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Intermediate dimensions of products

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Abstract

We construct under the Continuum Hypothesis an example of a compact space no finite power of which contains an infinite closed subset that is of finite dimension greater than 0. This is a partial answer to a question of the third author.

MSC: 55M10

Keywords: Continuum Hypothesis; Infinite-dimensional product; Resolution; Intermediate dimension

1. Introduction

In 1965, D.W. Henderson [12] constructed the first infinite-dimensional metrizable compactum with no closed subspaces of finite positive dimension. Simpler examples were presented subsequently by R.H. Bing [3], D.W. Henderson [13], A.V. Zarelua [25], and others. In 1979, J.J. Walsh [23] improved Henderson's result by the construction of a metrizable compactum all of whose subspaces are either zero- or infinite-dimensional (see also Rubin [18]).

For non-metrizable spaces, there are similar results. In 1973, V.V. Fedorchuk [9] constructed for any integer $n \geqslant 2$ a first countable separable n-dimensional compactum all of whose closed subsets of positive dimension are n-dimensional. Then in 1975, Fedorchuk [10], assuming the Continuum Hypothesis (abbreviated CH), constructed for any positive integer n an infinite separable compactum all of whose infinite closed subsets are n-dimensional and have cardinality $2^{\mathfrak{c}}$. In 1978, A.V. Ivanov [14], under the assumption of Jensen's principle $\diamondsuit$, constructed for any positive integer n a hereditarily separable n-dimensional compactum Y_n with the following property: for every $m \in \mathbb{N}$ and infinite closed subspace F of Y_n^m, the dimension of F is one of $n, 2n, \ldots, mn$.

The following question was asked by J. van Mill [15, Remark 5.7] (see also Pol [17, Question 8.1]):

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Question 1.1. Does there exist an infinite-dimensional metrizable compactum X none of whose finite powers contains a one-dimensional subset?

Its solution is currently beyond (our) reach. We consequently concentrate on the following problem, which is a weakening of the original question in two ways: the example is not required to be metrizable, and the subsets that we are interested in are closed.

Question 1.2. Does there exist an infinite-dimensional compactum X none of whose finite powers contains a one-dimensional *closed* subset?

Theorem 7.1 of our paper gives a positive answer (under CH) to Question 1.2. Theorem 6.1 is related to Fedorchuk's results on compacta without intermediate dimensions. The example constructed there is similar to Ivanov's Example that was mentioned above. Its properties are weaker, but its construction requires CH instead of the more powerful axiom $\diamondsuit$. Question 1.1 remains unsolved. As for non-metrizable spaces, Question 1.2 has several related versions. One of them is:

Question 1.3. Does there exist in ZFC an n-dimensional compactum Z_n, $n \geqslant 1$, such that for every $m \geqslant 2$, every non-empty closed subset F of Z_n^m has dimension kn, where k is some integer between 0 and m?

2. Preliminaries

Unless otherwise stated, all spaces are assumed to be Tychonov, and all mappings are continuous.

If $f : X \to Y$ is a function, and $A \subseteq Y$, then we say that f *is one-to-one over* A provided that $|f^{-1}(y)| = 1$ for every $y \in A$. If $x \in X$ then f is said to be *one-to-one at* x provided that f is one-to-one over $\{f(x)\}$. For a function $f : X \to Y$ and a set $A \subseteq X$ we let $f^\#(A)$ denote the *small image* of A, i.e.,

$$f^\#(A) = Y \setminus f(X \setminus A).$$

A *k-face* of a finite product $\prod_{i=1}^m X_i$ is a subset of the form $\prod_{i=1}^m A_i$, for which there exists a set $B \subseteq \{1, 2, \dots, m\}$ of size $m-k$ such that A_i is a singleton if $i \in B$ and $A_i = X_i$ if $i \notin B$. So a 0-face is a singleton, and an m-face is the whole product.

If $A \subseteq \prod_{i=1}^m X_i$ then the least integer k such that A is contained in the union of a *finite* family of k-faces, will be denoted by $\mathrm{f}(A)$. Clearly, $0 \leqslant \mathrm{f}(A) \leqslant m$. Observe that $\mathrm{f}(A) = 0$ if and only if A is finite.

Let C be a subset of a finite product $\prod_{k=1}^m Y_k$ of sets. We say that C is *in general position in* $\prod_{k=1}^m Y_k$ if each projection $\pi_i : \prod_{k=1}^m Y_k \to Y_i$, $i \leqslant m$, is one-to-one on C.

If X is a set, and κ is a cardinal number, then $[X]^{<\kappa}$ denotes the collection of all subsets of X of size smaller than κ.

For a space X, we let τX denote its topology. If A is a subset of a space X, then $\mathrm{Fr}\, A$ denotes its boundary. A *compactum* is a compact Hausdorff space. A *continuum* is a connected compactum.

We say that f is *fully closed* at the point $y \in Y$, Fedorchuk [8], if $f^{-1}(y) \neq \emptyset$ and for any finite cover $\mathcal{U}$ of $f^{-1}(y)$ by open sets in X the set

$$\{y\} \cup \bigcup_{U \in \mathcal{U}} f^\#(U)$$

is a neighborhood of y. We say that f is *fully closed* if f is fully closed at every $y \in Y$. It is easy to see that a fully closed map is a closed surjection. The converse need not be true.

By the symbol "$X \approx Y$" we mean that X and Y are homeomorphic topological spaces.

Let $f : X \to Y$ be continuous. We say that f is *atomic* provided that for every closed set A in X such that $f(A)$ is a non-degenerate continuum, we have that $A = f^{-1}(f(A))$. Also, f is *irreducible* if $f(A) \neq Y$ for every proper closed subset $A \subseteq X$. It is clear that if f is atomic and Y is a continuum, then f is irreducible. Indeed, if A is a closed subset of X such that $f(A) = Y$, then since f is atomic, $A = f^{-1}(f(A)) = f^{-1}(Y) = X$.

We let $\mathbb{I}$ denote the interval [0, 1].

