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ON n-TO-ONE CONTINUOUS IMAGES OF $\beta\mathbb{N}$

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Abstract

Eric van Douwen [5] produced a maximal crowded extremally disconnected regular space and showed that its Čech–Stone compactification is an at most two-to-one image of $\beta\mathbb{N}$. We construct for any $n \geqq 3$ an example of a compact crowded space X_n that is an image of $\beta\mathbb{N}$ under a map all of whose fibers have either size n or $n-1$. We also show that under CH this is best possible.

1. Introduction

All spaces considered here are Tychonoff. A function $f : X \to Y$ is $(\leqq)n$-*to-one* if for each $y \in Y$, there are $(\leqq)n$ points of X that map to y. Levy [15] asked whether there is a separable 2-to-one image of $\mathbb{N}^*$, the Stone–Čech remainder of the discrete space of natural numbers $\mathbb{N}$. It was shown recently by Dow and Techanie [11] that a 2-to-one continuous image of $\mathbb{N}^*$ must be $\mathbb{N}^*$ under CH. Dow proved in [7] that under PFA, all 2-to-one images of $\mathbb{N}^*$ are trivial.

Levy's question was partially answered by a striking result of van Douwen [5]: there exists a crowded $\leqq 2$-to-one continuous image of $\beta\mathbb{N}$. The restriction of the van Douwen map to $\mathbb{N}^*$ is a $\leqq 2$-to-one map from $\mathbb{N}^*$ onto a

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separable crowded space. This result is highly counter-intuitive since $\mathbb{N}^*$ is big and a $\leqq$ 2-to-one map should not make a big space small. The restriction of the van Douwen map to $\mathbb{N}$ is a 1-to-one map onto a crowded space X whose topology is maximal among all crowded topologies on X. Such a space is a countable *van Douwen* space. That this is striking, is obvious once one realizes that X is regular. Additional results on van Douwen spaces were obtained recently by Dow [6]. He proved among other things that it is consistent that there exist two van Douwen spaces whose Čech–Stone compactifications are not homeomorphic (in fact one of them, in ZFC, is the absolute of the Cantor cube $2^{\mathfrak{c}}$ and the other one, consistently, is ω-bounded).

In this paper we will present many van Douwen spaces that are dense subsets of the absolute $E(2^{\mathfrak{c}})$ of the Cantor cube $2^{\mathfrak{c}}$. In fact, we will characterize the countable dense subspaces of $2^{\mathfrak{c}}$ that can be 'lifted' to a van Douwen space. In the following result, let $p_{2^{\mathfrak{c}}}$ denote the 'unique' irreducible map $E(2^{\mathfrak{c}}) \twoheadrightarrow 2^{\mathfrak{c}}$.

THEOREM 1. *Let X be a countable dense subset of $2^{\mathfrak{c}}$. Then the following statements are equivalent:*

(1) *There is a van Douwen space X' in $E(2^{\mathfrak{c}})$ such that $p_{2^{\mathfrak{c}}}(X') = X$,*
(2) *X is open-hereditarily irresolvable.*

We will use this result to present for every $n \geqq 3$, an example of a $\leqq n$-to-one map from $\beta\mathbb{N}$ onto a crowded space. If fact, the fibers of the map have either size n or $n-1$. We also show that under CH, this is best possible by proving that if $f : \beta\mathbb{N} \to X$ is a $\leqq n$-to-one continuous surjection onto a crowded space X, then there is a point $x \in X$ such that $\left|f^{-1}(x)\right| \leqq n-1$.

2. Notation

All maps considered here are continuous. Let $f : X \to Y$ be a surjective map between compacta. As usual, we call f *irreducible* if for every proper closed subset A of X we have that $f(A)$ is a proper subset of Y. This is easily seen to be equivalent to the following statement: if U is any nonempty open subset of X, then there is an open subset V of Y such that $f^{-1}(V)$ is a dense subset of U. If $f : X \to Y$ is irreducible, then $D \subseteq X$ is nowhere dense in X if and only if $f(D)$ is nowhere dense in Y. Let $f : X \to Y$ be a surjective map between compacta. An easy Zorn's Lemma argument shows that a surjective map between compacta can always be restricted to a closed subset of its domain on which it is irreducible.

