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LOCAL PROPERTIES AND MAXIMAL TYCHONOFF CONNECTED SPACES

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Abstract. We prove that, if X is a Tychonoff connected space and $\chi(x, X) \leq \omega$ for some $x \in X$, then there exists a strictly stronger Tychonoff connected topology on the space X, i.e., the space X is not maximal Tychonoff connected. We also establish that if X is locally connected or σ-compact or has pointwise countable type then X cannot be maximal Tychonoff connected.

1. Introduction

A connected space X is called *maximal connected* if no strictly stronger topology on X is connected. The concept was introduced in [Tho], where examples of maximal connected T_1-spaces were constructed. Later, maximal connected spaces were studied in [GRS], [GS] and [GSW]. In this last paper and in [Si], maximal connected strengthenings of the usual topology on the real line **R** were constructed; the space **R** being Hausdorff, any strengthening is Hausdorff as well. In [NlW] it was shown that maximal connected T_1-spaces must be submaximal (i.e., all their dense subspaces are open); however, very few non-trivial examples of submaximal T_3-spaces without isolated points are known (see for example [vD]), and all known are (at least) totally disconnected. In particular, it is still an open question as to whether there exists a connected submaximal T_3-space. A lot of research has been done here; it is known, for example, that any infinite submaximal Tychonoff space which is either first countable, separable or

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compact is totally disconnected (see [AC, 4.8, 4.12 and 5.10], where even stronger results are given).

In view of these results it is natural to ask whether there is a maximal Tychonoff connected space, that is, an infinite connected Tychonoff space X such that any stronger *Tychonoff* topology on X is disconnected. Such spaces do not apparently have to be submaximal so the results of [AC] mentioned in the preceding paragraph do not apply. Even so, the folklore suspicion was that such spaces do not exist. The first steps of an attempted proof were taken in [Jo], where it was shown that there exists a connected *group* topology on the reals $\mathbf{R}$ which is stronger than the usual one (see also Example 2.10 of [ATTW]); in [TVs] stronger connected group topologies were constructed for certain Abelian topological groups.

It was later shown in [STTWW] that if X is a first countable or a separable or a locally Čech-complete infinite connected Tychonoff space, then it has a strictly stronger connected Tychonoff topology, that is, it is not maximal Tychonoff connected. The results were new even for the classes of metrizable or compact spaces.

A well-known class containing all first countable and locally Čech-complete spaces is the class of spaces of pointwise countable type. In this paper we prove that all spaces with this property are not maximal Tychonoff connected, answering positively Problem 2 from [STTWW]. Another result is that no Tychonoff locally connected connected space is maximal Tychonoff connected. We also establish that if a Tychonoff connected space has a point of countable character or is σ-compact then it cannot be maximal Tychonoff connected.

2. Notation and Terminology

All spaces are assumed to be Tychonoff. Given a space X the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. If $Y \subset X$ and $\mathscr{A}$ is a family of subsets of X then $\mathscr{A}|Y = \{A \cap Y : A \in \mathscr{A}\}$. We denote by $\mathbf{R}$ the set of the reals with its natural topology; $\mathbf{I} \subset \mathbf{R}$ is the set $[0, 1]$. If X is a space and $f : X \to Y$ is a map then $G(f) = \{(x, f(x)) : x \in X\} \subset X \times Y$ is its graph. A *connected space* is called *non-trivial* if it has more than one point; for technical reasons we consider that the empty space is not connected. If X is connected then $x \in X$ is called *an endpoint of* X if $X\backslash\{x\}$ is also connected.

A set $F \subset X$ is a *zero-set in the space* X if there is a continuous $f : X \to \mathbf{I}$ such that $F = f^{-1}(0)$. If X is a space and $F \subset X$ then a family $\mathscr{B} \subset \tau(X)$ is *an outer base of* F *in* X if for any $V \in \tau(X)$ with $F \subset V$ there is $U \in \mathscr{B}$ such that

$F \subset U \subset V$. The set F *has countable outer character in* X if it has a countable outer base in X. A space X is *of pointwise countable type* if it can be covered with compact subspaces of countable outer character in X. The symbol $\square$ denotes the end of the proofs of numbered statements; to indicate that a substatement's proof is finished, we use the symbol $\varDelta$.

If S is a set then $[S]^2 = \{T \subset S : |T| = 2\}$ is the set of all non-ordered pairs of the elements of S. If $F \subset [S]^2$ then a set $A \subset S$ is *homogeneous with respect to* F if $[A]^2 \subset F$. The rest of our notation is standard and can be found in [En].

3. General Properties of Maximal Tychonoff Connected Spaces

After a great deal of hard work on extending connected Tychonoff topologies the authors became convinced that it is highly probable that maximal Tychonoff connected spaces exist. They could not prove it, however, so all results about maximal Tychonoff connected spaces could announce properties of an empty class. On the other hand, while there is no proof that the class of maximal Tychonoff connected spaces is empty, looking at its properties might be of use when studying connected spaces.

3.1. PROPOSITION. *The following are equivalent for any Tychonoff space* X:

(1) X *is maximal Tychonoff connected*;

(2) *for any function* $f : X \to \mathbf{I}$ *its graph* $G(f) \subset X \times \mathbf{I}$ *is connected if and only if* f *is continuous.*

PROOF. There is a continuous one-to-one map of $G(f)$ onto X for any function $f : X \to \mathbf{I}$. If f is continuous then $G(f)$ is homeomorphic to X so $G(f)$ is connected. If f is discontinuous then the projection of $G(f)$ onto X is not a homeomorphism so $G(f)$ has to be disconnected because otherwise, by identifying the set $G(f)$ with X, we obtain a strictly stronger connected Tychonoff topology on X. This proves (1) $\Rightarrow$ (2).