Suppose that X is a topological space and that $\{Y_x\colon\ x \in X\}$ are topological spaces and, for each $x \in X$, $f_x : X \setminus \{x\} \to Y_x$ is a continuous function. We topologize

$$Z = \bigcup\bigl\{\{x\} \times Y_x\colon\ x \in X\bigr\}$$

as follows. If $x \in X$, U is an open neighborhood of x in X, and $W \subseteq Y_x$ is open, then

$$U \otimes_x W = \bigl(\{x\} \times W\bigr) \cup \bigcup\bigl\{\{x'\} \times Y_{x'}\colon\ x' \in U \cap f_x^{-1}(W)\bigr\}.$$

We usually omit the index x in the expression $U \otimes_x W$. The collection

$$\bigl\{U \otimes W\colon\ (x \in U \in \tau X) \,\&\, (W \in \tau Y_x)\bigr\}$$

is an open basis for Z. Topologized in this way, Z is called the *resolution of X at each point $x \in X$ into Y_x by the mapping f_x*.

Let $\pi_0 : Z \to X$ be the 'projection'. Then π_0 is continuous (this is clear), and fully closed if every Y_x is compact. Since the latter fact will be important in our construction, we will sketch the proof. Assume that $\mathcal{U}$ is a finite open cover of the fiber $\pi_0^{-1}(x)$, for certain $x \in X$. By compactness of Y_x, we may assume without loss of generality that $\mathcal{U}$ consists of basic open sets. So there are a finite open cover $\mathcal{W}$ of Y_x and for every $W \in \mathcal{W}$ an open neighborhood $U(W)$ of x in X such that

$$\mathcal{U} = \bigl\{W \otimes U(W)\colon\ W \in \mathcal{W}\bigr\}.$$

Put $U = \bigcap_{W \in \mathcal{W}} U(W)$. An easy verification shows that

$$U \setminus \{x\} \subseteq \bigcup_{W \in \mathcal{W}} \pi_0^{\#}\bigl(W \otimes U(W)\bigr),$$

as required.

Observe that for every $x \in X$ the set $\{x\} \times Y_x$ is a closed topological copy of Y_x in Z. If X is compact, and all Y_x are compact, then so is the resolution, being a perfect pre-image of a compact space. It is also clear that if X and all the Y_x are first countable, then so is the resolution.

Resolutions were introduced in Fedorchuk [7] (for details, see also Fedorchuk and Hart [11], and Watson [24]). They allow the replacement of each point in a space by a copy of some other space, possibly a different one for each point.

3. Inverse systems

Throughout, for some ordinal number γ, let $\mathbf{T} = \{X_\alpha, p^\alpha_\beta, \gamma\}$ be an inverse system of sets. If X_0 is not a singleton then for technical reasons it will be convenient to let X_{-1} denote $\{0\}$ and $p^0_{-1} : X_0 \to X_{-1}$ the constant function with value 0. This will not have any effect on the limit of the system under consideration. We let X_γ denote the inverse limit of $\mathbf{T}$, and $p^\gamma_\alpha : X_\gamma \to X_\alpha$ for every $\alpha \leqslant \gamma$ the natural projection. If $x \in X_\alpha$ and $\beta \leqslant \alpha \leqslant \gamma$, then x_β denotes $p^\alpha_\beta(x)$. So if $\beta \leqslant \delta \leqslant \alpha \leqslant \gamma$, then

$$x_\beta, \quad p^\alpha_\beta(x), \quad p^\delta_\beta\bigl(p^\alpha_\delta(x)\bigr), \quad p^\delta_\beta(x_\delta)$$

all represent the same point.

If $\delta \leqslant \gamma$, then $\mathbf{T}\restriction\delta$ denotes the restriction of $\mathbf{T}$ to δ, i.e., $\mathbf{T}\restriction\delta = \{X_\alpha, p^\alpha_\beta, \delta\}$. We say that $\mathbf{T}$ is *continuous* if for every limit ordinal $\delta < \gamma$ we have that X_δ is the inverse limit of the system $\{X_\alpha, p^\alpha_\beta, \delta\}$, and, for every $\beta \leqslant \delta$, $p^\delta_\beta : X_\delta \to X_\beta$ is the natural projection

$$\varprojlim \mathbf{T}\restriction\delta \to X_\beta.$$

We recall some details from Fedorchuk [10]. Assume that $\mathbf{T}$ is continuous, and let $\alpha \leqslant \gamma$ and $x \in X_\alpha$ be arbitrary. For $-1 \leqslant \beta \leqslant \alpha$, let

$$A^\alpha_{(\beta,x)} = \begin{cases} \emptyset & (\beta \text{ is a limit ordinal, or } \beta = -1), \\ (p^\beta_\delta)^{-1}(\{x_\delta\}) \setminus \{x_\beta\} & (\beta = \delta + 1). \end{cases}$$

Define the collection of subsets $\mathcal{M}_{(\alpha,x)}$ of X_α by

$$\mathcal{M}_{(\alpha,x)} = \{x\} \cup \bigcup_{\beta \leqslant \alpha} \bigl\{ \bigl(p^\alpha_\beta\bigr)^{-1}(z) \colon z \in A^\alpha_{(\beta,x)} \bigr\}.$$

It is not difficult to see that $\mathcal{M}_{(\alpha,x)}$ is a partition of X_α. Denote $X_\alpha/\mathcal{M}_{(\alpha,x)}$ by $X_{(\alpha,x)}$, and let $p_{(\alpha,x)} : X_\alpha \to X_{(\alpha,x)}$ be the natural decomposition function. Observe that $p_{(\alpha,x)}$ is one-to-one at x.

In the forthcoming applications, the system $\mathbf{T} = \{X_\alpha, p^\alpha_\beta, \gamma\}$ will consist of first countable compacta with continuous bonding maps p^α_β. The elements of the partition $\mathcal{M}_{(\alpha,x)}$ of X_α are then clearly closed G_δ-sets, hence $X_{(\alpha,x)}$ endowed with the quotient topology is T_1 and compact. Moreover, $X_{(\alpha,x)}$ is first countable if it is Hausdorff. In general, $X_{(\alpha,x)}$ need not be Hausdorff, so we need to be careful with this construction. For a simple example, consider the system of spaces $X_{-1} = \{0\}$, $X_0 = \mathbb{I}$, $X_1 = \mathbb{I}^2$, where $\mathbb{I} \to \{0\}$ is the constant function with value 0, and $\mathbb{I}^2 \to \mathbb{I}$ is the projection. If $x = \langle 0, 0\rangle$, then

$$\mathcal{M}_{(1,x)} = \bigl\{ \bigl\{ \langle 0, y\rangle \bigr\} \colon y \in \mathbb{I} \bigr\} \cup \bigl\{ \{p\} \times \mathbb{I} \colon 0 < p \leqslant 1 \bigr\},$$

hence $X_{(1,x)}$ is not Hausdorff. In the framework of resolutions there are fortunately no problems with this construction.