If X is a compact space, then there is an extremally disconnected compact space $E(X)$ which admits an irreducible map $p_X : E(X) \to X$. The space $E(X)$ is called the *absolute* of X, and its is known to be the topologically unique extremally disconnected compactum admitting an irreducible map on X. For details, see e.g., Porter and Woods [17].

A nonempty space is *crowded* if it has no isolated points. And a space is *nodec* if all of its nowhere dense subsets are closed (and hence discrete). A space X is *irresolvable* if it is crowded and no dense subset has dense complement. A space is *open-hereditarily irresolvable* if it is crowded and every nonempty open subset is irresolvable. Finally, a space is *hereditarily irresolvable* if it is crowded and every crowded subspace of it is irresolvable.

A crowded space X is called *van Douwen* if its topology is maximal among all crowded topologies (no separation axioms required on these topologies) on X. (The tricky thing about van Douwen spaces is of course that their topologies are regular.) The following result characterizes the van Douwen spaces. It nicely splits the maximality of the topology in three 'independent' pieces. (We remind the reader that all spaces considered here are Tychonoff.)

THEOREM 2 (van Douwen [5, Theorem 2.2]). *For a crowded space X the following are equivalent:*

(1) *X is van Douwen,*

(2) *X is extremally disconnected, open-hereditarily irresolvable, and nodec.*

By using an interesting and nontrivial Zorn's Lemma argument, van Douwen [5, Example 3.3] proved that there are countable van Douwen spaces. In §3 of the present paper, we will prove by a different technique that there are 'many' van Douwen spaces which are dense in the absolute of the Cantor cube $2^{\mathfrak{c}}$. It is interesting to note that no dense subspace of the Cantor cube itself can be extremally disconnected.

If Y is a space with subspace X then

$$N(X) = \bigcup\{\overline{D} : D \text{ is a countable discrete subset of } X\}.$$

Observe that $X \subseteqq N(X)$. We call a point of $N(X)$ a *near-point* of X. Put $F(X) = Y \setminus N(X)$. Then $F(X)$ consists of all the *far-points* of X.

If Y is a space with subspace X, then a point $y \in Y \setminus X$ is said to be *remote* from X provided that $x \notin \overline{E}$ for any nowhere dense subset E of X. Observe that if X is countable and nodec, then remote and far are equivalent notions. Also observe that if x is remote from X, then x is remote from any subspace of X.

The following triviality is the key to our construction.

LEMMA 3. *Let X be a space. Suppose that every $x \in X$ is remote from $X \setminus \{x\}$. Then X is nodec.*

In the context of Čech–Stone compactifications, near-points and remote points are very well studied in the literature. Van Douwen [4] and, independently, Chae and Smith [2] proved that if X is a nonpseudocompact space with countable π-weight, then $X^* = \beta X \setminus X$ contains a point that is remote from X. This result was generalized to products of such spaces by Dow [8]. There are also many spaces without remote points, but it is not the place here to go into that.

3. Proof of Theorem 1

We will now present the proof of Theorem 1. It will be convenient to break it up into several pieces. We first prove the easiest part of the theorem, i.e., the necessity of X being open-hereditarily irresolvable if X' is van Douwen.

LEMMA 4. *Let $X' \subseteqq E(2^{\mathfrak{c}})$ be such that $X = p_{2^{\mathfrak{c}}}(X')$ is dense in $2^{\mathfrak{c}}$. Then X' is open-hereditarily irresolvable if and only if X is.*

PROOF. Since $p_{2^{\mathfrak{c}}}$ is irreducible, X' is dense in $E(2^{\mathfrak{c}})$, and hence crowded. Assume now first that X' is open-hereditarily irresolvable. Let U be a nonempty open subset of X. If A is a dense subset of U, then $p_{2^{\mathfrak{c}}}^{-1}(A) \cap X'$ is a dense subset of $p_{2^{\mathfrak{c}}}^{-1}(U) \cap X'$ by irreducibility of $p_{2^{\mathfrak{c}}}$, hence $(p_{2^{\mathfrak{c}}}^{-1}(U) \cap X') \setminus p_{2^{\mathfrak{c}}}^{-1}(A)$ is not dense in X', which means that $U \setminus A$ is not dense in U, again by irreducibility. This proof obviously works both ways. □

COROLLARY 5. *Let $X' \subseteqq E(2^{\mathfrak{c}})$ be countable and nodec such that $X = p_{2^{\mathfrak{c}}}(X')$ is dense in $2^{\mathfrak{c}}$. If X is open-hereditarily irresolvable then X' is van Douwen.*

PROOF. Clearly, X' is dense, since X is and $p_{2^{\mathfrak{c}}}$ is irreducible. Therefore X' is extremally disconnected since it is a dense subspace of an extremally disconnected space. So we are done by Theorem 2 and Lemma 4. □

We now prove the other part of our theorem. In fact, we prove a slightly more general result than strictly needed. However, this will be precisely what we need in §4. To begin with, we first prove a result that in our opinion is of independent interest.