Now assume that X is not maximal Tychonoff connected. Then there is a connected Tychonoff topology τ' on X such that $\tau' \neq \tau \subset \tau'$. Since τ' and τ are Tychonoff, there is a function $f : X \to \mathbf{I}$ such that f is τ'-continuous and not τ-continuous. Let $X' = (X, \tau')$ and denote by $i : X' \to X$ the identity map. It is clear that the graph $G' = G(f)$ considered as a subspace of the space $X' \times \mathbf{I}$ is homeomorphic to X' so G' is connected. Now, if $j : \mathbf{I} \to \mathbf{I}$ is the identity map then $h = i \times j$ maps $X' \times \mathbf{I}$ continuously onto $X \times \mathbf{I}$ and $h(G') = G(f)$. Thus $f : X \to \mathbf{I}$ is a discontinuous function whose graph is connected; this contradicts (2) and hence (2) $\Rightarrow$ (1). $\square$

3.2. LEMMA. *Given a connected space X and a continuous function $f : X \to \mathbf{I}$ assume that the set $F = f^{-1}(0)$ is non-empty and $U \neq \emptyset$ is a clopen subspace of $X \backslash F$. Then $(0, \varepsilon) \subset f(U)$ for some $\varepsilon > 0$.*

PROOF. If our statement is not true then we can choose a strictly decreasing sequence $\{r_n : n \in \omega\} \subset (0,1) \backslash f(U)$ such that $r_n \to 0$. Pick an arbitrary $y \in U$ and $n \in \omega$ such that $r_n < f(y)$. Then $y \in V = U \backslash f^{-1}([0, r_n]) = U \backslash f^{-1}([0, r_n))$ and therefore V is a clopen non-empty proper subset of X which contradicts the connectedness of X. □

3.3. THEOREM. *Let X be a connected space for which there exists a continuous surjective function $\varphi : X \to \mathbf{I}$ with the following properties:*

(a) *the set $S = \varphi^{-1}(0)$ is connected;*

(b) *if $O_n = \varphi^{-1}\left([0, \frac{1}{n})\right)$ for all $n \in \mathbf{N}$ then the family $\mathcal{O} = \{O_n : n \in \mathbf{N}\}$ is an outer base of S in X.*

Then X has a strictly stronger connected Tychonoff topology.

PROOF. Let $Y = X \backslash S$; since φ is surjective, $B_n = \varphi^{-1}\left(\frac{1}{n}\right) \neq \emptyset$ for all $n \in \mathbf{N}$. It follows from Lemma 3.2 that

($*$) if C is clopen in Y and $C \neq \emptyset$ then $C \cap B_n \neq \emptyset$ for all but finitely many n.

The function $\eta = \frac{1}{\varphi}$ is continuous on Y and $\{\eta(B_n) : n \in \mathbf{N}\}$ is a discrete family of singletons in $\mathbf{R}$. An evident consequence is that there is a continuous function $\alpha : Y \to \mathbf{I}$ such that $\alpha(B_n) = 0$ if n is even and $\alpha(B_n) = 1$ for each odd $n \in \mathbf{N}$. Define $f : X \to \mathbf{I}$ by $f(x) = 0$ for any $x \in S$ and $f|Y = \alpha$; it follows from (b) of our hypothesis that f is discontinuous. By Proposition 3.1, all there remains to prove is that $G(f)$ is connected.

Assume the contrary and denote by $\pi : G(f) \to X$ the natural projection; let $S' = S \times \{0\} \subset G(f)$ and pick a clopen non-empty proper subset D of $G(f)$ with $D \cap S' \neq \emptyset$. Since the set S' is connected, we have $S' \subset D$ so $C = G(f) \backslash D$ is a clopen non-empty proper subset of $G(f)$ for which $C \cap S' = \emptyset$. Since α is a continuous map and $G(\alpha) = G(f) \backslash S'$, the map $\pi | G(\alpha) : G(\alpha) \to Y$ is a homeomorphism so $U = \pi(C)$ is a non-empty clopen subset of Y.

It follows from ($*$) that there is $m \in \mathbf{N}$ such that $U \cap B_n \neq \emptyset$ for all $n \geq m$; choose a point $x_n \in U \cap B_n$ for any $n \in N_0 = \{k \in \mathbf{N} : k \geq m \text{ and } k \text{ is even}\}$. Since the family $\mathcal{O}$ is an outer base of S in X, there is a cluster point $x \in S$ for the sequence $\{x_n : n \in N_0\}$. Then $P = \{(x_n, 0) : n \in N_0\} \subset C$ and $(x, 0) \in \bar{C} \cap D$ which is a contradiction. Thus $G(f)$ is connected so X admits a strictly stronger connected topology by Proposition 3.1. □

The following result, often referred to as Kuratowski's Lemma, is part of the folklore; its proof can be found in [Ko, Chapter 2, §5].

3.4. THEOREM. *Let X be a connected space with a connected subspace C. If S is a component of $X\backslash C$, then $X\backslash S$ is connected.*

From now on, for any space X we denote by $\mathcal{U}_X$ the collection of all connected open $U \subset X$ such that $|\bar{U}\backslash U| = 1$ and $X\backslash \bar{U} \neq \varnothing$. The unique point in $\bar{U}\backslash U$ will be denoted by x_U.