Observe that for every $\beta \leqslant \alpha$, there is a natural function

$$p^{(\alpha,x)}_{(\beta,x_\beta)} : X_{(\alpha,x)} \to X_{(\beta,x_\beta)}$$

such that the diagram

$$\begin{tikzcd}
X_\beta \arrow[d, "p_{(\beta,x_\beta)}"'] & X_\alpha \arrow[l, "p^\alpha_\beta"'] \arrow[d, "p_{(\alpha,x)}"] \\
X_{(\beta,x_\beta)} & X_{(\alpha,x)} \arrow[l, "p^{(\alpha,x)}_{(\beta,x_\beta)}"]
\end{tikzcd}$$

commutes. Also observe that x_β is the only element of the set $X_{(\beta,x_\beta)}$ that possibly has a non-degenerate fiber under the function $p^{(\alpha,x)}_{(\beta,x_\beta)}$. Hence $p^{(\alpha,x)}_{(\beta,x_\beta)}$ is one-to-one over $X_{(\beta,x_\beta)} \setminus \{x_\beta\}$. Also observe that if $\beta \leqslant \delta \leqslant \alpha \leqslant \gamma$ and $x \in X_\alpha$, then

$$p^{(\alpha,x)}_{(\beta,x)} = p^{(\delta,x)}_{(\beta,x)} \circ p^{(\alpha,x)}_{(\delta,x)},$$

from which it easily follows that

$$\bigl\{ X_{(\alpha,x)}, p^{(\alpha,x)}_{(\beta,x_\beta)}, \gamma \bigr\}$$

is a continuous inverse system.

4. Dimension theory

If $f : X \to Y$ is continuous, then, as usual,

$$\dim f = \sup\bigl\{ \dim f^{-1}(y) \colon y \in Y \bigr\}$$

denotes the *dimension* of f.

Theorem 4.1. (Fedorchuk [8]) *If $f : X \to Y$ is fully closed, where X is normal and Y is paracompact, then* $\dim X \leqslant \max\{\dim Y, \dim f\}$.

The following corollary is implicitly contained in Fedorchuk [10].

Corollary 4.2. *Let $\mathbf{S} = \{X_\alpha, p^\alpha_\beta, \gamma\}$ be a continuous inverse system of compacta such that*

(1) $\dim X_0 \leqslant n$,

and, for all $\alpha < \gamma$,

(2) $p_\alpha^{\alpha+1}: X_{\alpha+1} \to X_\alpha$ *is fully closed,*
(3) $\dim p_\alpha^{\alpha+1} \leqslant n$.

Then $\dim \varprojlim \mathbf{S} \leqslant n$.

Proof. It suffices to prove that $\dim X_\alpha \leqslant n$ for every $\alpha < \gamma$. This can be proved by transfinite induction, as follows. If γ is a limit, then there is nothing to prove since $\mathbf{S}$ is continuous. If γ is a successor, then there is nothing to prove by Theorem 4.1. $\square$

A locally compact Hausdorff space X is *acyclic* if $H^i(X) = 0$ for each integer $i \geqslant 0$; here $H^i(Z)$ denotes the ith cohomology group with compact supports of the locally compact space Z. See e.g., [4] for details. A surjective mapping between compacta is *acyclic* if all of its fibers are acyclic.

The following result is a consequence of the famous Vietoris–Begle–Skljarenko Theorem (see [1,2,20–22]).

Theorem 4.3. *If $f: X \to Y$ is a continuous surjection between compacta, and Y and f are acyclic, then so is X.*

Let $\mathbf{S} = \{X_\alpha, p_\beta^\alpha, \gamma\}$ be a continuous inverse system of compacta. Then $H^i(\varprojlim \mathbf{S})$ is isomorphic to the direct limit of the system $\{H^i(X_\alpha), (p_\beta^\alpha)^\star, \gamma\}$ (see [4]). Hence, by Theorem 4.3, we obtain the following result.

Theorem 4.4. *Let $\mathbf{S} = \{X_\alpha, p_\beta^\alpha, \gamma\}$ be a continuous inverse system of compacta. Assume that $p_\alpha^{\alpha+1}$ is acyclic for every $\alpha < \gamma$. Then the projections $p_\alpha^\gamma: \varprojlim \mathbf{S} \to X_\alpha$, $\alpha < \gamma$, are acyclic as well.*

Theorem 4.5. (Dyer [5], Skljarenko [21]) *If X and Y are compacta with Y finite-dimensional, and $f: X \to Y$ is acyclic, then* $\dim X \geqslant \dim Y$.

Dyer [5] proved this for metrizable compacta, and Skljarenko [21] generalized it for the case where X is paracompact and f is closed.

Let $f: X \to Y$ be continuous. As usual, f is said to be *monotone* provided that every fiber $f^{-1}(y)$ of f is connected.

The following lemma is well-known.

Lemma 4.6. *Let $f: X \to Y$ be a monotone, continuous surjection between the compacta X and Y. If A and B are disjoint closed sets in Y, and S is a partition in X between $f^{-1}(A)$ and $f^{-1}(B)$, then $f(S)$ is a partition in Y between A and B.*

A continuum X is *unicoherent* if for all subcontinua A and B with $A \cup B = X$ we have that $A \cap B$ is connected. The Hilbert cube $\mathbb{I}^\infty$ is unicoherent, as is any contractible metrizable continuum [16, A.12.10]. This implies by [16, Exercises 3.10.2 and 3.10.3], that a partition S between a pair of opposite faces A and B of $\mathbb{I}^\infty$ contains a connected partition between A and B. This will be used in the proof of Theorem 7.1.

Let X be a space. We say that X is *strongly infinite-dimensional* if it has an infinite essential family of pairs of disjoint closed subsets. Also, X is *weakly infinite-dimensional* if it is not strongly infinite-dimensional.

Theorem 4.7. (Skljarenko [19]) *Let $f: X \to Y$ be a continuous surjection between compacta, where X is weakly infinite-dimensional, and $|f^{-1}(y)| < \mathfrak{c}$ for every $y \in Y$. Then Y is weakly infinite-dimensional.*

For more information on dimension theory, see [6] and [16].

5. The inverse system S

In this section we put topologies on the elements of certain inverse systems. The inverse limits of these systems will be used in the next section for the construction of our examples. We assume CH throughout.

