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On one-point connectifications

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Abstract

In the first part of this note we explore the relationship between connectibility and cohesiveness, including showing that the concepts do not coincide in the class of totally disconnected spaces. We introduce the concept of strong cohesion which fits between cohesion and connectibility. Several examples demonstrate the sharpness of the obtained results. In the second part of this note we investigate when certain one-point connectifications have the fixed point property. In particular, we prove this property for the canonical one-point connectification of Erdős space. This result was claimed earlier in the literature but was withdrawn recently.

MSC: 54H25; 54D35; 54G20

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1. Introduction

All spaces in this note are assumed to be separable metric. We call a space X *connectible* if the space has a *one-point connectification*, that is, there is a connected space Y that contains X such that $Y \setminus X$ is a singleton. A space is called *cohesive* if it has an open covering no element of which contains a nonempty clopen subset of the space. Erdős space $\mathfrak{E}$ and complete Erdős space $\mathfrak{E}_c$ are important examples of cohesive spaces (see [15]) and the concept plays a crucial role in characterizing $\mathfrak{E}$, $\mathfrak{E}_c$, and $\mathfrak{E}_c^\omega$; see Dijkstra and van Mill [8–10] and Dijkstra [7]. Both $\mathfrak{E}$ and $\mathfrak{E}_c$ belong to (and are even universal elements of) the class of almost zero-dimensional spaces; see [23,19,9]. This concept lies between zero-dimensionality and total disconnectedness. It is shown in [9, Lemma 6.5] that every connectible space is cohesive and that for almost zero-dimensional spaces the concepts coincide. In Section 3 we give a useful intrinsic characterization of connectibility and we explore the relationship between connectible and cohesive spaces further. In particular, we show that there exists a totally disconnected space that is cohesive but not connectible. We also introduce the concept of strong cohesion which fits between cohesion and connectibility. We show that strong cohesion is equivalent to cohesion for discontinuous spaces and equivalent to connectibility for locally compact or locally connected spaces.

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If Y is a one-point connectification of a hereditarily disconnected space X, then we say that the point $p \in Y \setminus X$ is a *dispersion point* of Y. If X is totally disconnected, then we call p an *explosion point*. A space Y is said to have the *fixed point property* if every continuous map $f : Y \to Y$ has a fixed point. The issue of the connection between dispersion points and the fixed point property was raised by Cobb and Voxman [2]. They proved that the original example of a dispersion point space, the Knaster–Kuratowski fan [21], has the fixed point property. In Section 4 we consider a class of explosion point spaces that have the fixed point property. More specific, we show that the canonical one-point connectification of a space $\mathcal{E}$ of 'Erdős type' has the fixed point property. In particular, both $\mathfrak{E}$ and $\mathfrak{E}_c$ have one-point connectifications with the fixed point property and also the end-point set plus the base point of the Lelek fan [22] has this property.

2. Preliminaries

We recall some definitions. A space X is called *totally disconnected* if for every two distinct points x and y there is a clopen subset of X that contains x and misses y. A space is called *hereditarily disconnected* if its components are singletons. A space is called *almost zero-dimensional* if every point has a neighbourhood basis consisting of sets that are intersections of clopen sets. This concept was originally introduced by Oversteegen and Tymchatyn [23]; see also [1,11]. Every zero-dimensional space is almost zero-dimensional, every almost zero-dimensional space is totally disconnected, and every totally disconnected space is hereditarily disconnected. A connectible and totally disconnected space is called *pulverized*; see [12,19].

It is observed in [11] and [9, Remark 6.2] that connectibility and cohesion are open hereditary and that the product of an arbitrary space with a connectible (cohesive) space is also connectible (cohesive). Every connectible space is cohesive and every almost zero-dimensional cohesive space is connectible; see [9, Lemma 6.5]. A cohesive space is at least one-dimensional at every point but the converse is not true; see [5]. The spaces $\mathfrak{E}$ and $\mathfrak{E}_c$ show that cohesive spaces can be almost zero-dimensional.

If X is a non-compact and connected space, then X is connectible since we can select a point p from any compactification K of X and put $Y = X \cup \{p\}$. Also note that no nonempty compact space can be connectible. So a connected space X is connectible if and only if X is not compact. It is clear that every non-degenerate connected space is cohesive.

3. Connectible and cohesive spaces

Knaster [20] gives an intrinsic characterization of connectibility as an answer to a question asked by P. Alexandroff. We have the following characterization of connectibility. If $\mathcal{U}$ is a collection of subsets of X and $A \subset X$, then we say that A is *finitely coverable by* $\mathcal{U}$ if there exists a finite $\mathcal{U}' \subset \mathcal{U}$ such that $A \subset \bigcup \mathcal{U}'$.