LEMMA 6. *Let S be a closed nowhere dense G_δ-set in $2^{\mathfrak{c}}$. Then for every $x \in S$ there is an element $x' \in p_{2^{\mathfrak{c}}}^{-1}(x)$ such that x' is remote from $p_{2^{\mathfrak{c}}}^{-1}(2^{\mathfrak{c}} \setminus S)$.*

PROOF. We will first prove the weaker statement that there are $y \in S$ and $y' \in p_{2^{\mathfrak{c}}}^{-1}(y)$ such that y' is remote from $p_{2^{\mathfrak{c}}}^{-1}(X)$, where $X = 2^{\mathfrak{c}} \setminus S$. Indeed, there are a countable set $A \subseteqq \mathfrak{c}$ and a closed subset $T \subseteqq 2^A$ such that $S = \pi_A^{-1}(T)$. Here $\pi_A : 2^{\mathfrak{c}} \to 2^A$ denotes the projection. Observe that T is nowhere dense in 2^A, and that

$$(2^A \setminus T) \times 2^{\mathfrak{c} \setminus A} = X.$$

Hence, by Dow [8, Theorem 2.5], there is a point $z \in \beta X$ that is remote from X. Let $f : \beta X \to 2^{\mathfrak{c}}$ be the natural map. Observe that f is irreducible, hence $E(\beta X) = E(2^{\mathfrak{c}})$. In fact, since the composition of two irreducible maps is irreducible, it is clear that

$$p_{2^{\mathfrak{c}}} = f \circ p_{\beta X}.$$

Pick an arbitrary point $y' \in p_{\beta X}^{-1}(z)$ (in fact, since z is remote from X, it is easily seen that $p_{\beta X}^{-1}(z)$ is a single point). Observe that y' is remote from $p_{\beta X}^{-1}(X) = p_{2^{\mathfrak{c}}}^{-1}(X)$ since $p_{\beta X}$ is irreducible. So we conclude that if $y = f(z)$, then $y \in S$, $y' \in p_{2^{\mathfrak{c}}}^{-1}(y)$ and is remote from $p_{2^{\mathfrak{c}}}^{-1}(X)$.

To end the proof, we first claim that we may assume without loss of generality that $T \approx 2^{\omega}$. This can be achieved quite easily by, if necessary, enlarging A with countably infinitely many elements and by using the trivial fact that the product of any compact zero-dimensional metrizable space with 2^{ω} is homeomorphic to 2^{ω}.

Next, we claim that for every $s \in S$ there is a homeomorphism $\xi : 2^{\mathfrak{c}} \to 2^{\mathfrak{c}}$ such that $\xi(S) = S$ and $\xi(s) = y$. Indeed, let $s_A = \pi_A(s)$ and $y_A = \pi_A(y)$, respectively. Since $T \approx 2^{\omega}$, there is a homeomorphism $\eta : T \to T$ such that $\eta(s_A) = y_A$. Since T is nowhere dense in 2^A, by a well-known homeomorphism extension theorem by Knaster and Reichbach [13], we may extend η to a homeomorphism $\bar{\eta} : 2^A \to 2^A$. Now let $B = \mathfrak{c} \setminus A$, and $\pi_B : 2^{\mathfrak{c}} \to 2^B$ be the projection. If $s_B = \pi_B(s)$ and $y_B = \pi_B(y)$, then there clearly is a homeomorphism $\theta : 2^B \to 2^B$ such that $\theta(s_B) = y_B$. Then $\xi = \bar{\eta} \times \theta$ is as required.