Throughout, we fix a compact metrizable Absolute Retract Z containing more than one point. In the applications in the forthcoming sections, Z will be $\mathbb{I}^n$ for some n, or the Hilbert cube $\mathbb{I}^\infty$.

Let $n \in \mathbb{N}$ be fixed. For an ordinal number $\alpha \geqslant 1$ we let Z^α denote the set of all functions $\alpha \to Z$. If $\beta \leqslant \alpha$ then p^α_β denotes the natural projection $Z^\alpha \to Z^\beta$ defined by $p^\alpha_\beta(f) = f \restriction \beta$. We let $Z^0 = \{0\}$, and $p^1_0 : Z^1 \to Z^0$ the constant function with value 0.

The family

$$\mathbf{S} = \left\{Z^\alpha, p^\alpha_\beta, \omega_1\right\} \tag{†}$$

is a continuous inverse system of sets and will play a crucial role in the remaining part of this paper. Observe that the functions p^α_β are all surjective.

In our construction we need countably many copies of $\mathbf{S}$. So for $i \in \mathbb{N}$, $\beta \leqslant \alpha \leqslant \omega_1$ and $x \in X_\alpha$, we put

$$X^i_\alpha = Z^\alpha, \qquad p^{i,\alpha}_\beta = p^\alpha_\beta, \qquad X^i_{(\alpha,x)} = X_{(\alpha,x)}, \qquad p^i_{(\alpha,x)} = p_{(\alpha,x)}, \qquad p^{i,(\alpha,x)}_{(\beta,x_\beta)} = p^{(\alpha,x)}_{(\beta,x_\beta)},$$

and define

$$\mathbf{S}_i = \left\{X^i_\alpha, p^{i,\alpha}_\beta, \omega_1\right\}.$$

For $\alpha < \omega_1$, we let $\mathcal{E}_\alpha$ denote the collection of all countably infinite sets C for which there exists an $F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$ such that C is a subset of $\prod_{i\in F} X^i_\alpha$ in general position. It is clear that $|\mathcal{E}_\alpha| = \mathfrak{c}$, hence, by CH, we may fix a bijection

$$f : \omega_1 \to \bigcup_{\alpha<\omega_1} \mathcal{E}_\alpha,$$

such that for every $\alpha < \omega_1$ there exists $\beta \leqslant \alpha$ such that $f(\alpha) \in \mathcal{E}_\beta$. For every $\alpha < \omega_1$, put $C_\alpha = f(\alpha)$.

Let Q denote a fixed countable dense set in Z, and enumerate it by $\{q(j): j < \omega\}$. This enumeration is fixed throughout the forthcoming constructions.

Our aim is to topologize all elements of the systems $\mathbf{S}_i$. We use the technique of resolvable spectra invented by Fedorchuk [10] and further developed thereafter, and ideas and methods of Ivanov's paper [14].

By transfinite induction on $\alpha < \omega_1$, we will topologize the elements of the inverse systems $\mathbf{S}_i$, $i \in \mathbb{N}$, such that the following conditions are satisfied:

(1) X^i_1 is homeomorphic to Z for every $i \in \mathbb{N}$;
(2) the inverse systems

$$\mathbf{S}_i = \left\{X^i_\alpha, p^{i,\alpha}_\beta, \omega_1\right\}, \quad i \in \mathbb{N},$$

are continuous and consist of first-countable compacta;
(3) if $\alpha \leqslant \omega_1$, $i \in \mathbb{N}$, and $x \in X^i_\alpha$, then the quotient space $X^i_{(\alpha,x)}$ is Hausdorff.

Moreover, if $\alpha < \omega_1$, and $i \in \mathbb{N}$, then

(4) $(p^{i,\alpha+1}_\alpha)^{-1}(x)$ is, for every $x \in X^i_\alpha$, homeomorphic to Z;
(5) the projection $p^{i,\alpha+1}_\alpha$ is fully closed;
(6) the projection $p^{i,\alpha+1}_\alpha$ is atomic;
(7) if $C_\alpha \in \mathcal{E}_\beta$, where $\beta \leqslant \alpha$, say C_α is in general position in $\prod_{i\in F} X^i_\beta$ for certain $F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$, and $A \subseteq \prod_{i\in F} X^i_{\alpha+1}$ is such that

$$C_\alpha \subseteq \left(\prod_{i\in F} p^{i,\alpha+1}_\beta\right)(A),$$

then there is a point $z \in \prod_{i\in F} X^i_\alpha$ with

$$\left(\prod_{i\in F} p^{i,\alpha+1}_\alpha\right)^{-1}(z) \subseteq \overline{A}.$$

All finite products of the spaces X_α are endowed with the Tychonov product topology. In addition, if $\alpha \leqslant \omega_1$ and $F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$, then for every $i \in F$,

$$\pi^i_\alpha : \prod_{i\in F} X^i_\alpha \to X^i_\alpha$$

denotes the projection.

For all $i \in \mathbb{N}$, X^i_1 can be identified with Z in a natural way; we topologize X^i_1 in such a way that it is homeomorphic to Z. Suppose for some ordinal $1 \leqslant \gamma < \omega_1$, we defined the desired topologies on all the sets X^i_α for $i \in \mathbb{N}$ and $\alpha < \gamma$.

Assume first that γ is a limit ordinal. As to be expected, for every $i \in \mathbb{N}$ we put $X^i_\gamma = \varprojlim\{X^i_\alpha, p^{i,\alpha}_\beta, \gamma\}$, and $p^{i,\gamma}_\beta : X_\gamma \to X_\beta$ for $\beta < \gamma$ the natural projection. It is easy to see that this is as required (see Section 3).