Theorem 1. *The following statements are equivalent*:

(1) *X is connectible.*
(2) *X can be embedded in a connected space Y as a proper open subset.*
(3) *There exists a metric d on X such that all nonempty clopen subsets of X are unbounded with respect to d.*
(4) *There exists an open covering $\mathcal{U}$ of X such that any nonempty clopen subset of X is not finitely coverable by $\mathcal{U}$.*

Note that criterion (4) gives an intrinsic characterization that highlights the connection with cohesion. The equivalence of (1) and (2) is already contained in [20].

Proof. (1) $\Rightarrow$ (2) is trivial and (3) $\Rightarrow$ (4) is easy; use, for instance, a cover consisting of open 1-balls.

(2) $\Rightarrow$ (3). Assume (2) and let d be an admissible metric on Y. Put $A = Y \setminus X$ and define the following metric on X,

$$\rho(x, y) = d(x, y) + \left| \frac{1}{d(x, A)} - \frac{1}{d(y, A)} \right| \quad \text{for } x, y \in X.$$

It is easy to see that ρ is an admissible metric on X. Let C be a nonempty clopen subset of X. Then C is open in Y but C is not closed in Y because $C \neq Y$. Thus $d(C, A) = 0$ and hence C has infinite ρ-diameter.

(4) $\Rightarrow$ (1). Let $\mathcal{U}$ be an open covering of X that satisfies condition (4) and let $\mathcal{B}$ be a countable basis of X that refines $\mathcal{U}$. Consider the countable set $\mathcal{D} = \{(B_1, B_2) \in \mathcal{B}^2\colon\ \overline{B_1} \subset B_2\}$ and select for every $D = (B_1, B_2) \in \mathcal{D}$ a continuous function $f_D : X \to [0, 1]$ such that $f_D(B_1) \subset \{1\}$ and $f_D(X \setminus B_2) \subset \{0\}$. Let h be the Alexandroff–Urysohn imbedding of X into the Hilbert cube $[0, 1]^{\mathcal{D}}$, given by $h(x)_D = f_D(x)$. Let $Y = h(X) \cup \{\mathbf{0}\}$, where $\mathbf{0}$ represents the element of $[0, 1]^{\mathcal{D}}$ with all coordinates equal to 0. Let C be a clopen subset of Y that does not contain $\mathbf{0}$. Then there exist a finite subset $\{D_1, \ldots, D_n\}$ of $\mathcal{D}$ such that the set $h^{-1}(C)$ is contained in $\bigcup_{i=1}^{n} f_{D_i}^{-1}((0, 1])$. Thus the clopen set $h^{-1}(C)$ is contained in the union of n elements of $\mathcal{B}$ and must be empty. Consequently, $C = \emptyset$ and Y is connected, which means that X is connectible. $\square$

It is easily seen that the implication (2) $\Rightarrow$ (4) is valid for regular spaces and that (4) $\Rightarrow$ (1) is true for Tychonoff spaces (use a hypercube instead of a Hilbert cube).

Since in the class of almost zero-dimensional spaces connectibility and cohesion coincide it is a natural question whether they also coincide in the class of totally disconnected spaces. The following result shows that the answer is no.

Proposition 2. *There exists a totally disconnected space that is cohesive but not connectible.*

Proof. Consider the compact space $Y = \Delta \times \mathbb{I}$ where Δ is a Cantor set and $\mathbb{I}$ denotes the interval $[0, 1]$. Let $\psi_1 : Y \to \Delta$ and $\psi_2 : Y \to \mathbb{I}$ be the projections. Consider the collection $\mathcal{C}$ of all closed subsets C of Y such that $|\psi_1(C)| = |\Delta| = \mathfrak{c}$. Noting that $|\mathcal{C}| = \mathfrak{c}$ we can write $\mathcal{C} = \{C_\alpha\colon\ \alpha < \mathfrak{c}\}$. We construct by transfinite recursion subsets X_α of Y such that for each $\alpha \leqslant \mathfrak{c}$ we have,

(1) $X_\beta \subset X_\alpha$ for each $\beta < \alpha$,
(2) $C_\beta \cap X_\alpha \neq \emptyset$ for each $\beta < \alpha$,
(3) $|X_\alpha| \leqslant |\alpha|$, and
(4) $\psi_1 \restriction X_\alpha$ is one-to-one.

