of whose closed subsets of positive dimension are \( n \)-dimensional. Then in 1975, Fedorchuk [10], assuming the Continuum Hypothesis (abbreviated CH), constructed for any positive integer \( n \) an infinite separable compactum all of whose infinite closed subsets are \( n \)-dimensional and have cardinality \( 2^c \). In 1978, A.V. Ivanov [14], under the assumption of Jensen’s principle \( \Diamond \), constructed for any positive integer \( n \) a hereditarily separable \( n \)-dimensional compactum \( Y_n \) with the following property: for every \( m \in \mathbb{N} \) and infinite closed subspace \( F \) of \( Y_n^m \), the dimension of \( F \) is one of \( n, 2n, \ldots, mn \).

The following question was asked by J. van Mill [15, Remark 5.7] (see also Pol [17, Question 8.1]):

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Question 1.1. Does there exist an infinite-dimensional metrizable compactum $X$ none of whose finite powers contains a one-dimensional subset?

Its solution is currently beyond (our) reach. We consequently concentrate on the following problem, which is a weakening of the original question in two ways: the example is not required to be metrizable, and the subsets that we are interested in are closed.

Question 1.2. Does there exist an infinite-dimensional compactum $X$ none of whose finite powers contains a one-dimensional closed subset?

Theorem 7.1 of our paper gives a positive answer (under CH) to Question 1.2. Theorem 6.1 is related to Fedorchuk’s results on compacta without intermediate dimensions. The example constructed there is similar to Ivanov’s Example that was mentioned above. Its properties are weaker, but its construction requires CH instead of the more powerful axiom $\Diamond$. Question 1.1 remains unsolved. As for non-metrizable spaces, Question 1.2 has several related versions. One of them is:

Question 1.3. Does there exist in ZFC an $n$-dimensional compactum $Z_n$, $n \geq 1$, such that for every $m \geq 2$, every non-empty closed subset $F$ of $Z_n^m$ has dimension $kn$, where $k$ is some integer between 0 and $m$?

2. Preliminaries

Unless otherwise stated, all spaces are assumed to be Tychonov, and all mappings are continuous.

If $f : X \to Y$ is a function, and $A \subseteq Y$, then we say that $f$ is one-to-one over $A$ provided that $|f^{-1}(y)| = 1$ for every $y \in A$. If $x \in X$ then $f$ is said to be one-to-one at $x$ provided that $f$ is one-to-one over $\{f(x)\}$. For a function $f : X \to Y$ and a set $A \subseteq X$ we let $f^\#(A)$ denote the small image of $A$, i.e.,

$$f^\#(A) = Y \setminus f(X \setminus A).$$

A $k$-face of a finite product $\prod_{i=1}^m X_i$ is a subset of the form $\prod_{i=1}^m A_i$, for which there exists a set $B \subseteq \{1, 2, \ldots, m\}$ of size $m-k$ such that $A_i$ is a singleton if $i \in B$ and $A_i = X_i$ if $i \notin B$. So a 0-face is a singleton, and an $m$-face is the whole product.

If $A \subseteq \prod_{i=1}^m X_i$ then the least integer $k$ such that $A$ is contained in the union of a finite family of $k$-faces, will be denoted by $f(A)$. Clearly, $0 \leq f(A) \leq m$. Observe that $f(A) = 0$ if and only if $A$ is finite.

Let $C$ be a subset of a finite product $\prod_{k=1}^m Y_k$ of sets. We say that $C$ is in general position in $\prod_{k=1}^m Y_k$ if each projection $\pi_i : \prod_{k=1}^m Y_k \to Y_i$, $i \leq m$, is one-to-one on $C$.

If $X$ is a set, and $\kappa$ is a cardinal number, then $[X]^{<\kappa}$ denotes the collection of all subsets of $X$ of size smaller than $\kappa$.

For a space $X$, we let $\tau X$ denote its topology. If $A$ is a subset of a space $X$, then $\text{Fr}A$ denotes its boundary. A compactum is a compact Hausdorff space. A continuum is a connected compactum.

We say that $f$ is fully closed at the point $y \in Y$, Fedorchuk [8], if $f^{-1}(y) \neq \emptyset$ and for any finite cover $\mathcal{U}$ of $f^{-1}(y)$ by open sets in $X$ the set

$$\{y\} \cup \bigcup_{U \in \mathcal{U}} f^\#(U)$$

is a neighborhood of $y$. We say that $f$ is fully closed if $f$ is fully closed at every $y \in Y$. It is easy to see that a fully closed map is a closed surjection. The converse need not be true.

By the symbol “$X \approx Y$” we mean that $X$ and $Y$ are homeomorphic topological spaces.

Let $f : X \to Y$ be continuous. We say that $f$ is atomic provided that for every closed set $A$ in $X$ such that $f(A)$ is a non-degenerate continuum, we have that $A = f^{-1}(f(A))$. Also, $f$ is irreducible if $f(A) \neq Y$ for every proper closed subset $A \subseteq X$. It is clear that if $f$ is atomic and $Y$ is a continuum, then $f$ is irreducible. Indeed, if $A$ is a closed subset of $X$ such that $f(A) = Y$, then since $f$ is atomic, $A = f^{-1}(f(A)) = f^{-1}(Y) = X$.

We let $[0, 1]$ denote the interval $[0, 1]$. 

Suppose that $X$ is a topological space and that $\{Y_x: x \in X\}$ are topological spaces and, for each $x \in X$, $f_x: X \setminus \{x\} \to Y_x$ is a continuous function. We topologize

$$Z = \bigcup \{(x) \times Y_x: x \in X\}$$

as follows. If $x \in X$, $U$ is an open neighborhood of $x$ in $X$, and $W \subseteq Y_x$ is open, then

$$U \otimes_x W = (\{(x) \times W\} \cup \bigcup \{(x') \times Y_{x'}: x' \in U \cap f_x^{-1}(W)\}).$$

We usually omit the index $x$ in the expression $U \otimes_x W$. The collection

$$\{U \otimes W: (x \in U \in \tau X) \& (W \in \tau Y_x)\}$$

is an open basis for $Z$. Topologized in this way, $Z$ is called the resolution of $X$ at each point $x \in X$ into $Y_x$ by the mapping $f_x$.