3.5. PROPOSITION. *Let X be a connected space. Then,*

(1) *if $U \in \mathcal{U}_X$ then U is a component of $X\backslash\{x_U\}$ and $X\backslash U$ is connected;*

(2) *if $U, V \in \mathcal{U}_X$ and $x_V \in U$ then either $\bar{V} \subset U$ or $U \cup V = X$;*

(3) *if $U \in \tau(X)$ and $|\bar{U}\backslash U| = 1$ then $\bar{U}$ is connected;*

(4) *if $\{U_0, \dots, U_n\} \subset \tau(X)$, the family $\{\bar{U}_0, \dots, \bar{U}_n\}$ is disjoint and $|\bar{U}_i\backslash U_i| = 1$ for each $i \le n$, then $F = X\backslash \bigcup\{U_i : i \le n\}$ is connected and $U_i \cup F$ is connected for every $i \le n$.*

PROOF. The first part of (1) is trivial because $X\backslash\{x_U\}$ is the union of the disjoint open sets U and $X\backslash\bar{U}$ while U is connected. The second part of (1) follows from the first part and Theorem 3.4 (with $C = \{x_U\}$ and $S = U$).

To prove (2) note that $X\backslash U$ is connected by (1) so it is either contained in V or in $X\backslash\bar{V}$, i.e., $U \cup V = X$ or $\bar{V} \subset U$.

For (3), there is $x \in X$ such that $\bar{U}\backslash U = \{x\}$; if $\bar{U}$ is disconnected then $\bar{U} = E \cup F$ for some non-empty disjoint closed sets E and F. If $x \in E$ then F is clopen in X, and if $x \in F$ then E is clopen in X which contradicts connectedness of X.

As to (4), let $\bar{U}_i\backslash U_i = \{x_i\}$ and $F_i = X\backslash U_i$ for every $i \le n$. Assume that $F = \bigcap_{i \le n} F_i$ is not connected; then $F = E_0 \cup E_1$ for some disjoint non-empty closed sets E_0 and E_1. Observe that $\{x_0, \dots, x_n\} \subset F$ and let $\hat{E}_j = E_j \cup \bigcup\{\bar{U}_i : x_i \in E_j\}$ for every $j \in \{0, 1\}$. It is immediate that $\hat{E}_0$ and $\hat{E}_1$ form a partition of X into disjoint closed sets, which contradicts the connectedness of X; this shows that F is connected.

For the second part of (4) observe that $\bar{U}_i$ is connected by (3) and $x_i \in \bar{U}_i \cap F$ so $U_i \cup F$ is connected being a union of two connected non-disjoint sets. □

3.6. PROPOSITION. *Let X be a maximal Tychonoff connected space. Then,*

(1) *for any disjoint non-trivial connected sets $A, B \subset X$ there is a set $U \in \mathcal{U}_X$ such that either $A \subset U$ and $B \subset X\backslash U$ or vice versa;*

(2) *if $A, B \subset X$ are connected disjoint sets then* $|\bar{A} \cap \bar{B}| \leq 1$;

(3) *if connected sets $A, B \subset X$ are disjoint and $A \cup B = X$ then $\bar{A} \cap \bar{B}$ is a singleton.*

PROOF. (1) Let B' be the component of $X \setminus A$ that contains B, and $A' = X \setminus B'$. Then the set A' is connected by Theorem 3.4 and we have $A \subset A'$, $B \subset B'$ while $A' \cup B' = X$ and $A' \cap B' = \emptyset$. Since X is connected, we have $\overline{A'} \cap \overline{B'} \neq \emptyset$.

If $|\overline{A'} \cap \overline{B'}| > 1$, fix distinct points $a, b \in \overline{A'} \cap \overline{B'}$. We can assume, without loss of generality, that $a \in A'$. Take a continuous $g : X \to \mathbf{I}$ such that $g(a) = 0$ and $g(b) = 1$ and define a function $f : X \to \mathbf{I}$ as follows: $f(x) = 0$ for all $x \in B'$ and $f(x) = g(x)$ for every $x \in A'$. It is evident that f is discontinuous at b independently of whether $b \in A'$ or not.

Let us show that the graph $G(f)$ of the function f is connected. Indeed, the set $G_{B'} = \{(x, 0) : x \in B' \cup \{a\}\}$ is homeomorphic to $B' \cup \{a\}$ and hence connected. Analogously, $f|A'$ is continuous so $G_{A'} = G(f) \cap (A' \times \mathbf{I})$ is also connected. Thus $G(f) = G_{A'} \cup G_{B'}$ is a union of two connected subspaces with a non-empty intersection. Therefore $G(f)$ is connected which in light of Proposition 3.1 shows that X is not maximal Tychonoff connected. This contradiction proves that $\overline{A'} \cap \overline{B'} = \{x\}$ for some $x \in X$; if $x \in A'$ then A' is closed in X and hence $U = B' \in \mathcal{U}_X$ is as promised. Analogously, if $x \in B'$ then $U = A' \in \mathcal{U}_X$ while $A \subset U$ and $B \subset B' = X \setminus U$. This settles (1).

To prove (2) apply (1) to find $U \in \mathcal{U}_X$ such that $A \subset U$ and $B \subset X \setminus U$ or vice versa. It is immediate that in both cases we have $\bar{A} \cap \bar{B} \subset \bar{U} \cap \overline{X \setminus U} = \{x_U\}$.

As to the statement of (3), it follows from (2) that $|\bar{A} \cap \bar{B}| \leq 1$; since A and B are non-empty (recall that we consider that empty spaces are not connected), an immediate consequence of the connectedness of X is that $\bar{A} \cap \bar{B} \neq \emptyset$ so we have $|\bar{A} \cap \bar{B}| = 1$. □

3.7. PROPOSITION. *If X is a maximal Tychonoff space and $U \subset X$ is a non-trivial open connected subset of X then U (with the topology inherited from X) is also a maximal Tychonoff connected space.*

PROOF. Let τ be the topology of X and assume that there is a Tychonoff connected topology μ on U such that $\mu \neq \tau|U \subset \mu$. Let τ' be the topology on X generated by the family $\tau \cup \mu$ as a subbase; then $\tau \subset \tau'$ and $\tau' \neq \tau$.