Now assume that γ is a successor ordinal, say $\gamma = \alpha + 1$. Consider the set C_α, and assume it is in general position in $\prod_{i\in F} X^i_\beta$, where $F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$ and $\beta \leqslant \alpha$. Since X^i_β is first countable for every $i \in F$, there is a sequence

$$C'_\alpha \subseteq C_\alpha \tag{i}$$

converging to some point

$$z(\beta) = \left(z^i_\beta\right)_{i\in F} \in \prod_{i\in F} X^i_\beta. \tag{ii}$$

Since C_α is in general position, it is clear that we may assume that for every $i \in F$, $z^i_\beta \notin \pi^i_\beta(C'_\alpha)$, i.e.,

$$C'_\alpha \subseteq \left(\prod_{i\in F} X^i_\beta\right) \Big\backslash \bigcup_{i\in F} \left(\pi^i_\beta\right)^{-1}\left(z^i_\beta\right). \tag{iii}$$

Consider the continuous surjection

$$f = \prod_{i\in F} p^{i,\alpha}_\beta : \prod_{i\in F} X^i_\alpha \to \prod_{i\in F} X^i_\beta.$$

For every $c \in C'_\alpha$, pick an arbitrary element $x(c)$ in $f^{-1}(c)$. Since $\prod_{i\in F} X^i_\alpha$ is first countable, there is an infinite subset

$$C''_\alpha \subseteq C'_\alpha \subseteq C_\alpha \tag{iv}$$

such that the sequence $\{x(c)\colon c \in C''_\alpha\}$ converges to, say

$$z(\alpha) = \left(z^i_\alpha\right)_{i\in F} \in \prod_{i\in F} X^i_\alpha. \tag{v}$$

Observe that $z(\alpha) \in f^{-1}(z(\beta))$.

Consider for $i \in F$ the commutative diagram

$$\begin{tikzcd} X^i_\beta \arrow[d, "p^i_{(\beta,z^i_\beta)}"'] & X^i_\alpha \arrow[l, "p^{i,\alpha}_\beta"'] \arrow[d, "p^i_{(\alpha,z^i_\alpha)}"] \\ X^i_{(\beta,z^i_\beta)} & X^i_{(\alpha,z^i_\alpha)} \arrow[l, "p^{i,(\alpha,z^i_\alpha)}_{(\beta,z^i_\beta)}"] \end{tikzcd} \tag{vi}$$

and put

$$\Theta_i = \left(p^{i,(\alpha,z^i_\alpha)}_{(\beta,z^i_\beta)}\right)^{-1}\left(p^i_{(\beta,z^i_\beta)}\left(\pi^i_\beta(C''_\alpha)\right)\right). \tag{vii}$$

First observe that since $z^i_\beta \notin \pi^i_\beta(C''_\alpha)$ and $p^i_{(\beta,z^i_\beta)}$ is one-to-one at z^i_β, we have that

$$p^i_{(\beta,z^i_\beta)}\left(z^i_\beta\right) \notin p^i_{(\beta,z^i_\beta)}\left(\pi^i_\beta(C''_\alpha)\right).$$

Next observe that $p^{i,(\alpha,z^i_\alpha)}_{(\beta,z^i_\beta)}$ is one-to-one over $X^i_{(\beta,z^i_\beta)} \setminus \{p^i_{(\beta,z^i_\beta)}(z^i_\beta)\}$. Hence Θ_i is a sequence in $X^i_{(\alpha,z^i_\alpha)}$ having all of its limit points in the set

$$\big(p^{i,(\alpha,z^i_\alpha)}_{(\beta,z^i_\beta)}\big)^{-1}\big(p^i_{(\beta,z^i_\beta)}\big(z^i_\beta\big)\big).$$

Since the sequence $\{x(c)\colon c \in C''_\alpha\}$ converges to $z(\alpha) = (z^i_\alpha)_{i\in F}$ and (vi) commutes, this implies that

$$\Theta_i \text{ is a sequence converging to the point } p^i_{(\alpha,z^i_\alpha)}\big(z^i_\alpha\big). \tag{viii}$$

Let $m = |F|$.

We will now construct for every $i \in \mathbb{N}$ the space $X^i_{\alpha+1}$. It will be a resolution of X^i_α at each point by a copy of Z. So our task is to specify for all $y \in X^i_\alpha$ and $i \in \mathbb{N}$ a certain continuous function $h_y : X^i_\alpha \setminus \{y\} \to Z$. This function will have the form

$$h_y = h'_y \circ \big(p^i_{(\alpha,y)} \restriction X^i_\alpha \setminus \{y\}\big),$$

where $h'_y : X^i_{(\alpha,y)} \setminus \{p^i_{(\alpha,y)}(y)\} \to Z$, is constructed in two steps below.

Step 1. Consider first a fixed $i \in \mathbb{N}$ such that either $i \notin F$, or $i \in F$ but $y \neq z^i_\alpha$.

Since $X^i_{(\alpha,y)}$ is first countable, we can fix a neighborhood base $\{O_j\colon j \in \mathbb{N}\}$ at $p^i_{(\alpha,y)}(y)$ such that $\overline{O}_{j+1} \subseteq O_j$ for every j. Let, for every j, $\psi_j : \operatorname{Fr} O_j \to Z$ be the constant function with value $q(j)$. Evidently, $\bigcup_j \operatorname{Fr} O_j$ is closed in $X^i_{(\alpha,y)} \setminus \{p^i_{(\alpha,y)}(y)\}$, and $X^i_{(\alpha,y)} \setminus \{p^i_{(\alpha,y)}(y)\}$ is normal (being σ-compact). We can therefore extend $\bigcup_j \psi_j$ to a continuous function $h'_y : X^i_{(\alpha,y)} \setminus \{p^i_{(\alpha,y)}(y)\} \to Z$. It is clear that we can arrange h'_y to be surjective.

Step 2. We now consider all remaining cases simultaneously. That is, we consider the cases for which $i \in F$, and $y = z^i_\alpha$. As above, let $\{O^i_j\colon j \in \mathbb{N}\}$ be a neighborhood base at $p^i_{(\alpha,z^i_\alpha)}(z^i_\alpha)$ such that $\overline{O}^i_{j+1} \subseteq O^i_j$ for every j. We may assume without loss of generality that $\operatorname{Fr} O^i_j \cap \Theta_i = \emptyset$ for every j. Let $\psi^i_j : \operatorname{Fr} O^i_j \to Z$ for every j be the constant function with value $q(j)$. Now observe that formula (vii) defines for every $i \in F$ a bijection from C''_α onto Θ_i. So we may pick for every $i \in F$ a function $\psi_i : \Theta_i \to \mathcal{Q}$ such that the following holds: for every $f \in \mathcal{Q}^F$ there is an element $c \in C''_\alpha$ such that for every $i \in F$,

$$\big(p^{i,(\alpha,z^i_\alpha)}_{(\beta,z^i_\beta)}\big)^{-1}\big(p^i_{(\beta,z^i_\beta)}\big(\pi^i_\beta(c)\big)\big) = f(i). \tag{ix}$$

As in case 1, we can extend, for every $i \in F$, the function $\{\psi_i\} \cup \bigcup_j \psi^i_j$ to a continuous surjection $h'_{z^i_\alpha} : X^i_{(\alpha,z^i_\alpha)} \setminus \{p^i_{(\alpha,z^i_\alpha)}(z^i_\alpha)\} \to Z$.