Let $\pi_0: Z \to X$ be the ‘projection’. Then $\pi_0$ is continuous (this is clear), and fully closed if every $Y_x$ is compact. Since the latter fact will be important in our construction, we will sketch the proof. Assume that $\mathcal{U}$ is a finite open cover of the fiber $\pi_0^{-1}(x)$, for certain $x \in X$. By compactness of $Y_x$, we may assume without loss of generality that $\mathcal{U}$ consists of basic open sets. So there are a finite open cover $\mathcal{W}$ of $Y_x$, and for every $W \in \mathcal{W}$ an open neighborhood $U(W)$ of $x$ in $X$ such that

$$\mathcal{U} = \{W \otimes U(W): W \in \mathcal{W}\}.$$ 

Put $U = \bigcap_{W \in \mathcal{W}} U(W)$. An easy verification shows that

$$U \setminus \{x\} \subseteq \bigcup_{W \in \mathcal{W}} \pi_0^\#(W \otimes U(W)),$$

as required.

Observe that for every $x \in X$ the set $\{(x) \times Y_x\}$ is a closed topological copy of $Y_x$ in $Z$. If $X$ is compact, and all $Y_x$ are compact, then so is the resolution, being a perfect pre-image of a compact space. It is also clear that if $X$ and all the $Y_x$ are first countable, then so is the resolution.

Resolutions were introduced in Fedorchuk [7] (for details, see also Fedorchuk and Hart [11], and Watson [24]). They allow the replacement of each point in a space by a copy of some other space, possibly a different one for each point.

3. Inverse systems

Throughout, for some ordinal number $\gamma$, let $T = \{X_\alpha, p_\beta^\alpha, \gamma\}$ be an inverse system of sets. If $X_0$ is not a singleton then for technical reasons it will be convenient to let $X_{-1}$ denote $\{0\}$ and $p_{-1}^0: X_0 \to X_{-1}$ the constant function with value 0. This will not have any effect on the limit of the system under consideration. We let $X_{\gamma}$ denote the inverse limit of $T$, and $p_{\delta}^\alpha: X_\gamma \to X_\alpha$ for every $\alpha \leq \gamma$ the natural projection. If $x \in X_\alpha$ and $\beta \leq \alpha \leq \gamma$, then $x_\beta$ denotes $p_\beta^\alpha(x)$. So if $\gamma \leq \delta \leq \alpha \leq \gamma$, then

$$x_\beta, \quad p_\delta^\alpha(x), \quad p_\delta^\alpha(p_\beta^\alpha(x)), \quad p_\delta^\beta(x_\delta)$$

all represent the same point.

If $\delta \leq \gamma$, then $T|\delta$ denotes the restriction of $T$ to $\delta$, i.e., $T|\delta = \{X_\alpha, p_\beta^\alpha, \delta\}$. We say that $T$ is continuous if for every limit ordinal $\delta < \gamma$ we have that $X_\delta$ is the inverse limit of the system $\{X_\alpha, p_\beta^\alpha, \delta\}$, and, for every $\beta \leq \delta$, $p_\delta^\beta: X_\delta \to X_\beta$ is the natural projection

$$\lim_{\alpha \to \delta} T|\delta \to X_\beta.$$

We recall some details from Fedorchuk [10]. Assume that $T$ is continuous, and let $\alpha \leq \gamma$ and $x \in X_\alpha$ be arbitrary. For $-1 \leq \beta \leq \alpha$, let

$$A_{(\beta, \alpha)} = \begin{cases} \emptyset & (\beta \text{ is a limit ordinal, or } \beta = -1), \\ (p_0^\beta)^{-1}(\{x_\delta\}) \setminus \{x_\beta\} & (\beta = \delta + 1). \end{cases}$$
Define the collection of subsets \( \mathcal{M}(\alpha,x) \) of \( X_\alpha \) by
\[
\mathcal{M}(\alpha,x) = \{x\} \cup \bigcup_{\beta \leq \alpha} \{(p_\beta^\alpha)^{-1}(z) : z \in A_\beta\}
\]
It is not difficult to see that \( \mathcal{M}(\alpha,x) \) is a partition of \( X_\alpha \). Denote \( X_\alpha/\mathcal{M}(\alpha,x) \) by \( X(\alpha,x) \), and let \( p(\alpha,x) : X_\alpha \to X(\alpha,x) \) be the natural decomposition function. Observe that \( p(\alpha,x) \) is one-to-one at \( x \).

In the forthcoming applications, the system \( \mathcal{T} = \{X_\alpha, p_\beta^\alpha, \gamma\} \) will consist of first countable compacta with continuous bonding maps \( p_\beta^\alpha \). The elements of the partition \( \mathcal{M}(\alpha,x) \) of \( X_\alpha \) are then clearly closed \( G_\delta \)-sets, hence \( X(\alpha,x) \) endowed with the quotient topology is \( T_1 \) and compact. Moreover, \( X(\alpha,x) \) is first countable if it is Hausdorff. In general, \( X(\alpha,x) \) need not be Hausdorff, so we need to be careful with this construction. For a simple example, consider the system of spaces \( X_{-1} = \{0\} \), \( X_0 = I \), \( X_1 = I^2 \), where \( I \to \{0\} \) is the constant function with value 0, and \( I^2 \to I \) is the projection. If \( x = (0,0) \), then
\[
\mathcal{M}(1,x) = \{(0, y) : y \in I\} \cup \{(p) \times I : 0 < p \leq 1\},
\]
hence \( X(1,x) \) is not Hausdorff. In the framework of resolutions there are fortunately no problems with this construction.