For the basis step put $X_0 = \emptyset$. We note that the hypotheses are trivial or void for $\alpha = 0$. Assume now that X_β has been found for all $\beta < \alpha$. If α is a limit ordinal, then we put $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ and we note that the hypotheses are satisfied. If $\alpha = \gamma + 1$ then we note that $|\psi_1(X_\gamma)| \leqslant |\gamma| < \mathfrak{c} = |\psi_1(C_\gamma)|$. Select a $y \in C_\gamma$ such that $\psi_1(y) \notin \psi_1(X_\gamma)$. If we put $X_\alpha = X_\gamma \cup \{y\}$ then the hypotheses are trivially satisfied.

The induction being complete we consider the space $X_\mathfrak{c}$. Since $\psi_1 \restriction X_\mathfrak{c}$ is one-to-one and Δ is a Cantor set we have that $X_\mathfrak{c}$ is totally disconnected. Let U be a closed neighbourhood of a point $x \in \Delta$ and let $t \in \mathbb{I}$. Note that $U \times \{t\} \in \mathcal{C}$ so this set intersects $X_\mathfrak{c}$ by property (2). We may conclude that $X_\mathfrak{c}$ is dense in Y.

We show that $\psi_2(U)$ is dense in $\mathbb{I}$ for any nonempty clopen subset U of $X_\mathfrak{c}$ so that $\Delta \times [0, 2/3)$ and $\Delta \times (1/3, 1]$ form a cover that proves that $X_\mathfrak{c}$ is cohesive. To this end let U be a nonempty clopen subset of $X_\mathfrak{c}$ and let U' and V' be two (disjoint) open sets in Y such that $U = U' \cap X_\mathfrak{c}$ and $X_\mathfrak{c} \setminus U = V' \cap X_\mathfrak{c}$. So the compactum $B = Y \setminus (U' \cup V')$ is disjoint from $X_\mathfrak{c}$ and hence $|\psi_1(B)| < \mathfrak{c}$. This implies that $\psi_1(B)$ is nowhere dense in Δ. Since $\psi_1(U')$ is open and nonempty in Δ we may select an $x \in \psi_1(U') \setminus \psi_1(B)$. Thus the connected set $\{x\} \times \mathbb{I}$ is covered by the disjoint open sets U' and V' which means that $\{x\} \times \mathbb{I} \subset U'$. We have that $\psi_2(U') = \mathbb{I}$ and hence $\psi_2(U)$ is dense in $\mathbb{I}$ because $X_\mathfrak{c}$ is dense in Y.

We now prove that $X_\mathfrak{c}$ is not connectible by showing that the space does not satisfy condition (4) of Theorem 1. Let $\mathcal{U}$ be an arbitrary open covering of $X_\mathfrak{c}$ and put $\mathcal{U}' = \{O\colon\ O \text{ open in } Y,\ O \cap X_\mathfrak{c} \in \mathcal{U}\}$. So the set $B = Y \setminus \bigcup \mathcal{U}'$ is disjoint from $X_\mathfrak{c}$ thus $\psi_1(B)$ is a proper closed subset of Δ. Select a nonempty clopen subset C of Δ that is disjoint from $\psi_1(B)$. Then the compactum $C \times \mathbb{I}$ can be covered by finitely many elements of $\mathcal{U}'$. So the clopen subset $C' = X_\mathfrak{c} \cap (C \times \mathbb{I})$ in $X_\mathfrak{c}$ is finitely coverable by $\mathcal{U}$. Note that C' is nonempty since $X_\mathfrak{c}$ is dense in Y. $\square$

Let us say that a space X is *strongly cohesive* if it has an open covering $\mathcal{U}$ such that for every nonempty clopen subset C and every $U \in \mathcal{U}$ the set $C \setminus U$ is not compact. Clearly every strongly cohesive space is cohesive and by

(4) of Theorem 1 we have that every connectible space is strongly cohesive. It is easily seen that the example in Proposition 2 is strongly cohesive; cf. Theorem 9. As with cohesion and connectibility we have:

Proposition 3. *Strong cohesion is open hereditary and stable under products with arbitrary spaces.*

Proof. Let O be open in a strongly cohesive space X as witnessed by the cover $\mathcal{U}$. Let $\mathcal{V}$ consist of all open subsets V of O such that the closure $\overline{V}$ in X is contained in O and in some element of $\mathcal{U}$. Note that $\mathcal{V}$ covers O. Let C be a clopen nonempty subset of O and let $V \in \mathcal{V}$ be such that $C \setminus V$ is compact. Note that C is open in X. Also $C \cap \overline{V}$ is closed in X thus $C = (C \cap \overline{V}) \cup (C \setminus V)$ is clopen in X. Let $U \in \mathcal{U}$ be such that $V \subset U$ and thus $C \setminus U$ is compact, a contradiction.