To see that τ' is Tychonoff take any $x \in X$ and $V \in \tau'$ with $x \in V$. Assume first that $x \in U$ and take $W \in \tau$ such that $x \in W \subset \overline{W} \subset U$ (the bar denotes the closure in (X, τ)). Then $V' = V \cap W \in \mu$ and $x \in V'$ so there is a μ-continuous function $f : U \to \mathbf{I}$ such that $f(x) = 1$ and $f|(U\setminus V') \equiv 0$. Letting $g(y) = f(y)$ for all $y \in U$ and $g(y) = 0$ for all $y \in X\setminus U$ we obtain a function $g : X \to \mathbf{I}$ such that $g(x) = 1$ and $g|(X\setminus V) \equiv 0$. To see that g is τ'-continuous observe that it is τ'-continuous on U (because its restriction to U coincides with the μ-continuous function f) and constant on a τ-open set $X\setminus\overline{W}$. Therefore g is locally τ'-continuous and hence continuous on (X, τ').

Now, if $x \in X\setminus U$ then $\tau_x = \{U \in \tau : x \in U\}$ is a local base at x in (X, τ') so there is $W \in \tau$ such that $x \in W \subset V$. The space X being Tychonoff there is a τ-continuous function $f : X \to \mathbf{I}$ such that $f(x) = 1$ and $f|(X\setminus W) \equiv 0$. It is clear that f is continuous on (X, τ') and witnesses the Tychonoff property at x in (X, τ').

Finally, to see that the space $X' = (X, \tau')$ is connected take a τ'-clopen set W such that $W \cap U \neq \varnothing$. Since $\tau'|U = \mu$, the space $(U, \tau'|U)$ is connected so $U \subset W$. Let $W' = X\setminus W$; the family $\tau_y = \{G \in \tau : y \in G\}$ is a local base at y in X' for any $y \in X\setminus U$ which implies, together with $W' \subset X\setminus U$, that W' is open in X. Besides, any point of $W\setminus U$ has a τ-open neighbourhood contained in W; since U is also a τ-open neighbourhood of any element of U, we conclude that the set W is τ-open as well, i.e., W is a clopen subset of X. The space X being connected we have $W = X$ and hence X' is connected. Since X is a maximal Tychonoff connected space, we obtained a contradiction which proves that U is also maximal Tychonoff connected. □

3.8. Theorem. *Suppose that $\varphi : X \to Y$ is a continuous monotone open surjective map. If X is maximal Tychonoff connected then Y is also maximal Tychonoff connected.*

Proof. It is clear that Y is connected. If it is not maximal Tychonoff connected then it follows from Proposition 3.1 that there exists a discontinuous function $g : Y \to \mathbf{I}$ such that its graph $G(g)$ is connected. The function $f = g \circ \varphi$ is discontinuous because φ is a quotient map so it suffices by Proposition 3.1 to prove that $G(f)$ is connected.

If $\mathrm{id} : \mathbf{I} \to \mathbf{I}$ is the identity map then $\Phi = \varphi \times \mathrm{id} : X \times \mathbf{I} \to Y \times \mathbf{I}$ is open, monotone and $G(f) = \Phi^{-1}(G(g))$ so the mapping $\Phi' = \Phi|G(f)$ is open and monotone as well. Since the inverse image of a connected space under a monotone open map has to be connected, the set $G(f)$ is connected which contradicts maximal Tychonoff connectedness of X. □

We will now turn our attention to countably compact spaces. The methods of [STTWW] are not applicable to that class of spaces because of the existence of a dense-in-itself countably compact space all countable subspaces of which are scattered (see [JvM]).

3.9. THEOREM. *Let X be a normal, countably compact, maximal connected Tychonoff space. Then $\mathcal{U}_X$ is finite.*

PROOF. Assume, towards a contradiction, that $\mathcal{U}_X$ is infinite.

Claim 1. Let $\mathcal{U} \subset \tau^*(X)$ be an infinite pairwise disjoint collection. If $A = X \setminus \bigcup \mathcal{U}$ then there is $U \in \mathcal{U}$ such that $A \cup U$ is not connected.

To prove Claim 1 assume that $A \cup U$ is connected and pick a point $x_U \in U$ for every $U \in \mathcal{U}$. Since X is countably compact, the set $\{x_U : U \in \mathcal{U}\}$ has a cluster point $z \in A$. For every $U \in \mathcal{U}$ pick a continuous function $f_U : U \cup A \to \mathrm{I}$ such that $f_U(A) = 0$ and $f_U(x_U) = 1$. If $f = \bigcup_{U \in \mathcal{U}} f_U$, then f is discontinuous at the point z and $G(f) = \bigcup \{G(f_U) : U \in \mathcal{U}\}$ while $G(f_U)$ is homeomorphic to $A \cup U$ and hence connected for every $U \in \mathcal{U}$. Since the connected sets $G(f_U)$ have a non-empty intersection $G(f|A)$, the set $G(f)$ is connected, which, together with Proposition 3.1, contradicts the maximal Tychonoff connectedness of X. $\triangle$

Claim 2. If $x \in X$ then the family of all components of $X \setminus \{x\}$ contains at most finitely many elements that are open in X.