Observe that for every \( \beta \leq \alpha \), there is a natural function \( p(\alpha,x)(\beta,x) : X(\alpha,x) \to X(\beta,x) \) such that the diagram
\[
\begin{array}{ccc}
X_\beta & \xleftarrow{p_\beta^\alpha} & X_\alpha \\
\downarrow{p(\beta,x)} & & \downarrow{p(\alpha,x)} \\
X(\beta,x) & \xleftarrow{p(\alpha,x)(\beta,x)} & X(\alpha,x)
\end{array}
\]
commutes. Also observe that \( x_\beta \) is the only element of the set \( X(\beta,x) \) that possibly has a non-degenerate fiber under the function \( p(\alpha,x)(\beta,x) \). Hence \( p(\alpha,x)(\beta,x) \) is one-to-one over \( X(\beta,x) \setminus \{x_\beta\} \). Also observe that if \( \beta \leq \delta \leq \alpha \leq \gamma \) and \( x \in X_\alpha \), then
\[
p(\alpha,x)(\beta,x) = p(\delta,x) \circ p(\alpha,x)(\delta,x),
\]
from which it easily follows that
\[
\{X(\alpha,x), p(\alpha,x)(\beta,x), \gamma\}
\]
is a continuous inverse system.

4. Dimension theory

If \( f : X \to Y \) is continuous, then, as usual,
\[
\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}
\]
denotes the dimension of \( f \).

**Theorem 4.1.** (Fedorchuk [8]) If \( f : X \to Y \) is fully closed, where \( X \) is normal and \( Y \) is paracompact, then \( \dim X \leq \max\{\dim Y, \dim f\} \).

The following corollary is implicitly contained in Fedorchuk [10].

**Corollary 4.2.** Let \( S = \{X_\alpha, p_\beta^\alpha, \gamma\} \) be a continuous inverse system of compacta such that

1. \( \dim X_0 \leq n \),
and, for all \( \alpha < \gamma \),
(2) \( p_{\alpha}^{\alpha+1} : X_{\alpha+1} \to X_{\alpha} \) is fully closed,
(3) \( \dim p_{\alpha}^{\alpha+1} \leq n \).

Then \( \dim \lim S \leq n \).

**Proof.** It suffices to prove that \( \dim X_{\alpha} \leq n \) for every \( \alpha < \gamma \). This can be proved by transfinite induction, as follows. If \( \gamma \) is a limit, then there is nothing to prove since \( S \) is continuous. If \( \gamma \) is a successor, then there is nothing to prove by Theorem 4.1. \( \square \)

A locally compact Hausdorff space \( X \) is *acyclic* if \( H^i(X) = 0 \) for each integer \( i \geq 0 \); here \( H^i(Z) \) denotes the \( i \)-th cohomology group with compact supports of the locally compact space \( Z \). See e.g., [4] for details. A surjective mapping between compacta is *acyclic* if all of its fibers are acyclic.

The following result is a consequence of the famous Vietoris–Begle–Skljarenko Theorem (see [1,2,20–22]).

**Theorem 4.3.** If \( f : X \to Y \) is a continuous surjection between compacta, and \( Y \) and \( f \) are acyclic, then so is \( X \).

Let \( S = \{ X_{\alpha}, p_{\alpha}^{\alpha}, \gamma \} \) be a continuous inverse system of compacta. Then \( H^i(\lim S) \) is isomorphic to the direct limit of the system \( \{ H^i(X_{\alpha}), (p_{\alpha}^{\alpha})^*, \gamma \} \) (see [4]). Hence, by Theorem 4.3, we obtain the following result.

**Theorem 4.4.** Let \( S = \{ X_{\alpha}, p_{\alpha}^{\alpha}, \gamma \} \) be a continuous inverse system of compacta. Assume that \( p_{\alpha}^{\alpha+1} \) is acyclic for every \( \alpha < \gamma \). Then the projections \( p_{\alpha}^{\gamma} : \lim S \to X_{\alpha}, \alpha < \gamma \), are acyclic as well.

**Theorem 4.5.** (Dyer [5], Skljarenko [21]) If \( X \) and \( Y \) are compacta with \( Y \) finite-dimensional, and \( f : X \to Y \) is acyclic, then \( \dim X \geq \dim Y \).

Dyer [5] proved this for metrizable compacta, and Skljarenko [21] generalized it for the case where \( X \) is paracompact and \( f \) is closed.

Let \( f : X \to Y \) be continuous. As usual, \( f \) is said to be *monotone* provided that every fiber \( f^{-1}(y) \) of \( f \) is connected.

The following lemma is well-known.

**Lemma 4.6.** Let \( f : X \to Y \) be a monotone, continuous surjection between the compacta \( X \) and \( Y \). If \( A \) and \( B \) are disjoint closed sets in \( Y \), and \( S \) is a partition in \( X \) between \( f^{-1}(A) \) and \( f^{-1}(B) \), then \( f(S) \) is a partition in \( Y \) between \( A \) and \( B \).

A continuum \( X \) is *unicoherent* if for all subcontinua \( A \) and \( B \) with \( A \cup B = X \) we have that \( A \cap B \) is connected. The Hilbert cube \( I^\infty \) is unicoherent, as is any contractible metrizable continuum [16, A.12.10]. This implies by [16, Exercises 3.10.2 and 3.10.3], that a partition \( S \) between a pair of opposite faces \( A \) and \( B \) of \( I^\infty \) contains a connected partition between \( A \) and \( B \). This will be used in the proof of Theorem 7.1.

Let \( X \) be a space. We say that \( X \) is *strongly infinite-dimensional* if it has an infinite essential family of pairs of disjoint closed subsets. Also, \( X \) is *weakly infinite-dimensional* if it is not strongly infinite-dimensional.

**Theorem 4.7.** (Skljarenko [19]) Let \( f : X \to Y \) be a continuous surjection between compacta, where \( X \) is weakly infinite-dimensional, and \( |f^{-1}(y)| < c \) for every \( y \in Y \). Then \( Y \) is weakly infinite-dimensional.

For more information on dimension theory, see [6] and [16].