Let X be strongly cohesive (with witnessing cover $\mathcal{U}$) and let Y be an arbitrary space. Put $\mathcal{V} = \{U \times Y\colon\ U \in \mathcal{U}\}$. Let C be a clopen nonempty subset of $X \times Y$ and let $U \in \mathcal{U}$ be such that $C \setminus (U \times Y)$ is compact. Select an $(x, y) \in C$ and note that $C' = \{z \in X\colon\ (z, y) \in C\}$ is a nonempty clopen subset of X such that $C' \setminus U$ is compact. □

Remark 4. The product of non-cohesive spaces is evidently non-cohesive thus we have that a product is cohesive if and only if at least one of the factors is cohesive; cf. [10, Proposition 8]. An analogous statement is not valid for strong cohesion: the space $\mathbb{Q} \times \mathbb{I}$ is clearly a strongly cohesive space (cf. Proposition 7) but neither $\mathbb{Q}$ nor $\mathbb{I}$ is strongly cohesive. These observations lead us to the following question.

Question 5. Is the product of non-connectible spaces always non-connectible?

The following proposition gives a partial answer.

Proposition 6. *Let X be a non-connectible space. If Y is non-cohesive then $X \times Y$ is non-connectible. If Y contains a nonempty open and compact subset then $X \times Y$ is non-connectible.*

Proof. First, let Y be non-cohesive and select a $y \in Y$ such that every open neighbourhood of y contains a non-empty clopen subset of Y. Let $\mathcal{U}$ be an open covering for $X \times Y$ such that every element has the form $U \times V$. We have that $\mathcal{V} = \{U\colon\ U \times V \in \mathcal{U}$ and $y \in V$ for some $V\}$ is an open cover of X. Since X is not connectible there is a nonempty clopen subset C of X such that $C \subset \bigcup_{i=1}^{n} U_i$ for some $U_i \times V_i \in \mathcal{U}$ with $y \in V_i$; see Theorem 1. There is a non-empty clopen subset $C' \subset \bigcap_{i=1}^{n} V_i$ in Y and hence $C' \times C$ is a non-empty clopen subset of $X \times Y$ that is covered by finitely many elements of $\mathcal{U}$. By Theorem 1 we have that $X \times Y$ is not connectible.

Now let Y contain a nonempty open and compact subset, say K. Let $\mathcal{U}$ be an arbitrary open covering for $X \times Y$. With the Tube Lemma select for every $x \in X$ an open neighbourhood U_x of x such that $U_x \times K$ is covered by finitely many elements of $\mathcal{U}$ and let C be a nonempty clopen subset in X that is finitely coverable by the elements of the open cover $\{U_x\colon\ x \in X\}$. So $C \times K$ is finitely coverable by the elements of $\mathcal{U}$. □

Note that a strongly cohesive space cannot contain a nonempty open compactum. However, every nontrivial connected space is cohesive. Thus every non-degenerate continuum is cohesive but not strongly cohesive.

Proposition 7. *Let X be a space such that every compact subset is nowhere dense. If X is cohesive then it is strongly cohesive.*

Proof. Let $x \in X$ and U be an open neighbourhood of x that does not contain any nonempty clopen subset of X. Let V be an open neighbourhood of x such that $\overline{V} \subset U$. Let C be a nonempty clopen subset of X with $C \setminus V$ compact. By our assumption $\operatorname{int}(C \setminus V) = \emptyset$ and so $C \subset \overline{V} \subset U$ which contradicts the cohesion assumption. □

Proposition 8. *There exists a cohesive space that is not strongly cohesive and that has no nonempty open and compact subset.*

Proof. Consider the subspace

$$X = \bigcup_{n\in\mathbb{N}} \bigl((0, 1] \times \{1/n\}\bigr) \cup \bigl\{(0, 0)\bigr\}$$

of the plane which clearly has no nonempty open compact subset. Note that the projection of every nonempty clopen subset onto the x-axis contains the interval $(0, 1]$ and hence the space is cohesive. For every open neighbourhood U of $(0, 0)$ we find that there is an $n \in \mathbb{N}$ such that $((0, 1] \times (\{1/n\}) \setminus U$ is compact thus X is not strongly cohesive (and not connectible). □

The example in this proposition contains non-degenerate continua. The following result shows that this feature is necessary. Recall that X is said to be *punctiform* or *discontinuous* if X does not contain non-degenerate continua. We have that in realm of discontinuous spaces the concepts cohesive and strongly cohesive are equivalent.