Suppose that $\mathcal{C} = \{C_n : n \in \omega\} \subset \tau^*(X)$ is a fathfully indexed collection of components of $X \setminus \{x\}$. By Theorem 3.4 the set $F_0 = X \setminus C_0$ is connected. Now, assume that $n \in \omega$ and we proved that $F_i = X \setminus (\bigcup \{C_k : k \le i\})$ is connected for any $i \le n$. Since C_{n+1} is a component of $F_n \setminus \{x\}$, the set $F_{n+1} = F_n \setminus C_{n+1}$ is connected. Thus $F_n = X \setminus (\bigcup \{C_k : k \le n\})$ is connected for any $n \in \omega$ which implies, together with countable compactness and normality of X, that $Y = X \setminus \bigcup \mathcal{C}$ is connected. Since for every $C \in \mathcal{C}$ we have $x \in \bar{C}$, the set $C \cup Y$ is connected as well which is a contradiction with Claim 1. $\triangle$

Claim 3. There exists no infinite $\mathcal{V} \subset \mathcal{U}_X$ such that the family $\{\bar{V} : V \in \mathcal{V}\}$ is pairwise disjoint.

If Claim 3 is false then there is a family $\mathcal{V} = \{V_n : n \in \omega\} \subset \mathcal{U}_X$ such that $\bar{V}_n \cap \bar{V}_m = \varnothing$ whenever $n \neq m$. If $S_n = \bigcap \{X \setminus V_i : i \le n\}$ then S_n is connected for every $n \in \omega$ by Proposition 3.5. It follows from countable compactness and normality of X that $S = \bigcap \{S_n : n \in \omega\}$ is non-empty and connected. Observe that the components of $X \setminus S$ are precisely the sets V_n and $S \cup V_n$ is connected for every n because $x_{V_n} \in \bar{V}_n \cap S$. This gives a contradiction with Claim 1. $\triangle$

Fix an element $U \in \mathscr{U}_X$. By Claim 2, the collection $\{V \in \mathscr{U}_X : x_V = x_U\}$ is finite because if $V \in \mathscr{U}_X$ then V is a component of $X \setminus \{x_V\}$ (see Proposition 3.5). Since $\mathscr{U}_X$ is infinite, we can find a sequence $\{U_n : n \in \omega\} \subset \mathscr{U}_X$ such that if $n \neq m$ then $x_{U_n} \neq x_{U_m}$; for the sake of brevity let $x_n = x_{U_n}$ for all $n \in \omega$.

Claim 4. There is no infinite $A \subset \omega$ such that for all distinct $n, m \in A$ we have $U_n \cup U_m = X$.

If Claim 4 is false, then there is an infinite $A \subset \omega$ such that for any pair of distinct $m, n \in A$ we have $U_n \cup U_m = X$; let $V_n = X \setminus \bar{U}_n$ for any $n \in A$. The collection $\{\bar{V}_n : n \in A\}$ is pairwise disjoint so the set $S_n = X \setminus \bigcup\{V_i : i \in A, i \leq n\}$ is connected for every n by Proposition 3.5. It follows from countable compactness and normality of X that $S = X \setminus (\bigcup_{n \in A} V_n) = \bigcap_{n \in A} \bar{U}_n$ is connected. We have $X \setminus S = \bigcup_{n \in A} V_n$ and $V_n \cup \{x_n\} = \bar{V}_n$ is connected for every $n \in A$ (see Proposition 3.5). Now, $x_n \in S$ and hence $V_n \cup S$ is connected for every n; this is again a contradiction with Claim 1. Δ

In what follows we will need the sets $E_0 = \{p \in [\omega]^2$: if $m, n \in p$ and $m < n$ then $x_m \in U_n\}$ and $E_1 = \{p \in [\omega]^2$: if $m, n \in p$ and $m < n$ then $x_n \in U_m\}$ as well as $E_2 = [\omega]^2 \setminus (E_0 \cup E_1)$. The following three Claims show that, for any $i \in \{0, 1, 2\}$ there is no infinite homogeneous set for E_i which contradicts Ramsey's theorem (see [Ru, Chapter II, page 8]). Thus our proof will be complete after we establish Claims 5–7.

Claim 5. There is no infinite homogeneous set for E_0.

Indeed, if $A \subset \omega$ is an infinite homogeneous set for E_0 then it follows from Proposition 3.5 that $[A]^2 = E_{00} \cup E_{01}$ where $E_{00} = \{p \in [A]^2$: if $m, n \in p$ and $m < n$ then $\bar{U}_m \subset U_n\}$ and $E_{01} = \{p \in [A]^2$: if $m, n \in p$ and $m < n$ then $U_n \cup U_m = X\}$. Now apply Ramsey's theorem to find an infinite $B \subset A$ such that B is homogeneous either for E_{00} or for E_{01}.

If B is homogeneous for E_{00} then we have a collection $\{U_n : n \in B\}$ such that $\bar{U}_n \subset U_m$ whenever $n, m \in B$ and $n < m$. If $V_n = X \setminus \bar{U}_n$ then $\bar{V}_n \setminus V_n = \{x_n\}$ so $\bar{V}_m \subset V_n$ for all $m, n \in B$ such that $n < m$. As a consequence, we obtain a set $S = \bigcap\{V_n : n \in B\} = \bigcap\{\bar{V}_n : n \in B\}$ which is a connected zero-set of countable outer character in X. Using normality of X it is easy to construct a continuous function $\varphi : X \to \mathbf{I}$ as in Theorem 3.3 which is a contradiction with maximal Tychonoff connectedness of X.

Now, if $B \subset A$ is an infinite homogeneous set for E_{01} then $\{U_n : n \in B\}$ is an infinite family for which $U_n \cup U_m = X$ for any distinct $m, n \in B$; this contradicts Claim 4. Δ

Claim 6. There is no infinite homogeneous set for E_1.