5. The inverse system \( S \)

In this section we put topologies on the elements of certain inverse systems. The inverse limits of these systems will be used in the next section for the construction of our examples. We assume CH throughout.
Throughout, we fix a compact metrizable Absolute Retract $Z$ containing more than one point. In the applications in the forthcoming sections, $Z$ will be $\mathbb{I}^n$ for some $n$, or the Hilbert cube $\mathbb{I}^\infty$.

Let $n \in \mathbb{N}$ be fixed. For an ordinal number $\alpha \geq 1$ we let $Z^\alpha$ denote the set of all functions $\alpha \to Z$. If $\beta \leq \alpha$ then $p_\beta^\alpha$ denotes the natural projection $Z^\alpha \to Z^\beta$ defined by $p_\beta^\alpha(f) = f|\beta$. We let $Z^0 = \{0\}$, and $p_0^1: Z^1 \to Z^0$ the constant function with value 0.

The family

$$S = \{ Z^\alpha, p_\beta^\alpha, \omega_1 \} \quad (\dagger)$$

is a continuous inverse system of sets and will play a crucial role in the remaining part of this paper. Observe that the functions $p_\beta^\alpha$ are all surjective.

In our construction we need countably many copies of $S$. So for $i \in \mathbb{N}$, $\beta \leq \alpha \leq \omega_1$ and $x \in X_\alpha$, we put

$$X_\alpha^i = Z^\alpha, \quad p_\beta^\alpha = p_\beta^i, \quad X_\alpha^{(i,x)} = X_{(\alpha,x)}, \quad p_i(\alpha,x) = p(\alpha,x), \quad p_i(\beta,x) = p(\beta,x),$$

and define

$$S_i = \{ X_i^\alpha, p_i^\alpha, \omega_1 \}.$$}

For $\alpha < \omega_1$, we let $\mathcal{E}_\alpha$ denote the collection of all countably infinite sets $C$ for which there exists an $F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$ such that $C$ is a subset of $\prod_{i \in F} X_i^\alpha$ in general position. It is clear that $|\mathcal{E}_\alpha| = c$, hence, by CH, we may fix a bijection

$$f : \omega_1 \to \bigcup_{\alpha < \omega_1} \mathcal{E}_\alpha,$$

such that for every $\alpha < \omega_1$ there exists $\beta \leq \alpha$ such that $f(\alpha) \in \mathcal{E}_\beta$. For every $\alpha < \omega_1$, put $C_\alpha = f(\alpha)$.

Let $Q$ denote a fixed countable dense set in $Z$, and enumerate it by $\{q(j) : j < \omega\}$. This enumeration is fixed throughout the forthcoming constructions.

Our aim is to topologize all elements of the systems $S_i$, $i \in \mathbb{N}$. We use the technique of resolvable spectra invented by Fedorchuk [10] and further developed thereafter, and ideas and methods of Ivanov’s paper [14].

By transfinite induction on $\alpha < \omega_1$, we will topologize the elements of the inverse systems $S_i$, $i \in \mathbb{N}$, such that the following conditions are satisfied:

1. $X_1^i$ is homeomorphic to $Z$ for every $i \in \mathbb{N}$;
2. the inverse systems

$$S_i = \{ X_i^\alpha, p_i^\alpha, \omega_1 \}, \quad i \in \mathbb{N},$$

are continuous and consist of first-countable compacta;
3. if $\alpha \leq \omega_1$, $i \in \mathbb{N}$, and $x \in X_i^\alpha$, then the quotient space $X_i^{(\alpha,x)}$ is Hausdorff.

Moreover, if $\alpha < \omega_1$, and $i \in \mathbb{N}$, then

4. $(p_i^{1,\alpha+1})^{-1}(x)$ is, for every $x \in X_i^\alpha$, homeomorphic to $Z$;
5. the projection $p_i^{1,\alpha+1}$ is fully closed;
6. the projection $p_i^{1,\alpha+1}$ is atomic;
7. if $C_\alpha \in \mathcal{E}_\beta$, where $\beta \leq \alpha$, say $C_\alpha$ is in general position in $\prod_{i \in F} X_i^\beta$ for certain $F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$, and $A \subseteq \prod_{i \in F} X_i^\beta$ is such that

$$C_\alpha \subseteq \left( \prod_{i \in F} p_i^{1,\alpha+1} \right)(A),$$

then there is a point $z \in \prod_{i \in F} X_i^\alpha$ with

$$\left( \prod_{i \in F} p_i^{1,\alpha+1} \right)^{-1}(z) \subseteq \overline{A}.$$
All finite products of the spaces $X_\alpha$ are endowed with the Tychonov product topology. In addition, if $\alpha \leq \omega_1$ and $F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$, then for every $i \in F$,

$$\pi^i_\alpha : \prod_{i \in F} X^i_\alpha \to X^i_\alpha$$

denotes the projection.

For all $i \in \mathbb{N}$, $X^i_1$ can be identified with $Z$ in a natural way; we topologize $X^i_1$ in such a way that it is homeomorphic to $Z$. Suppose for some ordinal $1 \leq \gamma < \omega_1$, we defined the desired topologies on all the sets $X^i_\alpha$ for $i \in \mathbb{N}$ and $\alpha < \gamma$.

Assume first that $\gamma$ is a limit ordinal. As to be expected, for every $i \in \mathbb{N}$ we put $X^i_\gamma = \lim_{\leftarrow} \{ X^i_\alpha, p^{i,\alpha}_{\beta, \gamma} \}$, and $p^{i,\gamma}_{\beta} : X_\gamma \to X_\beta$ for $\beta < \gamma$ the natural projection. It is easy to see that this is as required (see Section 3).