Theorem 9. *Suppose that X is discontinuous. If X is cohesive then X is strongly cohesive because it has an open cover $\mathcal{U}$ such that for every nonempty clopen set C and every $U \in \mathcal{U}$ the set $C \setminus U$ is not σ-compact.*

Proof. Let $\mathcal{V}$ be an open cover of X such that for every nonempty clopen set C and every $V \in \mathcal{V}$ we have $C \setminus V \neq \emptyset$. Let $\mathcal{U}$ be an open cover of X such that $\{\overline{U}\colon U \in \mathcal{U}\}$ refines $\mathcal{V}$. Let C be a nonempty clopen set in X and let $U \in \mathcal{U}$ and assume that $C \setminus U$ is σ-compact. Since X is discontinuous we have that $C \setminus U$ is zero-dimensional. Thus the space X is zero-dimensional at every point of the open set $C \setminus \overline{U}$. Since X is cohesive we must have that $C \setminus \overline{U} = \emptyset$. Since $\overline{U}$ is contained in an element V of $\mathcal{V}$ we have $C \setminus V = \emptyset$, a contradiction. □

An example as in Proposition 8 cannot be locally connected as follows from the next result.

Proposition 10. *For a space X such that all components are open the following statements are equivalent*:

(1) *No component of X is compact.*
(2) *X has no nonempty open and compact subset.*
(3) *X is strongly cohesive.*
(4) *X is connectible.*

Proof. The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious.

Assume (1) and select for each component an open cover without a finite subcover. The union $\mathcal{U}$ of all these covers form an open cover of the space. Now, every nonempty clopen subset of X must contain at least one component thus it is not finitely coverable and X is connectible by Theorem 1. □

Every cohesive space is obviously dense in itself. For locally connected spaces that condition is also sufficient:

Proposition 11. *A space such that all components are open is cohesive if and only if it is dense in itself.*

Proof. If every component is open and there are no isolated points, then $\{C \setminus \{x\}\colon x \in X, C \text{ the component of } x\}$ is the required open cover of X. □

Fedeli and Le Donne [16] show that a subset of $\mathbb{R}$ is connectible if and only if it is locally connected and it has no compact components. We have the following characterization for cohesive sets in $\mathbb{R}$.

Proposition 12. *A subset of real line is cohesive if and only if it is locally connected and dense in itself.*

Proof. In view of Proposition 11 it suffices to show that cohesive subsets of $\mathbb{R}$ are locally connected. Assume that $X \subset \mathbb{R}$ is not locally connected. Then there is a component C of X that is not open. Let $x \in C$ be a point that is not an interior point. So for each $\varepsilon > 0$ the open interval $(x - \varepsilon, x + \varepsilon)$ should meet $\mathbb{R} \setminus X$. Let $\alpha \in \mathbb{R} \setminus X$ and without loss of generality we assume that $x - \varepsilon < \alpha < x$. There is a point $y \in X \setminus C$ such that $\alpha < y < x + \varepsilon$. Then x and y lie in different components of X and so there is a point $\beta \in \mathbb{R} \setminus X$ between them. Thus $(\alpha, \beta) \cap X$ is a nonempty clopen set that is contained in $(x - \varepsilon, x + \varepsilon)$. We have that X is not cohesive. □

An example as in Proposition 8 also cannot be locally compact by the next result.

Proposition 13. *For a locally compact space X the following statements are equivalent*:

(1) *No component of X is compact.*
(2) *X has no nonempty open and compact subset.*
(3) *X is strongly cohesive.*
(4) *X is connectible.*
(5) *The one-point compactification of X is connected.*

Proof. The implications (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1). Let $A \subset X$ be a component that is compact. Since X is locally compact there is an open neighbourhood U of A such that $\overline{U}$ is compact. Clearly A is a component of $\overline{U}$ and so there exists a clopen subset W in $\overline{U}$ such that $A \subset W \subset U$; see [13, Theorem 6.2.24]. W is compact and open in X since it is clopen in $\overline{U}$ and contained in the open set U.

(1) $\Rightarrow$ (5). Assume that the one-point compactification $\alpha X = X \cup \{\infty\}$ is disconnected. Then there is a nonempty clopen subset C of αX that does not contain ∞. Note that C is clopen and compact in X. Thus the components of C are compact components of X. $\square$

In view of the space $\mathbb{Q} \times \mathbb{I}$ of Remark 4 and the example of Proposition 8 we have that Proposition 13 does not admit an extension over the class of σ-compact spaces.