Now pick an arbitrary $s \in S$, and let the homeomorphism ξ be as above. Simply observe that there is a homeomorphism $E(\xi)$ of $E(2^{\mathfrak{c}})$ such that the diagram

$$\begin{array}{ccc} E(2^{\mathfrak{c}}) & \xrightarrow{E(\xi)} & E(2^{\mathfrak{c}}) \\ {\scriptstyle p_{2^{\mathfrak{c}}}}\downarrow & & \downarrow{\scriptstyle p_{2^{\mathfrak{c}}}} \\ 2^{\mathfrak{c}} & \xrightarrow[\xi]{} & 2^{\mathfrak{c}} \end{array}$$

commutes. Then $s' = E(\xi)^{-1}(x) \in p_{2^{\mathfrak{c}}}^{-1}(s)$ and is clearly remote from $p_{2^{\mathfrak{c}}}^{-1}(X)$ since $E(\xi)$ is a homeomorphism. $\square$

This leads us to the result we are after.

Corollary 7. *Let X be a countable dense subset of $2^{\mathfrak{c}}$. Then there is a dense nodec subspace $X' \subsetneqq E(2^{\mathfrak{c}})$ such that $p_{2^{\mathfrak{c}}}(X') = X$ and $p_{2^{\mathfrak{c}}} \restriction X'$ is 1-to-one.*

PROOF. Let $\{S_x : x \in X\}$ be a pairwise disjoint collection of closed G_δ-subsets of $2^{\mathfrak{c}}$ such that $x \in S_x$ for every $x \in X$. Observe that every S_x is evidently nowhere dense. Now for every $x \in X$, pick, by Lemma 6, a point $x' \in p_{2^{\mathfrak{c}}}^{-1}(x)$ such that x' is remote from $p_{2^{\mathfrak{c}}}^{-1}(2^{\mathfrak{c}} \setminus S_x)$. We claim that $X' = \{x' : x \in X\}$ is as required. To this end, pick an arbitrary $x \in X$. Then $K = \big\{ y' : y \in X \setminus \{x\} \big\}$ is a subset of $p_{2^{\mathfrak{c}}}^{-1}(2^{\mathfrak{c}} \setminus S_x)$, so x' is remote from K. Hence X' is nodec by Lemma 3. That X' is dense is obvious from the fact that $p_{2^{\mathfrak{c}}}$ is irreducible. $\square$

To see that the proof of Theorem 1 is indeed complete, let X be a countable open-hereditarily irresolvable subspace of $2^{\mathfrak{c}}$, and let X' be as in Corollary 7 for X. Then X' is nodec and open-hereditarily irresolvable by Lemma 4. So X' is a van Douwen space by Corollary 5. On the other hand, if X' is van Douwen, then it is open hereditarily-resolvable by Theorem 2, hence X is, again by Lemma 4.

Theorem 1 would be useless if there were no open-hereditarily irresolvable dense subsets of $2^{\mathfrak{c}}$. Fortunately, there are many such subspaces. Indeed, Zorn's Lemma implies that an independent family of infinite subsets of $\mathbb{N}$ can be extended to a maximal independent family. It is well-known that an independent family of cardinality $\mathfrak{c}$ corresponds directly to a countable dense subset of $2^{\mathfrak{c}}$, which, if the family is maximal, will be irresolvable. Also, Alas, Sanchis, Tkačenko, Tkachuk, and Wilson [1] present in Theorem 2.3 of their paper an example of a countable dense irresolvable subspace X of $2^{\mathfrak{c}}$. By van Douwen [5, Fact 3.1], X contains a nonempty open hereditarily irresolvable subspace, say U. Let U' be an open subspace of $2^{\mathfrak{c}}$ such that $U' \cap X = U$, and let C be a nonempty clopen subset of $2^{\mathfrak{c}}$ contained in U'. Since $C \approx 2^{\mathfrak{c}}$, we are done since $U \cap C$ is a countable dense hereditarily irresolvable subspace of C.