Indeed, if $A \subset \omega$ is an infinite homogeneous set for E_1 then it follows from Proposition 3.5 that $[A]^2 = E_{10} \cup E_{11}$ where $E_{10} = \{p \in [A]^2$: if $m, n \in p$ and $m < n$ then $\bar{U}_n \subset U_m\}$ and $E_{11} = \{p \in [A]^2$: if $m, n \in p$ and $m < n$ then $U_n \cup U_m = X\}$. Now apply Ramsey's theorem to find an infinite $B \subset A$ such that B is homogeneous either for E_{10} or for E_{11}.

If B is homogeneous for E_{10} then we have a collection $\{U_n : n \in B\}$ such that $\bar{U}_m \subset U_n$ whenever $n, m \in B$ and $n < m$. As a consequence, we obtain a set $S = \bigcap\{U_n : n \in B\} = \bigcap\{\bar{U}_n : n \in B\}$ which is a connected zero-set of countable outer character in X. Using normality of X it is easy to construct a continuous function $\varphi : X \to \mathbf{I}$ satisfying the hypothesis of Theorem 3.3 and hence X is not maximal Tychonoff connected.

Now, if $B \subset A$ is an infinite homogeneous set for E_{11} then again $\{U_n : n \in B\}$ is an infinite family for which $U_n \cup U_m = X$ for any distinct $m, n \in B$; this contradicts Claim 4. $\triangle$

Claim 7. There is no homogeneous infinite set for E_2.

Indeed, if $A \subset \omega$ is an infinite homogeneous set for E_2 then $x_n \notin U_m$ and $x_m \notin U_n$ for all distinct $m, n \in A$. Then the collection $\{\bar{U}_n : n \in A\}$ is pairwise disjoint for otherwise there are distinct $m, n \in A$ with $\bar{U}_n \cap \bar{U}_m \neq \varnothing$ and hence $U_n \cap U_m = \bar{U}_n \cap \bar{U}_m$ (we have to recall that $x_n \neq x_m$ and apply Proposition 3.6) is a clopen non-empty proper subset of X, which is impossible by connectedness of X. This contradiction with Claim 3 shows that Claim 7 is settled. $\triangle$

Thus we obtained a decomposition $[\omega]^2 = E_0 \cup E_1 \cup E_2$ with no homogeneous infinite set for all $i \in \{0, 1, 2\}$. This contradiction with Ramsey's theorem finishes our proof. $\square$

3.10. REMARK. Perhaps the reader feels that a better theorem would be that $\mathcal{U}_X$ is empty. This would indeed be the case. However, we do not know whether there is a normal, countably compact, maximal Tychonoff connected space. It is worth noting, however, that if there is a normal, countably compact, maximal Tychonoff connected space X with an endpoint then there is one for which $\mathcal{U}_X \neq \varnothing$.

PROOF. Let X be a normal, countably compact maximal Tychonoff connected space with a endpoint x. Consider the disjoint topological sum of two copies of X, i.e.,

$$(X \times \{0\}) \cup (X \times \{1\}),$$

and identify the points $(x, 0)$ and $(x, 1)$. The resulting space Y is again normal, countably compact, maximal Tychonoff connected and $\mathcal{U}_Y \neq \emptyset$. □

A similar trick cannot be repeated infinitely often, since then it is clear that one loses countable compactness. So it seems that, in a sense, Theorem 3.9 is the best possible.

3.11. COROLLARY. *Let X be a non-trivial, normal, countably compact, maximal Tychonoff connected space. Then every disjoint family of non-trivial connected subsets of X is finite.*

PROOF. Let $\mathcal{A}$ be a countably infinite family of pairwise disjoint non-trivial connected sets in X. By Theorem 3.9, the family $\mathcal{U}_X$ is finite, say $\mathcal{U}_X = \{U_0, \ldots, U_n\}$. If no U_i contains infinitely many elements from $\mathcal{A}$ then we may pick distinct $A, B \in \mathcal{A}$ such that $A \setminus U_i \neq \emptyset \neq B \setminus U_i$ for every $i \leq n$. Since this contradicts Proposition 3.6, we may assume, without loss of generality, that every element of $\mathcal{A}$ is contained in U_0. Proposition 3.6 implies that all the pairs of elements of $\mathcal{A}$ can be 'separated' by an element of the collection $\mathcal{U}_X \setminus \{U_0\}$ so we can repeat the same reasoning to throw out one more element of $\mathcal{U}_X$; after at most n steps this evidently leads to a contradiction. □

3.12. REMARK. Every non-trivial continuum X contains an infinite pairwise disjoint family of non-trivial subcontinua. To see it, take an infinite family $\mathcal{U} \subset \tau^*(X)$ such that the collection $\{\bar{U} : U \in \mathcal{U}\}$ is disjoint. For any $U \in \mathcal{U}$ choose $x_U \in U$ and observe that the component C_U of the point x_U in the space $\bar{U}$ is non-trivial because it has to intersect the boundary of U (see [En, Lemma 6.1.25]). Thus Corollary 3.11 implies that no non-trivial continuum is maximal Tychonoff connected providing, therefore, another method for the proof of Theorem 2 of [STTWW] for the compact case.

3.13. COROLLARY. *Let X be a non-trivial, normal, countably compact, maximal Tychonoff connected space. Then X contains a dense open totally disconnected subspace.*

PROOF. Every non-empty open subset of X contains an infinite disjoint family of non-empty open subsets of X. This means that one of them is totally disconnected by Corollary 3.11. So, a maximal family of totally disconnected open subspaces of X has dense union and is, clearly, totally disconnected. □

4. Classes in which Connected Tychonoff Topologies can be Strengthened

The results of this section show that many local properties of a connected space X imply that X is not maximal Tychonoff connected. The analogous global properties discovered so far imply, in some sense, that there exist "large" compact subsets in the space X.