Now assume that $\gamma$ is a successor ordinal, say $\gamma = \alpha + 1$. Consider the set $C_\alpha$, and assume it is in general position in $\prod_{i \in F} X^i_\beta$, where $F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$ and $\beta \leq \alpha$. Since $X^i_\beta$ is first countable for every $i \in F$, there is a sequence $C^\prime_\alpha \subseteq C_\alpha$ such that the sequence $(x(c))_{c \in C^\prime_\alpha}$ converges to, say $z(\alpha) = (z^i_\alpha)_{i \in F} \in \prod_{i \in F} X^i_\alpha$. (iv)

Observe that $z(\alpha) \in f^{-1}(z(\beta))$. 

Consider for $i \in F$ the commutative diagram

and put

$$\Theta_i = (p^{-1}_{(\beta, z^i_\beta)}(\pi^i_\gamma(C^\prime_\alpha)))$$.

(vii)

First observe that since $z^i_\beta \notin \pi^i_\beta(C^\prime_\alpha)$ and $p^i_{(\beta, z^i_\beta)}$ is one-to-one at $z^i_\beta$, we have that

$$p^i_{(\beta, z^i_\beta)}(z^i_\beta) \neq p^i_{(\beta, z^i_\beta)}(\pi^i_\beta(C^\prime_\alpha))$$. 

(vii)
Next observe that \( p_{i,(\alpha, z_i^\prime)}^j \) is one-to-one over \( X_{(\beta, z_i^\prime)} \backslash \{ \pi_{i,(\beta, z_i^\prime)}^j(z_j^\prime) \} \). Hence \( \Theta_i \) is a sequence in \( X_{(\alpha, z_i^\prime)} \) having all of its limit points in the set
\[
( p_{(\beta, z_i^\prime)}^j )^{-1} ( p_{(\beta, z_i^\prime)}^j(z_j^\prime) )).
\]
Since the sequence \( \{ x(c) : c \in C_\alpha' \} \) converges to \( z(\alpha) = (z_i^\prime)_{i \in F} \) and (vi) commutes, this implies that
\[
\Theta_i \text{ is a sequence converging to the point } p_{(\alpha, z_i^\prime)}^j(z_j^\prime).
\]
Let \( m = |F| \).

We will now construct for every \( i \in \mathbb{N} \) the space \( X_{\alpha + 1}^i \). It will be a resolution of \( X_{\alpha}^i \) at each point by a copy of \( Z \).

So our task is to specify for all \( y \in X_{\alpha}^i \) and \( i \in \mathbb{N} \) a certain continuous function \( h_y : X_{\alpha}^i \setminus \{ y \} \to Z \). This function will have the form
\[
h_y = h_y^i \circ (p_{(\alpha, y)}^i | X_{\alpha}^i \setminus \{ y \}),
\]
where \( h_y^i : X_{(\alpha,y)}^i \setminus \{ p_{(\alpha,y)}^i(y) \} \to Z \), is constructed in two steps below.

**Step 1.** Consider first a fixed \( i \in \mathbb{N} \) such that either \( i \notin F \), or \( i \in F \) but \( y \neq z_i^\prime \).

Since \( X_{\alpha}^i \) is first countable, we can fix a neighborhood base \( \{ O_j : j \in \mathbb{N} \} \) at \( p_{i,(\alpha, y)}^i(y) \) such that \( \overline{O}_j+1 \subseteq O_j \) for every \( j \). Let, for every \( j \), \( \psi_j : Fr O_j \to Z \) be the constant function with value \( q(j) \). Evidently, \( \bigcup_j Fr O_j \) is closed in \( X_{(\alpha,y)}^i \setminus \{ p_{(\alpha, y)}^i(y) \} \), and \( X_{(\alpha,y)}^i \setminus \{ p_{(\alpha, y)}^i(y) \} \) is normal (being \( \sigma \)-compact). We can therefore extend \( \bigcup_j \psi_j \) to a continuous function \( h_y^i : X_{(\alpha,y)}^i \setminus \{ p_{(\alpha, y)}^i(y) \} \to Z \). It is clear that we can arrange \( h_y^i \) to be surjective.

**Step 2.** We now consider all remaining cases simultaneously. That is, we consider the cases for which \( i \in F \), and \( y = z_i^\prime \). As above, let \( \{ O_j^i : j \in \mathbb{N} \} \) be a neighborhood base at \( p_{i,(\alpha, z_i^\prime)}^i(c) \) such that \( \overline{O}_j+1 \subseteq O_j^i \) for every \( j \). We may assume without loss of generality that \( Fr O_j^i \cap \Theta_i = \emptyset \) for every \( j \). Let \( \psi_j^i : Fr O_j^i \to Z \) for every \( j \) be the constant function with value \( q(j) \). Now observe that formula (vii) defines for every \( i \in F \) a bijection from \( C_\alpha' \) onto \( \Theta_i \). So we may pick for every \( i \in F \) a function \( \psi_i : \Theta_i \to Q \) such that the following holds: for every \( f \in Q^F \) there is an element \( c \in C_\alpha' \) such that for every \( i \in F \),
\[
( p_{(\beta, z_i^\prime)}^j )^{-1} ( p_{(\alpha, z_i^\prime)}^j(\pi_{i,(\beta, z_i^\prime)}^j(c)) ) = f(i).
\]
As in case 1, we can extend, for every \( i \in F \), the function \( \{ \psi_i \} \cup \bigcup_j \psi_j^i \) to a continuous surjection \( h_y^i : X_{(\alpha, z_i^\prime)}^i \setminus \{ p_{(\alpha, z_i^\prime)}^i(z_j^\prime) \} \to Z \).

So having all the mappings \( h_y \) constructed, we determined the spaces \( X_{\alpha + 1}^i \) for \( i \in \mathbb{N} \), and all there remains to verify is (2)_{\alpha+1} through (7)_{\alpha+1}.

It is clear that (2)_{\alpha+1}, (4)_{\alpha+1} and (5)_{\alpha+1} hold.