The proof of Corollary 7 would be simpler, if we could pick for every $x \in 2^{\mathfrak{c}}$ a point $x' \in p_{2^{\mathfrak{c}}}^{-1}(x)$ that is remote from $p_{2^{\mathfrak{c}}}^{-1}\big(2^{\mathfrak{c}} \setminus \{x\}\big)$. But this is unfortunately not possible, as the next argument shows (alternatively, use the main result in Terada [19]). Fix $x \in 2^{\mathfrak{c}}$, and let σ be a maximal cellular family of clopen subsets of $2^{\mathfrak{c}} \setminus \{x\}$. Put $S = 2^{\mathfrak{c}} \setminus \bigcup \sigma$. We claim that

$$p_{2^{\mathfrak{c}}}^{-1}(x) \subseteq \overline{p_{2^{\mathfrak{c}}}^{-1}\big(S \setminus \{x\}\big)},$$

which is as required since $p_{2^{\mathfrak{c}}}^{-1}\big(S \setminus \{x\}\big)$ is a nowhere dense subset of $p_{2^{\mathfrak{c}}}^{-1}\big(2^{\mathfrak{c}} \setminus \{x\}\big)$. To prove this, assume that U is a clopen subset of $E(2^{\mathfrak{c}})$ that intersects $p_{2^{\mathfrak{c}}}^{-1}(x)$ but misses $\overline{p_{2^{\mathfrak{c}}}^{-1}\big(S \setminus \{x\}\big)}$. Then $p_{2^{\mathfrak{c}}}(U)$ is a regular closed subset of $2^{\mathfrak{c}}$ that contains x but misses $S \setminus \{x\}$. Since σ is countable, S is a G_δ-subset of $2^{\mathfrak{c}}$. And so is $p_{2^{\mathfrak{c}}}(U)$ being a regular closed set in $2^{\mathfrak{c}}$. Since $S \cap p_{2^{\mathfrak{c}}}(U) = \{x\}$, this violates x having uncountable character.

Not every countable space can be 'lifted' to a nodec space in the absolute of its own Čech–Stone compactification, as the following trivial example shows. Let X be a countable van Douwen space, let $Y = \omega \times X$, and fix a point $x \in X$. In the space βY, pick any point $q \in Y^*$ that is a limit point of $\omega \times \{x\}$. Then $Y' = Y \cup \{q\}$ cannot be 'lifted' to a nodec space in $E(\beta Y')$. This becomes clear once one realizes that Y' is extremally disconnected, hence the absolute of $\beta Y' = \beta Y$ is βY.

4. The examples

Van Douwen [5] constructed his example by a Zorn's Lemma argument. We were unable to get our examples in a similar way. Instead, we use Theorem 1.

Our aim is to construct for every n a space having a family of n dense van Douwen subspaces $\mathcal{A}$ that are 'far' from one another, i.e, $N(A) \cap N(A') = \emptyset$ for all distinct $A, A' \in \mathcal{A}$.

Let X be any countable dense open-hereditarily irresolvable subspace of $2^{\mathfrak{c}}$, see §3. Let $x_0 = \underline{0}$, the point in $2^{\mathfrak{c}}$ having all coordinates equal to 0. Assume that $x_0, \ldots, x_i$ have been defined. Let x_{i+1} be any point in $2^{\mathfrak{c}} \setminus \bigcup_{j \leqq i}(x_i + X)$. Then

$$\mathcal{X} = \{x_i + X : i < \omega\}$$

is a pairwise disjoint collection dense homeomorphic copies of X in $2^{\mathfrak{c}}$ (in fact, one can continue in this way to get $2^{\mathfrak{c}}$ dense homeomorphic copies of X in $2^{\mathfrak{c}}$). Now fix n, and for every $i \leqq n-1$, let $X_i = x_i + X$. By Corollary 7, there is a countable nodec subspace Y of $E(2^{\mathfrak{c}})$ such that $\pi_{2^{\mathfrak{c}}}(Y) = \bigcup_{i<n} X_i$. Pick Y_i in Y such that $p_{2^{\mathfrak{c}}}(Y_i) = X_i$. Then Y_i is nodec, being a subspace of a nodec space. Hence Y_i is van Douwen by Corollary 5. To see that $N(Y_i) \cap N(Y_j) = \emptyset$ if $i \neq j$, simply observe that disjoint countable discrete subsets of Y will have disjoint closures in $\beta Y = E(2^{\mathfrak{c}})$ since Y is normal and nodec.

Now let $f_i : \mathbb{N} \to Y_i$ be a bijection, and let $\phi_i : \beta\mathbb{N} \to \beta Y$ be its Stone extension. Since $\beta Y_i = \beta Y$, and Y_i is van Douwen, it follows by van Douwen [5, Theorem 4.9] that

(I) if $x \in N(Y_i)$, then $\left|\phi_i^{-1}(x)\right| = 2$,

(II) if $x \in F(Y_i)$, then $\left|\phi_i^{-1}(x)\right| = 1$.