4.1. THEOREM. *Let X be a non-trivial connected Tychonoff space containing at least one point of countable character. Then X admits a strictly stronger connected Tychonoff topology.*

PROOF. Let X be first countable at the point x. It is easy to find a continuous surjective function $\varphi : X \to \mathbf{I}$ such that $\varphi(x) = 0$ and $\{U_n = \varphi^{-1}([0, 1/n)) : n \in \mathbf{N}\}$ is a local base at x in X. Now apply Theorem 3.3 to conclude the proof. □

4.2. THEOREM. *Let X be a non-trivial connected Tychonoff space that is the union of a family of fewer than $\mathfrak{c}$ compact subspaces. Then X admits a strictly stronger connected Tychonoff topology.*

PROOF. Let $\mathscr{K}$ be the family of all compact dense-in-themselves subspaces of X, and consider the set $F = \overline{\bigcup \mathscr{K}}$. If $F = X$ then we can apply Lemma 2 and Theorem 1 of [STTWW] to obtain the desired result. So assume that $U = X \backslash F$ is non-empty and pick a non-empty open subset V of X such that $\overline{V} \subset U$. Then $\overline{V}$ is the union of fewer than $\mathfrak{c}$ compact subspaces that all have to be scattered. We claim that $\overline{V}$ is zero-dimensional; to see it, pick an arbitrary continuous function $f : \overline{V} \to \mathbf{R}$. Since for every scattered compact $E \subset \overline{V}$ the set $f(E)$ is scattered and hence countable, the space $f(\overline{V})$ has size strictly less than $\mathfrak{c}$. The space $\overline{V}$ being Tychonoff, it has to be zero-dimensional. But no open subset of a connected space is zero-dimensional, which is a contradiction. □

4.3. COROLLARY. *Let X be a non-trivial connected σ-compact Tychonoff space. Then X admits a strictly stronger connected Tychonoff topology.*

Recall that a space X is of *pointwise countable type* if it is a union of a family of compact subspaces each having a countable outer base in X. It is not hard to prove that every Čech-complete space is of pointwise countable type. It was shown in Theorem 4 in [STTWW] that if X is a connected Tychonoff space of pointwise countable type and $c(X) = \omega$ then X admits a strictly stronger connected Tychonoff topology. The following result shows that there is no need to assume the Souslin property of X and answers Problem 2 of [STTWW].

4.4. THEOREM. *Let X be a non-trivial connected Tychonoff space of pointwise countable type. Then X admits a strictly stronger connected Tychonoff topology.*

PROOF. Again, let $\mathscr{K}$ be the family of all compact dense-in-themselves subspaces of X, and put $F = \overline{\bigcup \mathscr{K}}$. If $F = X$ then we can apply Lemma 2 and Theorem 1 of [STTWW] to obtain the desired result. So we assume that $U = X \backslash F$ is non-empty. Pick $x \in U$, and a compact subspace $K \ni x$ of X with a countable neighbourhood base. In addition, let V be an open neighbourhood of x such that $\bar{V} \subset U$. Then $\bar{V} \cap K$ is compact and scattered. Hence the non-empty open subspace $V \cap K$ of $\bar{V} \cap K$ has an isolated point, say y. But then y is an isolated point of K so X is first countable at y because K has a countable outer base in X. Now apply Theorem 4.1 to complete the proof. □

4.5. THEOREM. *If X is a non-trivial Tychonoff connected and locally connected space then there exists a strictly stronger connected Tychonoff topology on X.*

PROOF. Fix a continuous $\varphi : X \to \mathbf{I}$ such that $F = \varphi^{-1}(0) \neq \varnothing$ and $U = X \backslash F \neq \varnothing$. Consider the sets $B_n = \varphi^{-1}\left(\frac{1}{n}\right)$, $O_n = \varphi^{-1}\left(\left[0, \frac{1}{n}\right)\right)$ and $P_n = O_n \cup B_n$ for any $n \in \mathbf{N}$. We will also need the sets $C_n^m = \varphi^{-1}\left(\left(\frac{1}{m}, \frac{1}{n}\right)\right)$ and $D_n^m = \varphi^{-1}\left(\left[\frac{1}{m}, \frac{1}{n}\right]\right)$ for any $m, n \in \mathbf{N}$ with $n < m$. Furthermore, $N_0 = \{n : n \in \mathbf{N}$ and n is even$\}$ and $N_1 = \{n : n \in \mathbf{N}$ and n is odd$\}$. Observe first that

(i) if G is a clopen subset of U and $G \cap C_n^{n+1} \neq \varnothing$ then $G \cap B_k \neq \varnothing$ for any $k > n$,

because otherwise $G \backslash O_k = G \backslash P_k$ is a clopen non-empty proper subset of X. Furthermore,

(ii) if G is a clopen subset of U and $x \in \bar{G} \cap F$ then $x \in \overline{\bigcup \{G \cap B_i : i \in A\}}$ for any infinite $A \subset \mathbf{N}$.

To see that (ii) is true assume that W is an open connected neighbourhood of x such that $W \cap P = \varnothing$ where $P = \bigcup \{G \cap B_i : i \in A\}$. Take any $m \in A$; since $W \cap O_m$ is a neighbourhood of x, we have $O_m \cap W \cap G \neq \varnothing$ so there is $k \geq m$ and a point $y \in C_k^{k+1} \cap G \cap W$. Since A is infinite, we can take $l \in A$ with $k + 1 < l$. It is immediate that we have $y \in C_m^l \cap G \cap W = D_m^l \cap G \cap W$ so $W' = C_m^l \cap W \cap G$ is a non-empty proper clopen subset of $G \cap W$. Besides, $\overline{W'} \cap F = \varnothing$ and hence W' is a clopen non-empty proper subset of W; this contradiction with connectedness of W shows that (ii) holds.