So having all the mappings h_y constructed, we determined the spaces $X^i_{\alpha+1}$ for $i \in \mathbb{N}$, and all there remains to verify is $(2)_{\alpha+1}$ through $(7)_{\alpha+1}$.

It is clear that $(2)_{\alpha+1}$, $(4)_{\alpha+1}$ and $(5)_{\alpha+1}$ hold.

For $(3)_{\alpha+1}$, let $i \in \mathbb{N}$, and $x \in X^i_{\alpha+1}$ be arbitrary. If $y = p^{i,\alpha+1}_\alpha(x)$, then $X^i_{(\alpha+1,x)}$ is (homeomorphic to) the resolution of $X^i_{(\alpha,y)}$ at each point $z \in X^i_{(\alpha,y)}$ into Y_z by the mapping f_z, where

$$\begin{cases} Y_z = \{z\}, & f_z(z) = z \quad (z \neq p^i_{(\alpha,y)}(y)),\\ Y_z = Z, & f_z = h'_y \quad (z = p^i_{(\alpha,y)}(y)). \end{cases}$$

Hence $X^i_{(\alpha+1,x)}$ is a compactum.

For $(6)_{\alpha+1}$, let F be closed in $X^i_{\alpha+1}$ such that $G = p^{i,\alpha+1}_\alpha(F)$ is a non-degenerate continuum. Take an arbitrary $z \in G$, and put

$$G_z = p^i_{(\alpha,z)}(G).$$

The set G_z is clearly a continuum, and non-degenerate because $p^i_{(\alpha,z)}$ is one-to-one at z. As a consequence, all but finitely many of the boundaries of $p^i_{(\alpha,z)}(z)$ that were chosen to create h'_z meet G_z.

Claim 1. $(p_\alpha^{i,\alpha+1})^{-1}(z) \subseteq F$.

Proof. Take an arbitrary point $t \in P = (p_\alpha^{i,\alpha+1})^{-1}(z) \approx Z$, and an arbitrary basic neighborhood

$$U \otimes V = \big(\{z\} \times V\big) \cup \bigcup \big\{\{z'\} \times Z\colon\ z' \in U \cap h_z^{-1}(V)\big\}$$

of t in $X_{\alpha+1}^i$. Put

$$U_z = \big(p_{(\alpha,z)}^i\big)^{\#}(U).$$

Then $p_{(\alpha,z)}^i(z) \in U_z$, since $p_{(\alpha,z)}^i$ is one-to-one at z, and

$$h_z^{-1}(V) \cap U \supseteq \big(p_{(\alpha,z)}^i\big)^{-1}\big((h_z')^{-1}(V) \cap U_z\big).$$

The set V contains infinitely many points from $\mathcal{Q}$, hence $(h_z')^{-1}(V) \cap U_z$ contains infinitely many of the boundaries of the neighborhoods of $p_{(\alpha,z)}^i(z)$ that were chosen to create h_z', and, consequently, meets G_z. So we conclude that

$$\big(h_z^{-1}(V) \cap U\big) \cap G \neq \emptyset.$$

Hence, if $z' \in (h_z^{-1}(V) \cap U) \cap G$ then $(\{z'\} \times Z) \cap F \neq \emptyset$, i.e.,

$$(U \otimes V) \cap G \neq \emptyset.$$

Since $U \otimes V$ was arbitrary, we conclude that $t \in \overline{F} = F$, as desired. □

For $(7)_{\alpha+1}$, let $A \subseteq \prod_{i\in F} X_{\alpha+1}^i$ be such that

$$C_\alpha \subseteq \Big(\prod_{i\in F} p_\beta^{i,\alpha+1}\Big)(A).$$

Consider the point $z(\alpha) = (z_\alpha^i)_{i\in F} \in \prod_{i\in F} X_\alpha^i$ constructed in (v).

Claim 2. $[\prod_{i\in F}(p_\alpha^{i,\alpha+1})^{-1}](z(\alpha)) \subseteq \overline{A}$.

Proof. Enumerate C_α'' as $\{c_j\colon\ j < \omega\}$. For every $j < \omega$, pick $a_j \in A$ such that $(\prod_{i\in F} p_\beta^{i,\alpha+1})(a_j) = c_j$. Take an arbitrary element $x = (x_i)_{i\in F} \in \prod_{i\in F} X_{\alpha+1}^i$ in

$$\Big[\prod_{i\in F}(p_\alpha^{i,\alpha+1})^{-1}\Big]\big(z(\alpha)\big) = \prod_{i\in F}\big[(p_\alpha^{i,\alpha+1})^{-1}\big(z_\alpha^i\big)\big] \approx \prod_{i\in F}(Z)_i \approx Z^F.$$

For every $i \in F$, let

$$O_{x_i} = U_i \otimes V_i$$

be an arbitrary basic open neighborhood of x_i in $X_{\alpha+1}^i$; hence V_i is an open neighborhood of x_i in its fiber $(p_\alpha^{i,\alpha+1})^{-1}(p_\alpha^{i,\alpha+1}(x_i)) = Z$, and U_i is an open neighborhood of $p_\alpha^{i,\alpha+1}(x_i) = z_\alpha^i$ in X_α^i. Put $O_x = \prod_{i\in F} O_{x_i}$. We will prove that O_x meets A, and hence that $x \in \overline{A}$.