The topological sum of $n+1$ copies of $\beta\mathbb{N}$ is $\beta\mathbb{N}$, and consequently maps onto βY by a map g_n having the following properties:

(III) if $x \in \bigcup_{i \leq n} N(Y_i)$, then $\left|g_n^{-1}(x)\right| = n+2$,

(IV) if $x \in F(Y)$, then $\left|g_n^{-1}(x)\right| = n+1$.

This completes the construction of the examples.

Note that the construction ensures that $F(Y)$ is not empty. Therefore, we have actually proven something stronger.

THEOREM 8. *For each $n \geqq 2$, there is a map from $\beta\mathbb{N}$ onto $E(2^{\mathfrak{c}})$ such that every fiber has size either n or $n-1$ and each are realized.*

5. The nonexistence of n-to-one images

We now prove that, under CH, the examples constructed in the previous section are optimal.

THEOREM 9 (CH). *Let $f : \beta\mathbb{N} \to K$ be $(\leqq)n$-to-one, where K is crowded. Then there is a point $x \in K$ such that $\left|f^{-1}(x)\right| \leqq n-1$.*

PROOF. Put $X = f(\mathbb{N})$. It is clear that $f(\mathbb{N}^*) = K$, hence there is a closed subset E of $\mathbb{N}^*$ on which f is irreducible. Put $\phi = f{\restriction}E$, and $Q = \phi^{-1}(X)$. Since ϕ is irreducible, Q is dense in E. Also observe that E is crowded, hence so is Q.

Let S be a nonempty closed G_δ-subset of E which is contained in $E \setminus Q$. In addition, let $\mathcal{A}$ be a clopen partition of $\beta\mathbb{N}$ into nonempty sets such that $|\mathcal{A}| = n$. Put

$$N(\mathcal{A}) = f^{-1}\Big(\bigcap_{A\in\mathcal{A}} f(A)\Big).$$

CLAIM 1. $S \subsetneqq N(\mathcal{A})$.

Pick $A \in \mathcal{A}$ such that $S \cap A \neq \emptyset$. Replacing S by $S \cap A$, we may assume that $S \subseteqq A$. Since ϕ is irreducible, there is an open subset V of K such that $\phi^{-1}(V)$ is a dense open subset of $A \cap E$. Observe that $A = \overline{A \cap \mathbb{N}}$, hence

$$V \subseteqq f(A) = f(\overline{A \cap \mathbb{N}}) \subseteqq \overline{f(A \cap \mathbb{N})}. \tag{1}$$

Let $\{U_i : i < \omega\}$ be a decreasing sequence of clopen subsets of $A \cap E$ such that $\bigcap_{i<\omega} U_i = S$. For every i, the set $U_i \cap \phi^{-1}(V)$ is nonempty, hence there is a nonempty open subset U_i' of V such that $\phi^{-1}(U_i') \subseteqq U_i$. By (1), we may pick an element $a_i \in A$ such that $f(a_i) \in U_i'$. Then $\emptyset \neq \phi^{-1}\big(f(a_i)\big) \subseteqq U_i$, hence we may pick a point $d_i \in \phi^{-1}\big(f(a_i)\big)$. Put $D = \{d_i : i < \omega\}$.

We make a few observations.

(I) $f(d_i) = f(a_i)$ for every i. This is clear.

(II) $\overline{D} \setminus D \subseteqq S$. This is clear since the sequence of U_i's is decreasing. Hence D is a closed and discrete subset of Q. Since Q is crowded, this implies that $\overline{D}$ is a nowhere dense subset of E.

(III) $\phi(\overline{D})$ is nowhere dense in K. This is clear by (II) and the fact that ϕ is irreducible.

Put $V = \{a_i : i < \omega\}$. Then $f(V) = \phi(D)$. We will prove that $\overline{V} \cap E = \emptyset$. Striving for a contradiction, assume that $\overline{V} \cap E \neq \emptyset$. Since $\overline{V}$ is clopen in $\beta\mathbb{N}$, and ϕ is irreducible, there is an open subset W of K such that $\phi^{-1}(W)$ is a dense open subset of $\overline{V} \cap E$. But this is impossible since

$$\emptyset \neq W \subseteqq f(\overline{V}) \subseteqq \overline{f(V)} = \overline{\phi(D)} \subseteqq \phi(\overline{D}),$$

and $\phi(\overline{D})$ is nowhere dense in K by (III).