4. Fixed explosion points

Let p be a point in a space X. We say that *p is a fixed point of X* if for every non-constant continuous function $f: X \to X$ we have $f(p) = p$. It is clear that if a space contains a fixed point, then it has the fixed point property. On the other hand, every non-degenerate compact AR is an example of a space with the fixed point property but without a fixed point. Katsuura [18] constructed a dispersion point space such that the dispersion point is not a fixed point and Gutek [17] showed the existence of a dispersion point space without the fixed point property. The spaces in their examples are based on the Knaster–Kuratowski fan [21] and consequently the dispersion points are not explosion points. Dijkstra [6] constructed an explosion point space without the fixed point property. The issue of fixed dispersion points was originally raised by Cobb and Voxman [2] who proved that the dispersion point in the Knaster–Kuratowski fan is a fixed point.

Lemma 14. *Let p be a point in a space X such that $X \setminus \{p\}$ is hereditarily disconnected. If for every open neighbourhood U of p with $U \neq X$ the component of p in U is not closed in X, then p is a fixed point of X.*

Proof. First note that X is connected because every clopen neighbourhood of p has only components that are closed in X thus it must be the whole space. Assume that $f: X \to X$ is a non-constant continuous function and that $f(p) = q \neq p$. Note that $f(X)$ is a non-degenerate connected subspace of X and thus cannot be contained in $X \setminus \{p\}$. Thus $U = X \setminus f^{-1}(p)$ is an open neighbourhood of the point p which is not equal to X. Let C be the component of p in U and hence a closed subset of U. Then $f(C)$ is a connected subset of $X \setminus \{p\}$ and thus $f(C) = \{q\}$. So $C \subset f^{-1}(q) \subset U$. It follows that C is a closed subset of the fibre $f^{-1}(q)$ and hence C is closed in X in contradiction to the assumptions. $\square$

Lemma 14 corresponds to Theorem 2 in Katsuura [18]. We give the lemma and its proof because [18, Theorem 2] is misformulated (the condition $U \neq X$ is missing which makes the statement void).

Let $p > 0$ and consider the (quasi-)Banach space ℓ^p. This space consists of all sequences $z = (z_1, z_2, \dots)$ of real numbers such that $\sum_{i=1}^{\infty} |z_i|^p < \infty$. The topology on ℓ^p is generated by the (quasi-)norm $\|z\| = (\sum_{i=1}^{\infty} |z_i|^p)^{1/p}$. We extend the p-norm over $\mathbb{R}^{\mathbb{N}}$ by putting $\|z\| = \infty$ for each $z \in \mathbb{R}^{\mathbb{N}} \setminus \ell^p$. If $A \subset \mathbb{R}^{\mathbb{N}}$ then we put $\|A\| = \sup\{\|z\|:\ z \in A\}$.

For the remainder of this note let $E_1, E_2, \dots$ be a fixed sequence of zero-dimensional subsets of $\mathbb{R}$ and let the 'Erdős type' space $\mathcal{E}$ be given by

$$\mathcal{E} = \{z \in \ell^p:\ z_n \in E_n \text{ for every } n \in \mathbb{N}\}$$

as a subspace of some fixed ℓ^p. If we choose $p = 2$ and $E_n = \mathbb{Q}$ for every n, then $\mathcal{E}$ is called *Erdős space* $\mathfrak{E}$; see Erdős [15] who proved that this space is one-dimensional. The space $\mathcal{E}$ is easily seen to be almost zero-dimensional. We shall use the following result of Dijkstra [4].

Theorem 15. *If $\mathcal{E} \neq \emptyset$ then the following statements are equivalent*:

(1) *There exists an $x \in \prod_{n=1}^{\infty} E_n$ with $\|x\| = \infty$ and* $\lim_{n\to\infty} x_n = 0$.
(2) *Every nonempty clopen subset of $\mathcal{E}$ is unbounded.*
(3) *$\mathcal{E}$ is cohesive.*
(4) $\dim \mathcal{E} > 0$.

Compare item (2) of Theorem 15 with (3) in Theorem 1.

Let the space $\mathcal{E}^+ = \mathcal{E} \cup \{\infty\}$ be an extension of $\mathcal{E}$ such that for every neighbourhood U of ∞ in $\mathcal{E}^+$ we have that $\|\mathcal{E} \setminus U\| < \infty$.