For (3)_{\alpha+1}, let \( i \in \mathbb{N} \), and \( x \in X_{\alpha + 1}^i \) be arbitrary. If \( y = p_{i,(\alpha, x)}^i(\alpha, x) \), then \( X_{(\alpha+1,x)}^i \) is (homeomorphic to) the resolution of \( X_{(\alpha, y)}^i \) at each point \( z \in X_{(\alpha, y)}^i \) into \( Y_z \) by the mapping \( f_z \), where
\[
\begin{cases}
Y_z = \{ z \}, & f_z(z) = z \quad (z \neq p_{i,(\alpha, y)}^i(y)), \\
Y_z = Z, & f_z = h_y^i \quad (z = p_{i,(\alpha, y)}^i(y)).
\end{cases}
\]
Hence \( X_{i,(\alpha+1,x)}^i \) is a compactum.

For (6)_{\alpha+1}, let \( F \) be closed in \( X_{\alpha + 1}^i \) such that \( G = p_{i,(\alpha+1)}^i(F) \) is a non-degenerate continuum. Take an arbitrary \( z \in G \), and put
\[
G_z = p_{i,(\alpha, z)}^i(G).
\]
The set \( G_z \) is clearly a continuum, and non-degenerate because \( p_{i,(\alpha, z)}^i \) is one-to-one at \( z \). As a consequence, all but finitely many of the boundaries of \( p_{i,(\alpha, z)}^i(z) \) that were chosen to create \( h_y^i \) meet \( G_z \).
Claim 1. \((p^{i,α+1}_α)^{-1}(z) \subseteq F\).

Proof. Take an arbitrary point \(t \in P = (p^{i,α+1}_α)^{-1}(z) \approx Z\), and an arbitrary basic neighborhood
\[
U \otimes V = (\{z\} \times V) \cup \bigcup \{\{z'\} \times Z : z' \in U \cap h^{-1}_z(V)\}
\]
of \(t\) in \(X^{i}_{α+1}\). Put
\[
U_z = (p^{i,α+1}_α)^{(c)}(U).
\]
Then \(p^{i,α+1}_α(z) \in U_z\), since \(p^{i,α+1}_α\) is one-to-one at \(z\), and
\[
h^{-1}_z(V) \cap U \supseteq (p^{i,α+1}_α)^{-1}((h'_z)^{-1}(V) \cap U_z).
\]
The set \(V\) contains infinitely many points from \(Q\), hence \((h'_z)^{-1}(V) \cap U_z\) contains infinitely many of the boundaries of the neighborhoods of \(p^{i,α+1}_α(z)\) that were chosen to create \(h'_z\), and, consequently, meets \(G_z\). So we conclude that
\[
(h^{-1}_z(V) \cap U) \cap G \neq \emptyset.
\]
Hence, if \(z' \in (h'_z)^{-1}(V) \cap U \cap G\), then \(\{z'\} \times Z \cap F \neq \emptyset\), i.e.,
\[
(U \otimes V) \cap G \neq \emptyset.
\]
Since \(U \otimes V\) was arbitrary, we conclude that \(t \in F\), as desired. \(\square\)

For \((7)_{α+1}\), let \(A \subseteq \prod_{i\in F} X^{i}_{α+1}\) be such that
\[
C_α \subseteq \left( \prod_{i\in F} p^{i,α+1}_β \right)(A).
\]
Consider the point \(z(α) = (z^i_{α})_{i \in F} \in \prod_{i\in F} X^{i}_{α}\) constructed in (v).

Claim 2. \([\prod_{i\in F} (p^{i,α+1}_{α})^{-1}(z(α))] \leq A\).

Proof. Enumerate \(C_α''\) as \(\{c_j : j < ω\}\). For every \(j < ω\), pick \(a_j \in A\) such that \(\prod_{i\in F} p^{i,α+1}_β (a_j) = c_j\). Take an arbitrary element \(x = (x_i)_{i \in F} \in \prod_{i \in F} X^{i}_{α+1}\) in
\[
\left[\prod_{i \in F} (p^{i,α+1}_α)^{-1}\right](z(α)) = \prod_{i \in F} [(p^{i,α+1}_α)^{-1}(z^i_{α})] \approx \prod_{i \in F} (Z)_i \approx Z^F.
\]
For every \(i \in F\), let
\[O_{x_i} = U_i \otimes V_i\]
be an arbitrary basic open neighborhood of \(x_i\) in \(X^{i}_{α+1}\); hence \(V_i\) is an open neighborhood of \(x_i\) in its fiber \((p^{i,α+1}_α)^{-1}(p^{i,α+1}_α(x_i)) = Z\), and \(U_i\) is an open neighborhood of \(p^{i,α+1}_α(x_i) = z^i_{α}\) in \(X^{i}_{α}\). Put \(O_x = \prod_{i \in F} O_{x_i}\). We will prove that \(O_x\) meets \(A\), and hence that \(x \in A\).

For each \(i \in F\), we have that the set
\[W_i = (p^{i,α+1}_α)^{(c)}(U_i)\]
is a neighborhood of \((p^{i,α+1}_α)^{(c)}(z^i_{α})\) in \(X^{i}_{α+1}\), again since \(p^{i,α+1}_α\) is one-to-one at \(z^i_{α}\). For each \(i \in F\),
\[Θ_i = (p^{i,α+1}_α)^{(c)}(U_i) \supseteq (p^{i,α+1}_α)^{(c)}(C_α''(U_i))\]
is a sequence in \(X^{i}_{α+1}\) that converges to \(p^{i,α+1}_α(z^i_{α})\) (see (vii) and (viii)). Since \(F\) is finite, there consequently is an index \(j_0\) such that for all \(j > j_0\) we have that
\[
\forall i \in F: (p^{i,α+1}_α)^{(c)}(U_i) \supseteq (p^{i,α+1}_α)^{(c)}(C_α''(U_i)) \in W_i.
\]
For every $i \in F$ we pick an index $\ell_i$ such that $q_{\ell_i} \in V_i$. Since there are infinitely many such choices, we get by (ix) that there is an index $j > j_0$ such that
\[
(p_{(\beta, z^i_{\beta})}^{j, (\alpha, z^i_{\alpha})})(x_j) = q_{\ell_i} \in (h_{z^i_{\alpha}}^{i, (\alpha, z^i_{\alpha})})^{-1}(V_i) \quad (\forall i \in F).
\] (xi)