Now to finish the proof of the claim, pick an arbitrary point $p \in \overline{D} \setminus D$. Then $\phi(p) \in \overline{\phi(D)} = \overline{f(V)}$. Since $\overline{V} \cap E = \emptyset$, it consequently follows that $\left|f^{-1}\big(f(p)\big) \cap A\right| \geqq 2$. But this means that $p \notin N$, as required.

Since the weight of $\beta\mathbb{N}$ is $\mathfrak{c}$, we may by CH list all nonempty clopen subsets of $\beta\mathbb{N}$ as $\{B_\alpha : \alpha < \omega_1\}$. We assume without loss of generality that $B_0 = \beta\mathbb{N}$. In addition, we may list all the clopen partitions of $\beta\mathbb{N}$ into n nonempty sets as $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$. Let S_0 be any closed G_δ-subset of E that misses Q. By transfinite induction on $\alpha < \omega_1$, we will construct a nonempty closed G_δ-subset S_α of E having the following properties:

(1) if $\beta < \alpha$ then $S_\alpha \subseteqq S_\beta$,
(2) either $S_\alpha \subseteqq B_\alpha$ or $S_\alpha \cap B_\alpha = \emptyset$,
(3) $S_\alpha \cap N(\mathcal{A}_\alpha) = \emptyset$.

The construction is a triviality. Assume that S_β has been defined for all $\beta < \alpha$. Put $S = \bigcap_{\beta<\alpha} S_\beta$. Then S is a nonempty closed G_δ-subset of E. By Claim , there is an element $p \in S \setminus N(\mathcal{A}_\alpha)$. Since $N(\mathcal{A}_\alpha)$ is closed, there is a clopen neighborhood of p that misses $N(\mathcal{A}_\alpha)$ and has the property that it either misses B_α or is contained in it. Put $S_\alpha = C \cap S$. It is clear that S_α satisfies our inductive hypotheses.

By (1) and (2), there is a unique point $x \in \bigcap_{\alpha<\omega_1} S_\alpha$. We claim that $\left|f^{-1}\big(f(x)\big)\right| \leqq n-1$. If not, then there exists $\alpha < \omega_1$, such that $x \in N(\mathcal{A}_\alpha)$. But this contradicts (3) since $x \in S_\alpha$. □

6. Remarks

Reversing the order. The order in which the van Douwen's space is created by Zorn's Lemma is quite interesting. First a maximal regular space is constructed, and then some points are removed to get a maximal space (that is obviously regular being a subspace of a regular space). By doing things in opposite order, one can get into serious troubles, as the next example shows.

Let $\mathrm{Seq} = \mathbb{N}^{<\omega}$ be the set of finite sequences of elements of $\mathbb{N}$, and let $p \in \mathbb{N}^*$. Then a set $U \subseteqq \mathrm{Seq}$ is open if for every $t \in U$ the set $\{n \in \mathbb{N} :$

$t^\frown n \in U\}$ belongs to p. Let τ denote the topology on Seq. Then τ is regular, extremally disconnected, and crowded, see for example [9] for more information. There are a few statements about τ that are very useful. First, if $t \in \mathbb{N}^{<m}$, then the neighborhood trace of t on $\mathbb{N}^m$ is an ultrafilter. Second, if $Y \subseteqq \text{Seq}$ and $t \in \text{Seq}$ then t is in the τ-closure of A if and only if there is an n such that t is in the τ-closure of $\mathbb{N}^n \cap Y$.

Now we strengthen τ by also declaring, for each $A \notin p$, the set $\bigcup_{n \in A} \mathbb{N}^n$ to be closed. Let ρ denote the resulting topology. Then ρ is crowded, and evidently satisfies the following:

Lemma 10. *If $Y \subseteqq \text{Seq}$ and $t \in \text{Seq}$, then t is in ρ-closure Y if and only if $\{n : t \text{ is in } \tau\text{-closure of } \mathbb{N}^n \cap Y\} \in p$.*

Therefore, no $t \in \text{Seq}$ is in the ρ-closure of two disjoint subsets of Seq, i.e., ρ is maximal. But ρ cannot be used to construct a maximal topology that is regular.