Theorem 16. *The following statements about $\mathcal{E}^+$ are equivalent*:

(1) *∞ is a fixed point of $\mathcal{E}^+$.*
(2) *$\mathcal{E}^+$ has the fixed point property.*
(3) *$\mathcal{E}^+$ is connected.*
(4) $\dim \mathcal{E} \neq 0$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial and (3) $\Rightarrow$ (4) follows from [14, Corollary 1.5.6].

(4) $\Rightarrow$ (1). Assume that $\dim \mathcal{E} \neq 0$. Since every E_n is zero-dimensional we have that $\mathcal{E}$ is totally disconnected. In view of Lemma 14 let U be an open neighbourhood of ∞ in $\mathcal{E}^+$ such that $A = \mathcal{E} \setminus U \neq \emptyset$. By assumption $\|A\| < \infty$. Let C be the component of ∞ in U. We introduce some notation. Let $\xi_k : \mathbb{R}^{\mathbb{N}} \to \ell^p$ be given by $\xi_k(x) = (x_1, \ldots, x_k, 0, 0, \ldots)$ for $k \in \omega$ and $x \in \mathbb{R}^{\mathbb{N}}$. If $y \in \prod_{n=1}^{\infty} E_n$ and $k \in \omega$, then we define $Y_k(y) = \{z \in \mathcal{E}\colon\ \xi_k(z) = \xi_k(y)\}$. Note that if $\mathcal{E}$ satisfies condition (1) of Theorem 15, then so does $Y_k(y)$ because changing finitely many coordinates of x does not affect the properties $\|x\| = \infty$ and $\lim_{i\to\infty} x_i = 0$. Since $\dim \mathcal{E} > 0$ we have that every nonempty clopen subset of every $Y_k(y)$ is unbounded.

We construct inductively a sequence of points $x^0, x^1, \ldots$ in A and natural numbers $n_0 < n_1 < \cdots$ such that for $i \in \mathbb{N}$,

(a) $x^i \in Y_{n_{i-1}}(x^{i-1})$ and
(b) $\|\xi_{n_i}(x^i)\| > s_i - 2^{-i}$, where $s_i = \|A \cap Y_{n_{i-1}}(x^{i-1})\|$.

For the basis step choose $x^0 \in A$ and $n_0 = 1$ and note that the properties (a) and (b) do not apply to this case. Suppose that x^i and n_i have been found. Choose an $x^{i+1} \in A \cap Y_{n_i}(x^i)$ such that $\|x^{i+1}\| > \|A \cap Y_{n_i}(x^i)\| - 2^{-i-1} = s_{i+1} - 2^{-i-1}$. Let $n_{i+1} > n_i$ be such that $\|\xi_{n_{i+1}}(x^{i+1})\| > s_{i+1} - 2^{-i-1}$ and note that the properties (a) and (b) are satisfied.

By property (a) we can now define $x \in \prod_{n=1}^{\infty} E_n$ by $\xi_{n_i}(x) = \xi_{n_i}(x^i)$ for each $i \in \omega$. Note that $\|x\| = \lim_{i\to\infty} \|\xi_{n_i}(x)\| \leqslant \sup_{i\in\omega} \|x^i\| \leqslant \|A\| < \infty$ thus $x \in \mathcal{E}$. As is well known ℓ^p comes equipped with a Kadec norm, which means that the norm topology is the weakest topology that makes the coordinate projections and the norm function $\|\cdot\|$ continuous. Since $\|x\| = \lim_{i\to\infty} \|\xi_{n_i}(x^i)\|$ and clearly $x_j = \lim_{i\to\infty} (\xi_{n_i}(x^i))_j$ for each $j \in \mathbb{N}$, we have that $x = \lim_{i\to\infty} \xi_{n_i}(x^i)$ in ℓ^p. Note that for $i \in \mathbb{N}$, $\|x^i\| \geqslant \|\xi_{n_i}(x^i)\| > s_i - 2^{-i} \geqslant \|x^i\| - 2^{-i}$ by properties (a) and (b) thus with the same argument we have $x = \lim_{i\to\infty} \xi_{n_i}(x^i) = \lim_{i\to\infty} x^i$. Since A is closed we have $x \in A$.

