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### NOT ALL HOMOGENEOUS POLISH SPACES ARE PRODUCTS

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ABSTRACT. We prove that not every homogeneous Polish space is the product of one of its quasi-components and a totally disconnected space. This answers a question of Aarts and Oversteegen.

#### 1. INTRODUCTION

All spaces under discussion are separable and metrizable.

An interesting and unexpected consequence of the Effros Theorem [6] is that every homogeneous locally compact space is a product of two spaces, one of which is connected and the other of which is zero-dimensional. This result is for the compact case due to Mislove and Rogers [13, 14] and for the general case to Aarts and Oversteegen [1]. It was asked by Aarts and Oversteegen whether every homogeneous Polish space is the product of one of its quasi-components and a totally disconnected space. To put this question into perspective, observe that there are homogeneous, totally disconnected, 1-dimensional Polish spaces. An example of such a space is the so-called *complete Erdős space*  $E_c$ , that is, the set of vectors in Hilbert space  $\ell^2$  all coordinates of which are irrational. So the product  $E_c \times S^1$ , where  $S^1$  denotes the 1-sphere, is a homogeneous Polish space of which the components form an upper semi-continuous decomposition whose quotient space is not zero-dimensional (but is totally disconnected). This shows

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I am indebted to Lex Oversteegen for pointing out that my original construction could be significantly simplified by using powerful results of Lewis [12]. I am also indebted to Jan Dijkstra for some helpful comments.

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that for Polish spaces one should aim at totally disconnected instead of zerodimensional factors. The aim of this note is to answer the Aarts-Oversteegen question in the negative.

#### 2. Preliminaries

The example has two ingredients: the Lelek fan L and the pseudo-arc P. In this section we will among other things briefly discuss these spaces and state the results that we need from the literature.

A space is *Polish* if its topology is generated by a complete metric. By  $X \approx Y$  we mean that X and Y are homeomorphic spaces.

A space X is totally disconnected if all distinct points  $x, y \in X$  have disjoint clopen (= both closed and open) neighborhoods. Moreover, a space X is hereditarily disconnected if all components are singletons. A totally disconnected space is hereditarily disconnected. The converse is not true, as simple examples show.

The Lelek fan L, [11], is a subcontinuum of the cone over the Cantor set (the Cantor fan) having the set of its endpoints E dense in L. It is known that E is a 1-dimensional, totally disconnected,  $G_{\delta}$ -subset of L. Hence E is Polish. The uniqueness of the Lelek fan as proved by Bula and Oversteegen [3] and Charatonik [4] implicitly shows that E is homogeneous (this is well-known; for a generalization, see e.g. [5, Theorem 7.4]). In fact, if  $x, y \in E$  then there is a homeomorphism of L that maps E onto E and x onto y. It was shown by Kawamura, Oversteegen and Tymchatyn [10] that E is homeomorphic to the complete Erdős space  $E_c$  (see §1). Erdős [8] proved that  $E_c$  is 1-dimensional.

Let P denote the pseudo-arc in the plane. It is well-known that P is homogeneous. Lewis [12], building on work of Bing and Jones [2], proved that for every 1-dimensional continuum X there are a 1-dimensional continuum  $\tilde{X}$  and a continuous open surjection  $\pi: \tilde{X} \to X$ , such that

- (1) for all  $x \in X$ ,  $\pi^{-1}(x) \approx P$ ,
- (2) if  $f: X \to X$  is a homeomorphism, then there is a homeomorphism  $\tilde{f}: \tilde{X} \to \tilde{X}$  such that  $\pi \circ \tilde{f} = f \circ \pi$ ,
- (3) if for some  $x \in X$ ,  $g: \pi^{-1}(x) \to \pi^{-1}(x)$  is a homeomorphism, then there is a homeomorphism  $\tilde{g}: \tilde{X} \to \tilde{X}$  such that  $\tilde{g} \upharpoonright \pi^{-1}(x) = g$  and  $\tilde{g}(\pi^{-1}(y)) = \pi^{-1}(y)$  for every  $y \in X$ .

Observe that this result implies that  $\tilde{X}$  is homogeneous provided that X is.

#### 3. The construction

We adopt the notation in §2. Our example is  $Z = \pi^{-1}(E)$  (here  $\tilde{L}$  and  $\pi: \tilde{L} \to L$  are the space and the map given by the results of Lewis). The results mentioned in §2 imply that Z is homogeneous and Polish.

**Proposition 3.1.** If X is hereditarily disconnected and Y is connected, then Z and  $X \times Y$  are not homeomorphic.

PROOF. Let  $\xi: X \times Y \to X$  denote the projection. Striving for a contradiction, let  $h: Z \to X \times Y$  be a homeomorphism. Since E is totally disconnected,  $\{\pi^{-1}(u): u \in E\}$  is the collection of components of Z. In addition,  $\{\{x\} \times Y : x \in X\}$  is the collection of components of  $X \times Y$ . Since  $\pi$  is open, the assignment  $E \to X$  defined by

$$u \mapsto \pi^{-1}(u) \mapsto h(\pi^{-1}(u)) \mapsto \{\xi(h(\pi^{-1}(u)))\}$$

is continuous. By symmetry, it has a continuous inverse, hence  $E \approx X$ . Moreover,  $Y \approx P$  since every  $\pi^{-1}(u) \approx K$ . This means that  $Z \approx E \times P$ . But this is a contradiction since dim  $Z \leq 1$  being a subspace of the 1-dimensional space  $\tilde{L}$  (in fact, its dimension is obviously 1), and dim $(E \times P) = 2$ . The latter fact follows from the result due to Hurewicz [9] that the product of a 1-dimensional compactum and a 1-dimensional space is 2-dimensional (see also Engelking [7, 1.9.E]).

Since the quasi-components of Z coincide with its components and hence are continua, Proposition 3.1 solves negatively the Aarts-Oversteegen question that was mentioned in the introduction.

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