Next note that $\eta = \frac{1}{\varphi}$ is a continuous function on U such that $\{\eta(B_n) : n \in \mathbf{N}\}$

is a discrete family of singletons in $\mathbf{R}$. An evident consequence is that there exists a continuous function $g : U \to \mathbf{I}$ for which $g(B_i) = \{0\}$ for any $i \in N_0$ and $g(B_i) = \{1\}$ whenever $i \in N_1$. Let $f(x) = g(x)$ for any $x \in U$ and $f(x) = 0$ if $x \in F$.

The space X being connected, the set U is not closed in X so there is $x \in \overline{U} \cap F$. Applying (ii) with $G = U$ and $A = N_1$ we can see that $x \in \overline{\{B_i : i \in A\}}$ and hence the function f is discontinuous at the point x. We claim that the graph $G(f)$ of the function f is connected.

To arrive at a contradiction assume that E' and G' are non-empty disjoint clopen subsets of $G(f)$ such that $E' \cup G' = G(f)$. If $\pi : X \times \mathbf{I} \to X$ is the natural projection then it is a homeomorphism if restricted to any of the sets $F' = (F \times \mathbf{I}) \cap G(f)$ and $U' = (U \times \mathbf{I}) \cap G(f)$. Let $G = \pi(G')$ and $E = \pi(E')$; we can assume, without loss of generality, that $G \cap U \neq \varnothing$.

Suppose first that $U \subset G$; then $E \subset F$. Since $\pi|F' \to F$ is a homeomorphism, the set E is closed in F and hence in X. Therefore it is impossible that $E \subset \mathrm{Int}(F)$ because otherwise E is a clopen non-empty proper subset of X. Therefore we can take $x \in H = F \backslash \mathrm{Int}(F)$ such that $x \in E$. Applying (ii) to the set $B = \bigcup\{B_n : n \in N_0\}$ we conclude that $x \in \bar{B}$; the map $f|(B \cup F)$ is constant so $(x, 0) \in \overline{B \times \{0\}}$ which, together with $B \times \{0\} \subset G'$, shows that $(x, 0) \in E' \cap \overline{G'}$ which is a contradiction.

Thus we can assume that $E_1 = E \cap U \neq \varnothing$. Therefore $G_1 = G \cap U$ and E_1 are non-empty disjoint clopen subsets of the space U such that $E_1 \cup G_1 = U$. If $E_1 \cap O_n = \varnothing$ for some $n \in \mathbf{N}$ then $E_1 = E_1 \backslash O_{n+1} = E_1 \backslash P_{n+1}$ is a clopen non-empty proper subset of X, a contradiction. Analogously, it is impossible that $G_1 \cap O_n = \varnothing$ for some $n \in \mathbf{N}$.

Furthermore, $E \cap F$ and $G \cap F$ are disjoint closed subspaces of F and hence of X. It follows from connectedness of X that $\bar{E} \cap \bar{G} \neq \varnothing$ so either $\bar{G}_1 \cap E \neq \varnothing$ or $\bar{E}_1 \cap G \neq \varnothing$. The two cases are similar so take an arbitrary point $x \in \bar{G}_1 \cap E$. It is clear that $x \in F$ so we can apply (ii) again to conclude that $B \cap G_1$ contains x in its closure. But $B \cup F$ is homeomorphic to $(B \cup F) \times \{0\} \subset G(f)$ which shows that $(x, 0) \in E'$ is in the closure of $(B \cap G_1) \times \{0\} \subset G'$ which again provides a contradiction. Finally, apply Proposition 3.1 to conclude that X fails to be maximal Tychonoff connected. $\square$

4.6. REMARK. It is worth noting that the proof of Theorem 4.5 is valid for a slightly larger class of spaces than the locally connected ones. To see it observe that we only used local connectedness of X at the points of the boundary of the set F. Thus we have actually proved that *if X is a Tychonoff connected space in which there is a zero-set $F \subset X$ such that $\varnothing \neq F \neq X$ and X is locally connected at all points of $F \backslash \mathrm{Int}(F)$ then X is not maximal Tychonoff connected.*

5. Open Problems

The most intriguing problem is the existence of maximal Tychonoff connected spaces. Being convinced that they do exist, we also ask about their properties.

5.1. PROBLEM. *Does there exist a maximal Tychonoff connected space?*

5.2. PROBLEM. *Is it true that any connected space is a continuous image of a maximal Tychonoff connected space?*

5.3. PROBLEM. *Is it possible to strengthen the topology of* **R** *to a maximal Tychonoff connected topology? How about an arbitrary connected space?*

5.4. PROBLEM. *Does there exist a maximal Tychonoff connected countably compact space? Does it help to assume additionally that X is normal?*

5.5. PROBLEM. *Does there exist a maximal Tychonoff connected pseudocompact space?*

5.6. PROBLEM. *Does there exist a maximal Tychonoff connected Lindelöf space?*

5.7. PROBLEM. *Does there exist a maximal Tychonoff connected Fréchet–Urysohn space? How about maximal Tychonoff connected spaces which are sequential or k-spaces?*

5.8. PROBLEM. *Is it true that all compact subsets of a maximal Tychonoff connected space are zero-dimensional? Is it possible to prove, at least, that a maximal Tychonoff connected space cannot contain a copy of* **I**?

5.9. PROBLEM. *Let X be a maximal Tychonoff connected space. Does there exist a cut point in X, i.e., a point $x \in X$ for which $X \setminus \{x\}$ is disconnected?*

5.10. PROBLEM. *Must every maximal Tychonoff connected space be (strongly) σ-discrete?*

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