For each $i \in \mathbb{N}$ select a $y^i \in Y_{n_{i-1}}(x^{i-1})$ such that $\|y^i\| = \|x^i\| + 2^{-i}$. This is possible because if such a y^i does not exist, then $\{z \in Y_{n_{i-1}}(x^{i-1})\colon\ \|z\| < \|x^i\| + 2^{-i}\}$ is a bounded clopen subset of $Y_{n_{i-1}}(x^{i-1})$ that contains x^i. Note that $x = \lim_{i\to\infty} y^i$ again by the Kadec norm argument. We have $\|y^i\| \geqslant \|\xi_{n_i}(x^i)\| + 2^{-i} > s_i$. Select a $k > n_{i-1}$ such that $\|\xi_k(y^i)\| > s_i$ and note that $Y_k(y^i) \subset Y_{n_{i-1}}(x^{i-1}) \setminus A \subset U$. Consider the space $B = Y_k(y^i) \cup \{\infty\}$. If K is a clopen subset of B that does not contain ∞, then K is a clopen and bounded subset of $Y_k(y^i)$. Thus $K = \emptyset$ and we may conclude that B is connected. Since $B \subset U$ we have $y^i \in B \subset C$ for every $i \in \mathbb{N}$. We now have that x is a point in the closure of C that is not in U which means that we can apply Lemma 14 to obtain that ∞ is a fixed point of $\mathcal{E}^+$. □

If $\dim \mathcal{E} > 0$ then we call $\mathcal{E}^+$ the canonical one-point connectification of $\mathcal{E}$ if the neighbourhoods of ∞ are precisely the complements of the bounded subsets of $\mathcal{E}$.

Corollary 17. *The canonical one-point connectification of Erdős space* $\mathfrak{E}$ *has the fixed point property.*

Cobb and Voxman claim this result (without proof) in [2] using the representation of Roberts [24]. However, they [3] have withdrawn that claim.

In [4] it is shown that the space

$$\mathfrak{E}_c = \{x \in \ell^1\colon\ x_i \in \{0, 1/i\} \text{ for } i \in \mathbb{N}\}$$

is a representation of complete Erdős space such that the canonical one-point connectification corresponds to the end-point set plus the base point of the Lelek fan which leads to:

Corollary 18. *The end-point set together with the base point of the Lelek fan has the fixed point property.*

Example 19. Let $\mathcal{E}$ be such that $\dim \mathcal{E} > 0$ and let $\mathcal{E}^+$ denote the canonical one-point connectification. Define

$$\mathcal{E}^+_{\sin} = \{(x, \sin\|x\|)\colon\ x \in \mathcal{E}\} \cup \{(\infty, 0)\} \subset \mathcal{E}^+ \times [-1, 1].$$

Note that $\mathcal{E}^+_{\sin} \setminus \{p\}$ for $p = (\infty, 0)$ is homeomorphic to $\mathcal{E}$. We verify that $\mathcal{E}^+_{\sin}$ is connected. Let C be a clopen subset of $\mathcal{E}^+_{\sin}$ that contains p. Then C contains the set $\{(x, 0)\colon\ \|x\| > n,\ \sin\|x\| = 0\}$ for some $n \in \mathbb{N}$. Assume that $C \neq \mathcal{E}^+_{\sin}$ and select an $a \in \mathcal{E} \setminus C$. Let $k \in \mathbb{N}$ be such that $\pi k > \max\{\|a\|, n\}$. Then $\{x \in \mathcal{E} \setminus C\colon\ \|x\| < \pi k\} = \{x \in \mathcal{E} \setminus C\colon\ \|x\| \leqslant \pi k\}$ is a clopen bounded subset of $\mathcal{E}$ that contains a. This fact contradicts Theorem 15 and we may conclude that $\mathcal{E}^+_{\sin}$ is connected. Thus p is an explosion point of $\mathcal{E}^+_{\sin}$.

However, the proof of Theorem 16 does not work for $\mathcal{E}^+_{\sin}$ because Lemma 14 does not apply to $\mathcal{E}^+_{\sin}$. Consider the open neighbourhood $U = \mathcal{E}^+_{\sin} \cap (\mathcal{E}^+ \times (-1, 1))$ of p. Note that for every $k \in \omega$ the set $\{(x, t) \in U\colon\ \|x\| < \pi k + \frac{\pi}{2}\}$ is clopen in U. Thus we have that the component of p in U is $\{p\}$.

Question 20. Does the space $\mathcal{E}^+_{\sin}$ in Example 19 have the fixed point property, in particular when $\mathcal{E} = \mathfrak{E}$ or $\mathcal{E} = \mathfrak{E}_c$?

Dijkstra [6] has constructed a one-point connectification of a totally disconnected space that does not have the fixed point property. Since that example is not almost zero-dimensional one may ask:

Question 21. Is there a one-point connectification of an almost zero-dimensional space without the fixed point property?

Note that a negative answer to Question 20 is also a positive answer to Question 21.

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