For each $i \in F$, we have that the set

$$W_i = \big(p_{(\alpha,z_\alpha^i)}^i\big)^{\#}(U_i)$$

is a neighborhood of $(p_{(\alpha,z_\alpha^i)}^i)(z_\alpha^i)$ in $X_{(\alpha,z_\alpha^i)}^i$, again since $p_{(\alpha,z_\alpha^i)}^i$ is one-to-one at z_α^i. For each $i \in F$,

$$\Theta_i = \big(p_{(\beta,z_\beta^i)}^{i,(\alpha,z_\alpha^i)}\big)^{-1}\big(p_{(\beta,z_\beta^i)}^i\big(\pi_\beta^i(C_\alpha'')\big)\big)$$

is a sequence in $X_{(\alpha,z_\alpha^i)}^i$ that converges to $p_{(\alpha,z_\alpha^i)}^i(z_\alpha^i)$ (see (vii) and (viii)). Since F is finite, there consequently is an index j_0 such that for all $j > j_0$ we have that

$$\forall i \in F\colon\ \big(p_{(\beta,z_\beta^i)}^{i,(\alpha,z_\alpha^i)}\big)^{-1}\big(p_{(\beta,z_\beta^i)}^i\big(\pi_\beta^i(c_j)\big)\big) \in W_i. \tag{x}$$

For every $i \in F$ we pick an index ℓ_i such that $q_{\ell_i} \in V_i$. Since there are infinitely many such choices, we get by (ix) that there is an index $j > j_0$ such that

$$\big(p^{i,(\alpha,z^i_\alpha)}_{(\beta,z^i_\beta)}\big)^{-1}\big(p^i_{(\beta,z^i_\beta)}\big(\pi^i_\beta(c_j)\big)\big) = q_{\ell_i} \in \big(h'_{z^i_\alpha}\big)^{-1}(V_i) \quad (\forall i \in F). \tag{xi}$$

We will prove that $a_j \in O_x$. First observe that by (iii),

$$\bigg(\prod_{i\in F} p^{i,\alpha+1}_\beta\bigg)(a_j) = c_j \in C''_\alpha \subseteq C'_\alpha \subseteq \bigg(\prod_{i\in F} X^i_\beta\bigg)\Big\backslash \bigcup_{i\in F}\big(\pi^i_\beta\big)^{-1}\big(z^i_\beta\big). \tag{xii}$$

Take an arbitrary $i \in F$. It clearly suffices to check that

$$a^i_j = \pi^i_{\alpha+1}(a_j) \in O_{x_i} = U_i \otimes V_i. \tag{xiii}$$

By (xii) we have $p^{i,\alpha+1}_\beta(a^i_j) \neq z^i_\beta$, i.e., $p^{i,\alpha+1}_\alpha(a^i_j) \neq z^i_\alpha = p^{i,\alpha+1}_\alpha(x_i)$. This means that for (xiii) we have to check that

$$b^i_j = p^{i,\alpha+1}_\alpha\big(a^i_j\big) \in U_i \cap h^{-1}_{z^i_\alpha}(V_i).$$

Now observe that

$$p^{i,(\alpha,z^i_\alpha)}_{(\beta,z^i_\beta)}\big(p^i_{(\alpha,z^i_\alpha)}\big(b^i_j\big)\big) = p^i_{(\beta,z^i_\beta)}\big(p^{i,\alpha}_\beta\big(b^i_j\big)\big) = p^i_{(\beta,z^i_\beta)}\big(\pi^i_\beta(c_j)\big),$$

hence by (x),

$$p^i_{(\alpha,z^i_\alpha)}\big(b^i_j\big) \in W_i = \big(p^i_{(\alpha,z^i_\alpha)}\big)^{\#}(U_i),$$

so $b^i_j \in U_i$. Finally, by (xi),

$$\begin{aligned} h_{z^i_\alpha}\big(b^i_j\big) &= h'_{z^i_\alpha}\big(p^i_{(\alpha,z^i_\alpha)}\big(b^i_j\big)\big) \\ &= h'_{z^i_\alpha}\big(\big(p^{i,(\alpha,z^i_\alpha)}_{(\beta,z^i_\beta)}\big)^{-1}\big(p^i_{(\beta,z^i_\beta)}\big(\pi^i_\beta(c_j)\big)\big)\big) \\ &\in V_i, \end{aligned}$$

as required. □

This completes the inductive construction of the topologies on the elements of the inverse systems $\mathbf{S}_i$, $i \in \mathbb{N}$.

6. Products without intermediate dimensions

In this section we will formulate and prove our first main result. To begin with, we will first state some general facts about the inverse limits of the inverse systems considered in the previous section.

We put for every $i \in \mathbb{N}$,

$$X_i = \varprojlim \mathbf{S}_i = \varprojlim\big\{X^i_\alpha, p^{i,\alpha}_\beta, \omega_1\big\},$$

where the inverse system $\mathbf{S}_i$ is as in Section 5.

Let $F \subseteq \mathbb{N}$ be finite and non-empty, and let G be an infinite closed subset of $\prod_{i\in F} X_i$. Let k be the first integer such that G is contained in a finite union of k-faces. Then $k \geqslant 1$, since if $k = 0$ then G is finite. In this finite family of k-faces, there is at least one k-face Γ such that $G \cap \Gamma$ is not contained in a finite family of $(k-1)$-faces. There is a subset F' of F of size k such that Γ is homeomorphic to $\prod_{i\in F'} X_i$. By abuse of notation, we think of $G' = G \cap \Gamma$ as a subspace of $\prod_{i\in F'} X_i$. Since G' is not contained in a finite union of $(k-1)$-faces of $\prod_{i\in F'} X_i$, there is a countably infinite set $H \subseteq G'$ that is in general position in $\prod_{i\in F'} X_i$. For each $\alpha < \omega_1$, put

$$H_\alpha = \bigg(\prod_{i\in F'} p^{i,\omega_1}_\alpha\bigg)(H).$$

Since H is countable, we may pick $\beta < \omega_1$ such that H_β is infinite and in general position in $\prod_{i\in F'} X^i_\beta$. Then $H_\beta \in \mathcal{E}_\beta$, and, consequently, there is $\beta \leqslant \alpha < \omega_1$ such that $H_\beta = C_\alpha$. In view of condition $(7)_\alpha$, there is a point $z \in \prod_{i\in F'} X^i_\alpha$ such that

$$A = \left(\prod_{i\in F'} p^{i,\alpha+1}_\alpha\right)^{-1}(z) \subseteq \overline{H}_{\alpha+1} \subseteq \left(\prod_{i\in F'} p^{i,\omega_1}_{\alpha+1}\right)(G'). \tag{i}$$

Fix $i \in F'$ for a moment. Observe that the set $B = (p^{i,\alpha+1}_\alpha)^{-1}(z)$ is a continuum, being homeomorphic to Z. The restriction of $p^{i,\omega_1}_{\alpha+1}$ to the set $(p^{i,\omega_1}_{\alpha+1})^{-1}(B)$ is atomic by $(5)_\gamma$, for $\alpha+1 \leqslant \gamma \leqslant \omega_1$. Hence this restriction is irreducible. Since products of irreducible maps are irreducible, the restriction of $\prod_{i\in F'} p^{i,\omega_1}_{\alpha+1}$ to the set $(\prod_{i\in F'} p^{i,\omega_1}_{\alpha+1})^{-1}(A)$ is irreducible as well. So by (i), this implies that

$$\left(\prod_{i\in F'} p^{i,\omega_1}_\alpha\right)^{-1}(z) = \left(\prod_{i\in F'} p^{i,\omega_1}_{\alpha+1}\right)^{-1}(A) \subseteq G', \tag{ii}$$

hence $|G'| = 2^{\mathfrak{c}}$.

