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NOT ALL HOMOGENEOUS POLISH SPACES ARE PRODUCTS

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Abstract. We prove that not every homogeneous Polish space is the product of one of its quasi-components and a totally disconnected space. This answers a question of Aarts and Oversteegen.

1. Introduction

All spaces under discussion are separable and metrizable.

An interesting and unexpected consequence of the Effros Theorem [6] is that every homogeneous locally compact space is a product of two spaces, one of which is connected and the other of which is zero-dimensional. This result is for the compact case due to Mislove and Rogers [13, 14] and for the general case to Aarts and Oversteegen [1]. It was asked by Aarts and Oversteegen whether every homogeneous Polish space is the product of one of its quasi-components and a totally disconnected space. To put this question into perspective, observe that there are homogeneous, totally disconnected, 1-dimensional Polish spaces. An example of such a space is the so-called *complete Erdős space* E_c, that is, the set of vectors in Hilbert space ℓ^2 all coordinates of which are irrational. So the product $E_c \times \mathbb{S}^1$, where $\mathbb{S}^1$ denotes the 1-sphere, is a homogeneous Polish space of which the components form an upper semi-continuous decomposition whose quotient space is not zero-dimensional (but is totally disconnected). This shows

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I am indebted to Lex Oversteegen for pointing out that my original construction could be significantly simplified by using powerful results of Lewis [12]. I am also indebted to Jan Dijkstra for some helpful comments.

that for Polish spaces one should aim at totally disconnected instead of zero-dimensional factors. The aim of this note is to answer the Aarts-Oversteegen question in the negative.

2. Preliminaries

The example has two ingredients: the Lelek fan L and the pseudo-arc P. In this section we will among other things briefly discuss these spaces and state the results that we need from the literature.

A space is *Polish* if its topology is generated by a complete metric. By $X \approx Y$ we mean that X and Y are homeomorphic spaces.

A space X is *totally disconnected* if all distinct points $x, y \in X$ have disjoint clopen (= both closed and open) neighborhoods. Moreover, a space X is *hereditarily disconnected* if all components are singletons. A totally disconnected space is hereditarily disconnected. The converse is not true, as simple examples show.

The *Lelek fan* L, [11], is a subcontinuum of the cone over the Cantor set (the Cantor fan) having the set of its endpoints E dense in L. It is known that E is a 1-dimensional, totally disconnected, G_δ-subset of L. Hence E is Polish. The uniqueness of the Lelek fan as proved by Bula and Oversteegen [3] and Charatonik [4] implicitly shows that E is homogeneous (this is well-known; for a generalization, see e.g. [5, Theorem 7.4]). In fact, if $x, y \in E$ then there is a homeomorphism of L that maps E onto E and x onto y. It was shown by Kawamura, Oversteegen and Tymchatyn [10] that E is homeomorphic to the complete Erdős space E_c (see §1). Erdős [8] proved that E_c is 1-dimensional.

Let P denote the pseudo-arc in the plane. It is well-known that P is homogeneous. Lewis [12], building on work of Bing and Jones [2], proved that for every 1-dimensional continuum X there are a 1-dimensional continuum $\tilde{X}$ and a continuous open surjection $\pi\colon \tilde{X} \to X$, such that

(1) for all $x \in X$, $\pi^{-1}(x) \approx P$,
(2) if $f\colon X \to X$ is a homeomorphism, then there is a homeomorphism $\tilde{f}\colon \tilde{X} \to \tilde{X}$ such that $\pi \circ \tilde{f} = f \circ \pi$,
(3) if for some $x \in X$, $g\colon \pi^{-1}(x) \to \pi^{-1}(x)$ is a homeomorphism, then there is a homeomorphism $\tilde{g}\colon \tilde{X} \to \tilde{X}$ such that $\tilde{g} \restriction \pi^{-1}(x) = g$ and $\tilde{g}(\pi^{-1}(y)) = \pi^{-1}(y)$ for every $y \in X$.

Observe that this result implies that $\tilde{X}$ is homogeneous provided that X is.

3. The construction

We adopt the notation in §2. Our example is $Z = \pi^{-1}(E)$ (here $\tilde{L}$ and $\pi\colon \tilde{L} \to L$ are the space and the map given by the results of Lewis). The results mentioned in §2 imply that Z is homogeneous and Polish.

Proposition 3.1. *If X is hereditarily disconnected and Y is connected, then Z and $X \times Y$ are not homeomorphic.*

Proof. Let $\xi\colon X \times Y \to X$ denote the projection. Striving for a contradiction, let $h\colon Z \to X \times Y$ be a homeomorphism. Since E is totally disconnected, $\{\pi^{-1}(u) : u \in E\}$ is the collection of components of Z. In addition, $\{\{x\} \times Y : x \in X\}$ is the collection of components of $X \times Y$. Since π is open, the assignment $E \to X$ defined by

$$u \mapsto \pi^{-1}(u) \mapsto h(\pi^{-1}(u)) \mapsto \{\xi(h(\pi^{-1}(u)))\}$$

is continuous. By symmetry, it has a continuous inverse, hence $E \approx X$. Moreover, $Y \approx P$ since every $\pi^{-1}(u) \approx K$. This means that $Z \approx E \times P$. But this is a contradiction since $\dim Z \leq 1$ being a subspace of the 1-dimensional space $\tilde{L}$ (in fact, its dimension is obviously 1), and $\dim(E \times P) = 2$. The latter fact follows from the result due to Hurewicz [9] that the product of a 1-dimensional compactum and a 1-dimensional space is 2-dimensional (see also Engelking [7, 1.9.E]). □

Since the quasi-components of Z coincide with its components and hence are continua, Proposition 3.1 solves negatively the Aarts-Oversteegen question that was mentioned in the introduction.

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