Lemma 11. *No crowded subset of* Seq *is regular.*

Proof. Let $A \subseteqq \text{Seq}$ be crowded. Then A is open by maximality. Pick an arbitrary element $t \in A$. We may assume without loss of generality that $t = \emptyset$. Then $A' = A \setminus \mathbb{N}^1$ is a neighborhood of $\emptyset$. Let B be a neighborhood of $\emptyset$ that is contained in A'. We may assume that B has the form

$$C \setminus \bigcup_{n \in D} \mathbb{N}^n,$$

where C is a τ-open neighborhood of $\emptyset$ and $D \notin p$. There is an element $E \in p$ such that

$$\{\emptyset^\frown n : n \in E\} \subseteqq C.$$

Now take any $n \in E$. Then, clearly, $n = \emptyset^\frown n$ belongs to the τ-closure of $C \cap \mathbb{N}^m$, for every $m > 1$. Hence n belongs to the τ-closure of $C \cap \mathbb{N}^m$ for every $m \in \mathbb{N} \setminus \big(D \cup \{1\}\big) \in p$. But this implies by lemma 10 that n belongs to the ρ-closure of B. $\square$

Idempotents in $\beta\mathbb{N}$**.** Van Douwen spaces have surfaced at an unexpected place. It was shown by Protasov, that there is an ultrafilter $p \in \mathbb{N}^*$ such that, among other things, p is an idempotent such that the set $\{p + n : n \in \mathbb{N}\}$ is a van Douwen space. This is easily deduced (and known) from the comments immediately following Theorem 3.9 of Hindman and Strauss [12]. So there is an example of a homogeneous van Douwen space in ZFC.

Homogeneous van Douwen spaces. We do not know whether we can pick X in Corollary 7 in such a way that X' can be homogeneous. The first naive and obvious thing one thinks of is the following. Let X be a subgroup of

$2^{\mathfrak{c}}$, pick a point in $E(2^{\mathfrak{c}})$ remote in the sense of Lemma 6, and 'translate it around in $E(2^{\mathfrak{c}})$' to ensure homogeneity of X'. But this does not work since no infinite Abelian totally bounded topological group is irresolvable by a result of Comfort, Gladdines and van Mill [3]. This leads us to the following question: can there be a dense homogeneous van Douwen space in $E(2^{\mathfrak{c}})$? Observe that there are consistent examples of van Douwen groups by Malyhin [16]. And that there are homogeneous van Douwen spaces in ZFC by the result of Protasov just quoted.

The character of van Douwen spaces. The van Douwen spaces constructed in this paper are dense subspaces of $E(2^{\mathfrak{c}})$, and hence have character $\mathfrak{c}$. It would be interesting to have a consistent example of a van Douwen space with character less than $\mathfrak{c}$.

The π-character of van Douwen spaces. The van Douwen spaces constructed in this paper are dense subspaces of $E(2^{\mathfrak{c}})$, and hence have π-character $\mathfrak{c}$. There are consistent examples of van Douwen spaces that have π-character less than $\mathfrak{c}$. This follows from the following observations. As usual, $\mathfrak{i}$ denotes the least cardinal of a maximal independent family of infinite subsets of ω. By the same arguments as in §3, $2^{\mathfrak{i}}$ contains a countable dense hereditarily irresolvable space, hence $E(2^{\mathfrak{i}})$ contains a countable dense van Douwen space. Now all one needs to do is to apply the result in Kunen [14, VIII, Exercise A13] (see also Shelah [18] for a stronger result) that there is a model where $\mathfrak{i} < \mathfrak{c}$. So consistently, $2^{\mathfrak{c}}$ and $2^{\mathfrak{i}}$ are non-homeomorphic Čech–Stone compactifications of van Douwen spaces. For many more such examples, see Dow [6].

Uncountable van Douwen spaces. For which cardinals κ can there exist a ccc van Douwen space of density κ? The techniques in this paper can be modified to produce examples up to density $\mathfrak{c}$ and other uncountable van Douwen spaces with greater density [10].

Other point pre-images. It would be interesting to further explore finite to one maps from $\beta\mathbb{N}$. Specifically, it would be interesting to determine which finite sets $F \subsetneqq \mathbb{N}$ have the property that there is a map f from $\beta\mathbb{N}$ onto a crowded space K, such that $F = \left\{ \left| f^{-1}(y) \right| : y \in K \right\}$.

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