These observations lead us to our first main result.

Theorem 6.1. (CH) *For every $n \in \mathbb{N}$, there is a family of separable compacta X_i, $i \in \mathbb{N}$, such that for every non-empty finite subset F of $\mathbb{N}$ and every non-empty closed subset G of $\prod_{i\in F} X_i$ we have* $\dim G = \mathrm{f}(G)n$. *(In particular, if G is infinite, then* $\dim G \geqslant n$.*) Moreover, for each infinite closed $G \subseteq \prod_{i\in F} X_i$ we have $|G| = 2^{\mathfrak{c}}$.*

We now put $Z = \mathbb{I}^n$ as input in the inverse system $\mathbf{S}$ in Section 5, and we claim that the spaces X_i, $i \in \mathbb{N}$, we get from our construction, satisfy the conditions in Theorem 6.1. That every X_i is separable is clear. Simply observe that the bonding maps $p^{i,\alpha+1}_\alpha$ in the inverse system creating X_i are atomic for all $1 \leqslant \alpha \leqslant \omega_1$, hence irreducible, and that $X_\alpha = Z$ is separable.

To check the remaining properties of the spaces X_i, $i \in \mathbb{N}$, let $F \subseteq \mathbb{N}$ be finite and non-empty, and let G be an infinite closed subset of $\prod_{i\in F} X_i$. As above, let k be the first integer such that G is contained in a finite union of k-faces. Consider the sets F' and G', the ordinal number α, and the point z that were found above. By (ii), it suffices to prove that $\dim G' = kn$.

The inequality $\dim G' \leqslant kn$ is a consequence of the fact that $\dim X_i \leqslant n$, for every $i \in \mathbb{N}$. To see this, simply apply Corollary 4.2. To prove the converse, first observe that $(\prod_{i\in F'} p^{i,\omega_1}_\alpha)^{-1}(z)$ is the inverse limit of an inverse system of which the bonding maps are acyclic (Theorem 4.4). Observe that

$$\left(\prod_{i\in F'} p^{i,\alpha+1}_\alpha\right)^{-1}(z) \approx Z^{F'}$$

is nk-dimensional. Hence $\dim(\prod_{i\in F'} p^{i,\omega_1}_\alpha)^{-1}(z) \geqslant nk$ by 4.5. So by (ii), we conclude that $\dim G' \geqslant nk$, as required.

7. Hereditarily strongly infinite-dimensional products

We now come to our second main result.

Theorem 7.1. (CH) *There exists an infinite separable compactum X such that for any positive integer m, if G is an infinite closed subspace of X^m, then $|G| = 2^{\mathfrak{c}}$ and G is strongly infinite-dimensional.*

We now take $Z = \mathbb{I}^\infty$ as input for the inverse system $\mathbf{S}$ in Section 5. We claim that every individual X_i satisfies the conclusions of Theorem 7.1. As before, it follows that each X_i is separable, and that any infinite closed subspace of any finite product of the X_i's has cardinality $2^{\mathfrak{c}}$.

We will first prove that any infinite closed subspace of any finite product of the X_i's is strongly infinite-dimensional. Let $F \subseteq \mathbb{N}$ be finite and non-empty, and let G be an infinite closed subset of $\prod_{i\in F} X_i$. As above, let k be the first integer such that G is contained in a finite union of k-faces. Consider the sets F' and G', the ordinal number α, and the point $z = (z_i)_{i \in F'}$ that were found above. It suffices to prove that G' is strongly infinite-dimensional. And for that, it suffices by (ii) to prove that the fiber $S = (\prod_{i \in F'} p_\alpha^{i,\omega_1})^{-1}(z)$ is strongly infinite-dimensional. Since

$$S = \prod_{i \in F'} \left(p_\alpha^{i,\omega_1}\right)^{-1}(z_i),$$

it suffices to prove that one of its factors is strongly infinite-dimensional. In fact, all of its factors are strongly infinite-dimensional.

To prove this, fix an arbitrary $i \in F'$. Observe that the set $T_i = (p_\alpha^{i,\alpha+1})^{-1}(z_i)$ is a copy of $\mathbb{I}^\infty$, and that the restriction f of $p_{\alpha+1}^{i,\omega_1}$ to $S_i = (p_\alpha^{i,\omega_1})^{-1}(z_i)$ is atomic, and monotone.

Consider a pair A and B of opposite faces of T_i, and let P be a partition in S_i between $f^{-1}(A)$ and $f^{-1}(B)$. Since f is monotone, $f(P)$ is a partition between A and B by Lemma 4.6. As observed at the end of Section 4, $f(P)$ contains a connected partition C between A and B. Hence $f^{-1}(C) \cap P$ is a closed subset of S_i that is mapped by the atomic map f onto the continuum C. This means that P contains $f^{-1}(C)$. Since the collection of pairs of opposite faces of T_i is an essential collection, this shows that the collection of pre-images under f of that collection is essential in S_i. Hence S_i is strongly infinite-dimensional.

Now fix an arbitrary $i \in \mathbb{N}$. By induction on $m \in \mathbb{N}$, we will verify that for every infinite closed subset G of X_i^m we have that $|G| = 2^{\mathfrak{c}}$, and G is strongly infinite-dimensional. For $m = 1$, we are done by the above. So assume that we have what we want for $m-1$, $m \geqslant 2$, and let G be an infinite closed subset of X_i^m. Let $\pi : X_i^m \to X_i^{m-1}$ be the projection, and consider the function $q = \pi \restriction G : G \to \pi(G)$. If there is an element $x \in q(G)$ such that $q^{-1}(x)$ is infinite, then we are clearly done. So assume that q is finite-to-one. Then $q(G)$ is infinite, and consequently has size $2^{\mathfrak{c}}$ and is strongly infinite-dimensional by our inductive assumption. Hence, clearly, $|G| = 2^{\mathfrak{c}}$. Moreover, since q is finite-to-one, G is strongly infinite-dimensional by Theorem 4.7.

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