We will prove that $a_j \in O_x$. First observe that by (iii),
\[
\prod_{i \in F} p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(a_j) = c_j \in C^{\prime \prime}_{\alpha} \subseteq C^i_{\alpha} \subseteq \left( \prod_{i \in F} X^i_{\beta} \right) \backslash \bigcup_{i \in F} (\pi^i_{(\beta, z^i_{\beta})})^{-1}(z^i_{\beta}).
\] (xii)

Take an arbitrary $i \in F$. It clearly suffices to check that
\[
\begin{align*}
ai \cdot j &= \pi^{i, (\alpha, z^i_{\alpha})}(a_j) \in O_{x_i} = U_i \otimes V_i. \\
\end{align*}
\] (xiii)

By (xii) we have $\prod_{i \in F} p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(a_j) = c_j \in C^{\prime \prime}_{\alpha}$, i.e., $p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(a_j) = p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(x_i)$. This means that for (xiii) we have to check that
\[
bi \cdot j = \pi^i_{(\beta, z^i_{\beta})}(a_j) \in U_i \cap h_{z^i_{\alpha}}^{-1}(V_i).
\]

Now observe that
\[
p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(p_{(\alpha, z^i_{\alpha})}^{i, (\beta, z^i_{\beta})}(b_j)) = p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(b_j)) = p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(\pi^i_{(\beta, z^i_{\beta})}(c_j)),
\]

hence by (x),
\[
p_{(\alpha, z^i_{\alpha})}^{i, (\beta, z^i_{\beta})}(b_j) \in W_i = (p_{(\alpha, z^i_{\alpha})}^{i, (\beta, z^i_{\beta})}(U_i)),
\]

so $b_j \in U_i$. Finally, by (xii),
\[
h_{z^i_{\alpha}}^{i, (\alpha, z^i_{\alpha})}(b_j) = h_{z^i_{\alpha}}^{i, (\alpha, z^i_{\alpha})}(p_{(\alpha, z^i_{\alpha})}^{i, (\beta, z^i_{\beta})}(b_j))
\]
\[
= h_{z^i_{\alpha}}^{i, (\alpha, z^i_{\alpha})}(p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(\pi^i_{(\beta, z^i_{\beta})}(c_j)))
\]
\[
\in V_i,
\]
as required. □

This completes the inductive construction of the topologies on the elements of the inverse systems $S_i$, $i \in \mathbb{N}$.

6. Products without intermediate dimensions

In this section we will formulate and prove our first main result. To begin with, we will first state some general facts about the inverse limits of the inverse systems considered in the previous section.

We put for every $i \in \mathbb{N}$,
\[
X_i = \lim S_i = \lim \{ X^i_{\alpha}, p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}, \omega_1 \},
\]

where the inverse system $S_i$ is as in Section 5.

Let $F \subseteq \mathbb{N}$ be finite and non-empty, and let $G$ be an infinite closed subset of $\prod_{i \in F} X_i$. Let $k$ be the first integer such that $G$ is contained in a finite union of $k$-faces. Then $k \geq 1$, since if $k = 0$ then $G$ is finite. In this finite family of $k$-faces, there is at least one $k$-face $\Gamma$ such that $G \cap \Gamma$ is not contained in a finite family of $(k-1)$-faces. There is a subset $F'$ of $F$ of size $k$ such that $\Gamma = h_{z^i_{\alpha}}(F' \cap F \cup \{ i \})$. By abuse of notation, we think of $\Gamma = G \cap \Gamma$ as a subspace of $\prod_{i \in F'} X_i$. Since $\Gamma$ is not contained in a finite union of $(k-1)$-faces of $\prod_{i \in F'} X_i$, there is a countably infinite set $H \subseteq G' = \Gamma$ that is in general position in $\prod_{i \in F'} X_i$. For each $\alpha < \omega_1$, put
\[
H_\alpha = \left( \prod_{i \in F'} p_{(\beta, z^i_{\beta})}^{i, (\alpha, z^i_{\alpha})}(H) \right).
\]
Since $H$ is countable, we may pick $\beta < \omega_1$ such that $H_\beta$ is infinite and in general position in $\prod_{i \in F'} X^i_\beta$. Then $H_\beta \in \mathcal{E}_\beta$, and, consequently, there is $\beta \leq \alpha < \omega_1$ such that $H_\beta = C_\alpha$. In view of condition (7)$_\alpha$, there is a point $z \in \prod_{i \in F'} X^i_\alpha$ such that

$$A = \left( \prod_{i \in F'} p^{i,\alpha+1}_a \right)^{-1} (z) \subseteq \overline{\prod_{i \in F'} p^{i,\alpha+1}_a} (G').$$

Fix $i \in F'$ for a moment. Observe that the set $B = (p^{i,\alpha+1}_a)^{-1}(z)$ is a continuum, being homeomorphic to $Z$. The restriction of $p^{i,\alpha+1}_a$ to the set $(p^{i,\alpha+1}_a)^{-1}(B)$ is atomic by (5)$_\gamma$, for $\gamma + 1 \leq \gamma \leq \omega_1$. Hence this restriction is irreducible. Since products of irreducible maps are irreducible, the restriction of $\prod_{i \in F'} p^{i,\alpha+1}_a$ to the set $(\prod_{i \in F'} p^{i,\alpha+1}_a)^{-1}(A)$ is irreducible as well. So by (i), this implies that

$$\left( \prod_{i \in F'} p^{i,\alpha+1}_a \right)^{-1} (z) \subseteq \overline{\prod_{i \in F'} p^{i,\alpha+1}_a} (A) \subseteq G',$$

hence $|G'| = 2^\kappa$.

These observations lead us to our first main result.

**Theorem 6.1.** (CH) For every $n \in \mathbb{N}$, there is a family of separable compacta $X_i$, $i \in \mathbb{N}$, such that for every non-empty finite subset $F$ of $\mathbb{N}$ and every non-empty closed subset $G$ of $\prod_{i \in F} X_i$ we have $\dim G = \text{f}(G)n$. (In particular, if $G$ is infinite, then $\dim G \geq n$.) Moreover, for each infinite closed $G \subseteq \prod_{i \in F} X_i$ we have $|G| = 2^\kappa$.

We now put $Z = \mathbb{N}$ as input in the inverse system $\mathcal{S}$ in Section 5, and we claim that the spaces $X_i$, $i \in \mathbb{N}$, we get from our construction, satisfy the conditions in Theorem 6.1. That every $X_i$ is separable is clear. Simply observe that the bonding maps $p^{i,\alpha+1}_a$ in the inverse system creating $X_i$ are atomic for all $1 \leq \alpha \leq \omega_1$, hence irreducible, and that $X_\alpha = Z$ is separable.

To check the remaining properties of the spaces $X_i$, $i \in \mathbb{N}$, let $F \subseteq \mathbb{N}$ be finite and non-empty, and let $G$ be an infinite closed subset of $\prod_{i \in F} X_i$. As above, let $k$ be the first integer such that $G$ is contained in a finite union of $k$-faces. Consider the sets $F'$ and $G'$, the ordinal number $\alpha$, and the point $z$ that were found above. By (ii), it suffices to prove that $\dim G' = kn$.

The inequality $\dim G' \leq kn$ is a consequence of the fact that $\dim X_i \leq n$, for every $i \in \mathbb{N}$. To see this, simply apply Corollary 4.2. To prove the converse, first observe that $(\prod_{i \in F'} p^{i,\alpha+1}_a)^{-1}(z)$ is the inverse limit of an inverse system of which the bonding maps are acyclic (Theorem 4.4). Observe that

$$\left( \prod_{i \in F'} p^{i,\alpha+1}_a \right)^{-1} (z) \approx Z^{F'}$$

is $nk$-dimensional. Hence $\dim (\prod_{i \in F'} p^{i,\alpha+1}_a)^{-1}(z) \geq nk$ by 4.5. So by (ii), we conclude that $\dim G' \geq nk$, as required.

**7. Hereditarily strongly infinite-dimensional products**

We now come to our second main result.

**Theorem 7.1.** (CH) There exists an infinite separable compactum $X$ such that for any positive integer $m$, if $G$ is an infinite closed subspace of $X^m$, then $|G| = 2^\kappa$ and $G$ is strongly infinite-dimensional.

We now take $Z = \mathbb{N}^\infty$ as input for the inverse system $\mathcal{S}$ in Section 5. We claim that every individual $X_i$ satisfies the conclusions of Theorem 7.1. As before, it follows that each $X_i$ is separable, and that any infinite closed subspace of any finite product of the $X_i$'s has cardinality $2^\kappa$.

We will first prove that any infinite closed subspace of any finite product of the $X_i$'s is strongly infinite-dimensional. Let $F \subseteq \mathbb{N}$ be finite and non-empty, and let $G$ be an infinite closed subset of $\prod_{i \in F} X_i$. As above, let $k$ be the first integer such that $G$ is contained in a finite union of $k$-faces. Consider the sets $F'$ and $G'$, the ordinal number $\alpha$, and
the point $z = (z_i)_{i \in F'}$ that were found above. It suffices to prove that $G'$ is strongly infinite-dimensional. And for that, it suffices by (ii) to prove that the fiber $S = (\prod_{i \in F'} p_{\alpha}^{i})^{-1}(z_i)$ is strongly infinite-dimensional. Since

$$S = \prod_{i \in F'} (p_{\alpha}^{i})^{-1}(z_i),$$

it suffices to prove that one of its factors is strongly infinite-dimensional. In fact, all of its factors are strongly infinite-dimensional.

To prove this, fix an arbitrary $i \in F'$. Observe that the set $T_i = (p_{\alpha}^{i+1})^{-1}(z_i)$ is a copy of $\mathbb{I}^\infty$, and that the restriction $f$ of $p_{\alpha+1}^i$ to $S_i = (p_{\alpha}^{i})^{-1}(z_i)$ is atomic, and monotone.

Consider a pair $A$ and $B$ of opposite faces of $T_i$, and let $P$ be a partition in $S_i$ between $f^{-1}(A)$ and $f^{-1}(B)$. Since $f$ is monotone, $f(P)$ is a partition between $A$ and $B$ by Lemma 4.6. As observed at the end of Section 4, $f(P)$ contains a connected partition $C$ between $A$ and $B$. Hence $f^{-1}(C) \cap P$ is a closed subset of $S_i$ that is mapped by the atomic map $f$ onto the continuum $C$. This means that $P$ contains $f^{-1}(C)$. Since the collection of pairs of opposite faces of $T_i$ is an essential collection, this shows that the collection of pre-images under $f$ of that collection is essential in $S_i$. Hence $S_i$ is strongly infinite-dimensional.

Now fix an arbitrary $i \in \mathbb{N}$. By induction on $m \in \mathbb{N}$, we will verify that for every infinite closed subset $G$ of $X^m_i$ we have that $|G| = 2^c$, and $G$ is strongly infinite-dimensional. For $m = 1$, we are done by the above. So assume that we have what we want for $m-1$, $m \geq 2$, and let $G$ be an infinite closed subset of $X^m_i$. Let $\pi : X^m_i \rightarrow X^{m-1}_i$ be the projection, and consider the function $q = \pi | G : G \rightarrow \pi(G)$. If there is an element $x \in q(G)$ such that $q^{-1}(x)$ is infinite, then we are clearly done. So assume that $q$ is finite-to-one. Then $q(G)$ is infinite, and consequently has size $2^c$ and is strongly infinite-dimensional by our inductive assumption. Hence, clearly, $|G| = 2^c$. Moreover, since $q$ is finite-to-one, $G$ is strongly infinite-dimensional by Theorem 4.7.

References